PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 123, Number 10, October 1995

ON PRINCIPAL SECTIONS OF A PAIR OF FORMS

CHE-MAN CHENG

(Communicated by Lance W. Small)

ABSTRACT. Let H and C be $n \times n$ Hermitian matrices with C positive definite. Let $H(i_1, \dots, i_r)$ denote the submatrix of H formed by deleting the rows and columns $i_1, \dots, i_r$ of H. In this paper, with $r_1 + \cdots + r_k \leq n$, we study the roots of the determinantal equation $\det(\lambda C - H) = 0$ and those of

$$\det((\lambda C - H)(r_1 + \cdots + r_{i-1} + 1, \dots, r_1 + \cdots + r_i)) = 0$$

for $i = 1, \dots, k$.

1. INTRODUCTION

Let $\mathscr{M}_n$ denote the set of all $n \times n$ complex matrices. For $A \in \mathscr{M}_n$, let $A(i_1, \dots, i_r)$ denote the submatrix of A formed by deleting the rows and columns $i_1, \dots, i_r$ of A. Concerning the eigenvalues of a Hermitian matrix and those of its principal submatrices, we have the following well-known interlacing theorem (e.g. [2] or [3, pp. 185–188]).

Theorem A. *Let H be an $n \times n$ Hermitian matrix with eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$ and $1 \leq r \leq n-1$ be fixed. Then there exists a unitary matrix U such that $UHU^*(1, \dots, r)$ has eigenvalues $\eta_1 \leq \cdots \leq \eta_{n-r}$ if and only if*

$$\lambda_j \leq \eta_j \leq \lambda_{j+r}, \quad j = 1, \dots, n-r. \tag{1}$$

The eigenvalues of H are the roots of the equation $\det(\lambda I - H) = 0$. A more general setting is to consider the roots of

$$\det(\lambda C - H) = 0 \tag{2}$$

where C is positive definite. It is known (see [6] or Theorem 1 in Section 2) that there is a Hermitian H such that the roots of (2) are $\lambda_1 \leq \cdots \leq \lambda_n$ and the roots of

$$\det((\lambda C - H)(1, \dots, r)) = 0 \tag{3}$$

are $\eta_1 \leq \cdots \leq \eta_{n-r}$ if and only if (1) is satisfied.

Instead of just one, let us consider several principal submatrices at the same time. In [5], Thompson proved the result below when $r_1 = \cdots = r_k = 1$. It was generalized to the case when $r_1 = \cdots = r_k$ in [1], but actually the proof there can be modified so as to obtain the following theorem.

Received by the editors February 24, 1994.
1991 *Mathematics Subject Classification*. Primary 15A42.

Theorem B. *Let H be an $n \times n$ Hermitian matrix with eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$, and $r_1, \ldots, r_k$ $(k \geq 2)$ be positive integers such that $r_1 + \cdots + r_k \leq n$. Then as U varies over all unitary matrices, the eigenvalues $\eta_1^{(i)} \leq \cdots \leq \eta_{n-r_i}^{(i)}$ of the principal submatrix $UHU^*(r_1 + \cdots + r_{i-1} + 1, \ldots, r_1 + \cdots + r_i)$ of UHU^*, for $1 \leq i \leq k$, independently assume all values permitted by the interlacing inequalities*

$$\lambda_j \leq \eta_j^{(i)} \leq \lambda_{j+r_i}, \quad j = 1, \ldots, n - r_i, \quad i = 1, \ldots, k, \tag{4}$$

if and only if each distinct eigenvalue of H has multiplicity at least $r_1 + \cdots + r_k$.

In this paper, we consider the parallel problem for the pencil $\lambda C - H$. The result when $r_1 = \cdots = r_k = 1$ is given in [6]. Here, making use of Theorem B, we solve the problem for the general case (see Theorem 2 in Section 2).

In our discussion, let I and O be, respectively, the identity and zero matrices of appropriate order. The direct sum of matrices A and B is denoted by $A \oplus B$, and let $\operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ be the diagonal matrix with diagonal entries $\lambda_1, \ldots, \lambda_n$. Let $\langle\ ,\ \rangle$ denote the Euclidean inner product on $\mathbb{C}^n$.

2. Results

Throughout this section, let C be an $n \times n$ positive definite matrix and suppose $C = XX^*$ where X is invertible. As the roots of (2) are exactly the eigenvalues of $X^{-1}HX^{*-1}$, we easily obtain the following lemma.

Lemma 1. *For Hermitian H, (2) has roots $\lambda_1, \ldots, \lambda_n$ if and only if $H = XU\operatorname{diag}(\lambda_1, \ldots, \lambda_n)U^*X^*$ for some unitary U.*

For the sake of completeness, we prove the following theorem.

Theorem 1. *There is an $n \times n$ Hermitian matrix H such that (2) has roots $\lambda_1 \leq \cdots \leq \lambda_n$ and (3) has roots $\eta_1 \leq \cdots \leq \eta_{n-r}$ if and only if (1) holds.*

Proof. By the Gram-Schmidt process, we can find an upper triangular matrix T such that $X^{-1}T = W^*$ where W is unitary. Then $X = TW$. Let $T = \left[\begin{smallmatrix} T_{11} & T_{12} \\ O & T_{22} \end{smallmatrix}\right]$ where $T_{11} \in \mathscr{M}_r$, $T_{22} \in \mathscr{M}_{n-r}$. For any

$$H = XU\operatorname{diag}(\lambda_1, \ldots, \lambda_n)U^*X^*$$

with U unitary, we have

$$\begin{aligned}(\lambda C - H)(1, \ldots, r) \\ = T_{22}(\lambda I - WU\operatorname{diag}(\lambda_1, \ldots, \lambda_n)U^*W^*(1, \ldots, r))T_{22}^*.\end{aligned}$$

Hence, we know that the roots of (3) are exactly the eigenvalues of

$$WU\operatorname{diag}(\lambda_1, \ldots, \lambda_n)U^*W^*(1, \ldots, r).$$

The result follows from Lemma 1 and Theorem A. □

By Lemma 1, we easily obtain the following lemma.

Lemma 2. *Let H be an $n \times n$ Hermitian matrix, and suppose λ_0 is a root of (2) with multiplicity m. Then $\operatorname{rank}(\lambda_0 C - H) = n - m$.*

The following lemma can be deduced from [4, Lemma 4.1].

Lemma 3. *Let H be an $n \times n$ Hermitian matrix with eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$. If $H(1, \dots, r)$ has eigenvalues $\lambda_{r+1}, \dots, \lambda_n$, then $H = H_1 \oplus H_2$ where $H_1 \in \mathscr{M}_r$ has eigenvalues $\lambda_1, \dots, \lambda_r$, and $H_2 \in \mathscr{M}_{n-r}$ has eigenvalues $\lambda_{r+1}, \dots, \lambda_n$.*

Let x_j denote the j-th column of X^{-1}, i.e. $X^{-1} = [x_1 \vdots \cdots \vdots x_n]$.

Lemma 4. *Let $r_1 + r_2 \leq n$. If $\lambda_1 \leq \cdots \leq \lambda_n$ are real numbers with $\lambda_1 < \lambda_n$ such that for $i = 1, 2$, and for any choice of $\eta_1^{(i)} \leq \cdots \leq \eta_{n-r_i}^{(i)}$ satisfying*

$$\lambda_j \leq \eta_j^{(i)} \leq \lambda_{j+r_i}, \quad j = 1, \dots, n - r_i,$$

there is an $n \times n$ Hermitian matrix H such that (2) has roots $\lambda_1, \dots, \lambda_n$ and

$$\det((\lambda C - H)(r_{i-1} + 1, \dots, r_{i-1} + r_i)) = 0 \tag{5}$$

has roots $\eta_1^{(i)}, \dots, \eta_{n-r_i}^{(i)}$, then

(a) $\lambda_1 = \cdots = \lambda_{r_1+r_2}$,

(b) $\lambda_{n-r_1-r_2+1} = \cdots = \lambda_n$,

(c) $\langle x_p, x_q \rangle = 0$ *for* $p = 1, \dots, r_1$, *and* $q = r_1 + 1, \dots, r_1 + r_2$.

Proof. We first derive some preliminary results. Without loss of generality, we assume $r_1 \leq r_2$. Let T be an upper triangular matrix such that $X^{-1}T = W^*$ where W is unitary. Write

$$T = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ O & T_{22} & T_{23} \\ O & O & T_{33} \end{bmatrix} \tag{6}$$

where $T_{11} \in \mathscr{M}_{r_1}$, $T_{22} \in \mathscr{M}_{r_2}$, and the T_{ii}'s are invertible. By Lemma 1, for any Hermitian H such that (2) has roots $\lambda_1, \dots, \lambda_n$,

$$\lambda C - H = T(\lambda I - WU\operatorname{diag}(\lambda_1, \dots, \lambda_n)U^*W^*)T^*$$

for some unitary U. Let $A = WU\operatorname{diag}(\lambda_1, \dots, \lambda_n)U^*W^*$ so that it is a Hermitian matrix with eigenvalues $\lambda_1, \dots, \lambda_n$, and

$$\lambda C - H = T(\lambda I - A)T^*. \tag{7}$$

Now suppose H is chosen such that $\eta_1^{(1)} = \lambda_{r_1+1}, \dots, \eta_{n-r_1}^{(1)} = \lambda_n$. From (7), and as in the proof of Theorem 1, we deduce that $\det(\lambda I - A(1, \dots, r_1)) = 0$ has roots $\lambda_{r_1+1}, \dots, \lambda_n$, i.e. $A(1, \dots, r_1)$ has eigenvalues $\lambda_{r_1+1}, \dots, \lambda_n$. By Lemma 3, let

$$A = \begin{bmatrix} A_{11} & O & O \\ O & A_{22} & A_{23} \\ O & A_{32} & A_{33} \end{bmatrix}$$

where $A_{11} \in \mathscr{M}_{r_1}$ has eigenvalues $\lambda_1, \dots, \lambda_{r_1}$, $A_{22} \in \mathscr{M}_{r_2}$, $\left[\begin{smallmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{smallmatrix}\right]$ has eigenvalues $\lambda_{r_1+1}, \dots, \lambda_n$. This is the case irrespective of the choice of $\eta_1^{(2)}, \dots, \eta_{n-r_2}^{(2)}$. With the above H (and hence A),

$$\begin{aligned} &\det((\lambda C - H)(r_1 + 1, \dots, r_1 + r_2)) \\ &\quad = \det\left(\begin{bmatrix} T_{11}(\lambda I - A_{11})T_{11}^* & O \\ O & O \end{bmatrix} \right. \\ &\qquad \left. + \begin{bmatrix} T_{12} & T_{13} \\ O & T_{33} \end{bmatrix} \left(\lambda I - \begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix} \right) \begin{bmatrix} T_{12}^* & O \\ T_{13}^* & T_{33}^* \end{bmatrix} \right). \end{aligned}$$

Let

$$f_1(\lambda) = \begin{bmatrix} T_{11}(\lambda I - A_{11})T_{11}^* & O \\ O & O \end{bmatrix},$$

$$f_2(\lambda) = \begin{bmatrix} T_{12} & T_{13} \\ O & T_{33} \end{bmatrix} \left(\lambda I - \begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}\right) \begin{bmatrix} T_{12}^* & O \\ T_{13}^* & T_{33}^* \end{bmatrix}.$$

(a) Suppose H is chosen such that $\eta_1^{(2)} = \lambda_{r_2+1}, \dots, \eta_{n-r_2}^{(2)} = \lambda_n$. Then λ_n is a root of $\det(f_1(\lambda) + f_2(\lambda)) = 0$. If $\lambda_{r_1} = \lambda_n$, then (as $r_1 \le r_2$) $\eta_1^{(2)} = \cdots = \eta_{n-r_2}^{(2)} = \lambda_{r_1}$ and so by Lemma 2 $f_1(\lambda_{r_1}) + f_2(\lambda_{r_1}) = O$. Since $\left[\begin{smallmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{smallmatrix}\right]$ has eigenvalues $\lambda_{r_1+1}, \dots, \lambda_n$, we see that $f_2(\lambda_{r_1}) = O$ and hence $f_1(\lambda_{r_1}) = O$. Consequently, as T_{11} is invertible, $A_{11} = \lambda_{r_1} I$ and so $\lambda_1 = \cdots = \lambda_n$. This contradicts $\lambda_1 < \lambda_n$.

Now suppose $\lambda_{r_1} < \lambda_n$. Let $z = (z_1, z_2)$, where $z_1 \in \mathbb{C}^{r_1}$ and $z_2 \in \mathbb{C}^{n-r_1-r_2}$, be a nonzero vector such that

$$z(f_1(\lambda_n) + f_2(\lambda_n))z^* = 0.$$

As $T_{11}(\lambda_n I - A_{11})T_{11}^*$ is positive definite and $f_2(\lambda_n)$ is positive semidefinite, we deduce that $z_1 = 0$ and

$$z_2 T_{33}(\lambda_n I - A_{33})T_{33}^* z_2^* = 0.$$

If $r_1 + r_2 = n$ so that no A_{33} appears, then $z = z_1$ and thus $z_1 = 0$ gives a contradition. We now assume $r_1 + r_2 < n$. As $\lambda_n I - A_{33}$ is positive semidefinite and T_{33} is invertible, we conclude that λ_n is an eigenvalue of A_{33}. If $\lambda_{r_1} < \lambda_{n-1}$, let V be a unitary matrix such that $VA_{33}V^* = \operatorname{diag}(\alpha_1, \dots, \alpha_{n-r_1-r_2-1}, \lambda_n)$. Then by Lemma 3

$$(I \oplus V)\begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}(I \oplus V)^* = \begin{bmatrix} A_{22} & \tilde{A}_{23} & O \\ \tilde{A}_{32} & D & O \\ O & O & \lambda_n \end{bmatrix}$$

where

$$D = \operatorname{diag}(\alpha_1, \dots, \alpha_{n-r_1-r_2-1}).$$

By multiplying on the left a suitable matrix of the form $\left[\begin{smallmatrix} I & * \\ 0 & 1 \end{smallmatrix}\right]$ to the matrix $(I \oplus VT_{33}^{-1})(f_1(\lambda) + f_2(\lambda))(I \oplus VT_{33}^{-1})^*$, we see that

$$\det\left\{\begin{bmatrix} T_{11}(\lambda I - A_{11})T_{11}^* & O \\ O & O \end{bmatrix} + \begin{bmatrix} T_{12} & \widetilde{T}_{13} \\ O & I \end{bmatrix}\left(\lambda I - \begin{bmatrix} A_{22} & \tilde{A}_{23} \\ \tilde{A}_{32} & D \end{bmatrix}\right)\begin{bmatrix} T_{12}^* & O \\ \widetilde{T}_{13}^* & I \end{bmatrix}\right\} = 0$$

has root $\lambda_{r_1+1}, \dots, \lambda_{n-1}$. Here, $\widetilde{T}_{13}$ is the matrix obtained by deleting the last column of $T_{13}V^*$. Repeating the same argument as above, we see that $\lambda_{n-1} = \alpha_j$ for some j. Inductively, we conclude that if $\lambda_{r_1} < \lambda_\ell$, then λ_ℓ is an eigenvalue of A_{33} counting multiplicity. Consequently, we deduce that $\lambda_{r_1} = \cdots = \lambda_{r_1+r_2}$. Furthermore, by Theorem A, the eigenvalues of A_{33} are the $n - r_1 - r_2$ largest eigenvalues of $\left[\begin{smallmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{smallmatrix}\right]$. By Lemma 3, $A_{23} = O$, $A_{32} = O$, and $A_{22} = \lambda_{r_1} I$. Now suppose $\lambda_{r_1} = \cdots = \lambda_{r_1+r_2} = \cdots = \lambda_s < \lambda_{s+1}$ where

$s \geq r_1 + r_2$. Then, by Lemma 2, $\operatorname{rank}(f_1(\lambda_{r_1}) + f_2(\lambda_{r_1})) = n - s$. Now, as

$$f_1(\lambda_{r_1}) + f_2(\lambda_{r_1}) = T_{11}(\lambda_{r_1} I - A_{11})T_{11}^* \oplus T_{33}(\lambda_{r_1} I - A_{33})T_{33}^* ,$$

we conclude that $A_{11} = \lambda_{r_1} I$. Hence $\lambda_1 = \cdots = \lambda_{r_1+r_2}$.

(b) If $\lambda_1, \dots, \lambda_n$ satisfy the condition given in the hypothesis of the lemma, then so are $-\lambda_n, \dots, -\lambda_1$ because one can take $-H$ instead of H in the consideration of roots of (2) and (5). So, by part (a), we have $\lambda_{n-r_1-r_2+1} = \cdots = \lambda_n$.

(c) It suffices to show T_{12} given in (6) is the zero matrix. By (a) and (b), we now have $\lambda_1 = \cdots = \lambda_s < \lambda_{s+1} \leq \cdots \leq \lambda_n$ where $r_1 + r_2 \leq s \leq n - r_1 - r_2$. By Lemma 2, if we choose H such that $\eta_1^{(2)} = \lambda_1, \dots, \lambda_{n-r_2}^{(2)} = \lambda_{n-r_2}$, then $\operatorname{rank}(f_1(\lambda_1) + f_2(\lambda_1)) = n - r_2 - s$. Notice that now we have $f_1(\lambda_1) = O$. If $T_{12} \neq O$, then $\operatorname{rank}\left(\left[\begin{smallmatrix} T_{12} & T_{13} \\ O & T_{33} \end{smallmatrix}\right]\right) > n - r_1 - r_2$ and so $\operatorname{rank}(f_2(\lambda_1)) > n - r_1 - r_2 - (s - r_1) = n - s - r_2$. Thus we have a contradiction. Hence $T_{12} = O$. □

Theorem 2. *Let C be positive definite, $\lambda_1 \leq \cdots \leq \lambda_n$ be fixed real numbers with $\lambda_1 < \lambda_n$, and $r_1, \dots, r_k$ $(k \geq 2)$ be positive integers such that $r_1 + \cdots + r_k \leq n$. Then the following conditions* (I) *and* (II) *are equivalent.*

(I) *For any choice of $\eta_1^{(i)} \leq \cdots \leq \eta_{n-r_i}^{(i)}$, $i = 1, \dots, k$, satisfying* (4) *there is an $n \times n$ Hermitian H such that the roots of* (2) *are $\lambda_1, \dots, \lambda_n$ and, for $i = 1, \dots, k$, the roots of*

$$\det((\lambda C - H)(r_1 + \cdots + r_{i-1} + 1, \dots, r_1 + \cdots + r_i)) = 0 \tag{8}$$

are $\eta_1^{(i)}, \dots, \eta_{n-r_i}^{(i)}$.

(II) (i) *Each distinct number among $\lambda_1, \dots, \lambda_n$ has multiplicity at least $r_1 + \cdots + r_k$, and*

(ii) *if $C^{-1} = (M_{ij})_{i,j=1,\dots,k+1}$ with $M_{ii} \in \mathscr{M}_{r_i}$, $i = 1, \dots, k$, then $M_{ij} = O$ for $1 \leq i \neq j \leq k$.*

Proof. Let $N_1 = \{1, \dots, r_1\}, \dots, N_k = \{r_1 + \cdots + r_{k-1} + 1, \dots, r_1 + \cdots + r_k\}$. If we consider pairwise N_r, N_s and PCP^* for suitable permutation matrix P, then by Lemma 4 we can deduce that a necessary condition for (I) is that, with $X^{-1} = [x_1 \vdots \cdots \vdots x_n]$,

$$\langle x_p, x_q \rangle = 0 , \quad p \in N_r , \quad q \in N_s , \quad 1 \leq r \neq s \leq k .$$

As $C^{-1} = X^{-1*}X^{-1}$, this is equivalent to saying that C^{-1} satisfies (II)(ii). Now suppose this condition holds. Then we can find an upper triangular

$$T = \left[\begin{array}{cccc|c} T_{11} & O & \cdots & O & \\ O & T_{22} & \cdots & O & \bigstar \\ \vdots & \vdots & \ddots & \vdots & \\ O & O & \cdots & T_{kk} & \\ \hline O & O & \cdots & O & T_{k+1,k+1} \end{array}\right]$$

such that $X = TW$ where W is unitary. Then for any

$$H = XU\mathrm{diag}(\lambda_1, \dots, \lambda_n)U^*X^*$$

with U unitary, and $1 \le i \le k$,

$$\begin{aligned}
&\det((\lambda C - H)(r_1 + \cdots + r_{i-1} + 1, \dots, r_1 + \cdots + r_i)) \\
&\quad = \Big(\prod_{\substack{j=1 \\ j \ne i}}^{k+1} \det T_{jj} \det T_{jj}^*\Big) \\
&\qquad \times \det(\lambda I - WU\mathrm{diag}(\lambda_1, \dots, \lambda_n) \\
&\qquad\qquad \cdot U^*W^*(r_1 + \cdots + r_{i-1} + 1, \dots, r_1 + \cdots + r_i)) .
\end{aligned}$$

That is, the roots of (8) are exactly the eigenvalues of

$$WU\mathrm{diag}(\lambda_1, \dots, \lambda_n)U^*W^*(r_1 + \cdots + r_{i-1} + 1, \dots, r_1 + \cdots + r_i) .$$

The result follows from Lemma 1 and Theorem B. □

Finally, we state without proof the following theorem for completeness.

Theorem 3. *The roots of* (2) *satisfy* $\lambda_1 = \cdots = \lambda_n = \alpha$ *if and only if* $H = \alpha C$. *If this is so,*

$$\begin{aligned}
&\det((\lambda C - H)(r_1 + \cdots + r_{i-1} + 1, \dots, r_1 + \cdots + r_i)) \\
&\quad = \det(C(r_1 + \cdots + r_{i-1} + 1, \dots, r_1 + \cdots + r_i)) \quad (\lambda - \alpha)^{n - r_i} .
\end{aligned}$$

Acknowledgment

I would like to thank Professor R. C. Thompson for bringing my attention to [6] and suggesting the investigation of Theorem 2.

References

1. C. M. Cheng, *Independence of eigenvalues and independence of singular values of submatrices*, Linear Algebra Appl. **166** (1992), 115–129.
2. K. Fan and G. Pall, *Imbedding conditions for Hermitian and normal matrices*, Canad. J. Math. **9** (1958), 298–304.
3. R. A. Hom and C. R. Johnson, *Matrix analysis*, Cambridge Univ. Press, New York, 1991.
4. C. K. Li, *Matrices with some extremal properties*, Linear Algebra Appl. **101** (1988), 255–267.
5. R.C. Thompson, *Principal submatrices* IV. *On the independence of eigenvalues of different principal submatrices*, Linear Algebra Appl. **2** (1969), 355–374.
6. R. C. Thompson, *Principal submatrices* VIII. *Principal sections of a pair of forms*, Rocky Mountain J. Math. **2** (1972), 97–110.

Faculty of Science and Technology, University of Macau, Macau
E-mail address: fstcmc@uealab.umac.mo