ON PRINCIPAL SECTIONS OF A PAIR OF FORMS

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Abstract. Let $H$ and $C$ be $n \times n$ Hermitian matrices with $C$ positive definite. Let $H(i_1, \ldots, i_r)$ denote the submatrix of $H$ formed by deleting the rows and columns $i_1, \ldots, i_r$ of $H$. In this paper, with $r_1 + \cdots + r_k \leq n$, we study the roots of the determinantal equation $\det(\lambda C - H) = 0$ and those of

$$
\det((\lambda C - H)\left(r_1 + \cdots + r_i + 1, \ldots, r_1 + \cdots + r_i\right)) = 0
$$

for $i = 1, \ldots, k$.

1. Introduction

Let $\mathcal{M}_n$ denote the set of all $n \times n$ complex matrices. For $A \in \mathcal{M}_n$, let $A(i_1, \ldots, i_r)$ denote the submatrix of $A$ formed by deleting the rows and columns $i_1, \ldots, i_r$ of $A$. Concerning the eigenvalues of a Hermitian matrix and those of its principal submatrices, we have the following well-known interlacing theorem (e.g. [2] or [3, pp. 185–188]).

Theorem A. Let $H$ be an $n \times n$ Hermitian matrix with eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$ and $1 \leq r \leq n - 1$ be fixed. Then there exists a unitary matrix $U$ such that $UHU^*(1, \ldots, r)$ has eigenvalues $\eta_1 \leq \cdots \leq \eta_{n-r}$ if and only if

$$
\lambda_j \leq \eta_j \leq \lambda_{j+r}, \quad j = 1, \ldots, n-r.
$$

The eigenvalues of $H$ are the roots of the equation $\det(\lambda I - H) = 0$. A more general setting is to consider the roots of

$$
\det(\lambda C - H) = 0
$$

where $C$ is positive definite. It is known (see [6] or Theorem 1 in Section 2) that there is a Hermitian $H$ such that the roots of (2) are $\lambda_1 \leq \cdots \leq \lambda_n$ and the roots of

$$
\det((\lambda C - H)\left(1, \ldots, r\right)) = 0
$$

are $\eta_1 \leq \cdots \leq \eta_{n-r}$ if and only if (1) is satisfied.

Instead of just one, let us consider several principal submatrices at the same time. In [5], Thompson proved the result below when $r_1 = \cdots = r_k = 1$. It was generalized to the case when $r_1 = \cdots = r_k$ in [1], but actually the proof there can be modified so as to obtain the following theorem.

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Theorem B. Let $H$ be an $n \times n$ Hermitian matrix with eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$, and $r_1, \ldots, r_k$ ($k \geq 2$) be positive integers such that $r_1 + \cdots + r_k \leq n$. Then as $U$ varies over all unitary matrices, the eigenvalues $\eta_1^{(i)} \leq \cdots \leq \eta_{n-r_i}^{(i)}$ of the principal submatrix $U H U^* (r_1 + \cdots + r_{i-1} + 1, \ldots, r_1 + \cdots + r_i)$ of $U H U^*$, for $1 \leq i \leq k$, independently assume all values permitted by the interlacing inequalities

\begin{align*}
\lambda_j \leq \eta_j^{(i)} \leq \lambda_{j+r_i}, & \quad j = 1, \ldots, n-r_i, \quad i = 1, \ldots, k, \\
\end{align*}

if and only if each distinct eigenvalue of $H$ has multiplicity at least $r_1 + \cdots + r_k$.

In this paper, we consider the parallel problem for the pencil $\lambda C - H$. The result when $r_1 = \cdots = r_k = 1$ is given in [6]. Here, making use of Theorem B, we solve the problem for the general case (see Theorem 2 in Section 2).

In our discussion, let $I$ and $O$ be, respectively, the identity and zero matrices of appropriate order. The direct sum of matrices $A$ and $B$ is denoted by $A \oplus B$, and let $\text{diag}(\lambda_1, \ldots, \lambda_n)$ be the diagonal matrix with diagonal entries $\lambda_1, \ldots, \lambda_n$. Let $\langle \cdot, \cdot \rangle$ denote the Euclidean inner product on $\mathbb{C}^n$.

2. Results

Throughout this section, let $C$ be an $n \times n$ positive definite matrix and suppose $C = XX^*$ where $X$ is invertible. As the roots of (2) are exactly the eigenvalues of $X^{-1} H X X^{-1}$, we easily obtain the following lemma.

Lemma 1. For Hermitian $H$, (2) has roots $\lambda_1, \ldots, \lambda_n$ if and only if $H = X U \text{diag}(\lambda_1, \ldots, \lambda_n) U^* X^*$ for some unitary $U$.

For the sake of completeness, we prove the following theorem.

Theorem 1. There is an $n \times n$ Hermitian matrix $H$ such that (2) has roots $\lambda_1 \leq \cdots \leq \lambda_n$ and (3) has roots $\eta_1 \leq \cdots \leq \eta_{n-r}$ if and only if (1) holds.

Proof. By the Gram-Schmidt process, we can find an upper triangular matrix $T$ such that $X^{-1} T = W^*$ where $W$ is unitary. Then $X = T W$. Let $T = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix}$ where $T_{11} \in \mathbb{M}_r$, $T_{22} \in \mathbb{M}_{n-r}$. For any $H = X U \text{diag}(\lambda_1, \ldots, \lambda_n) U^* X^*$ with $U$ unitary, we have

\begin{align*}
(\lambda C - H)(1, \ldots, r) &= T_{22}(\lambda I - W U \text{diag}(\lambda_1, \ldots, \lambda_n) U^* W^*(1, \ldots, r)) T_{22}^*.
\end{align*}

Hence, we know that the roots of (3) are exactly the eigenvalues of $W U \text{diag}(\lambda_1, \ldots, \lambda_n) U^* W^*(1, \ldots, r)$.

The result follows from Lemma 1 and Theorem A. \hfill \Box

By Lemma 1, we easily obtain the following lemma.

Lemma 2. Let $H$ be an $n \times n$ Hermitian matrix, and suppose $\lambda_0$ is a root of (2) with multiplicity $m$. Then $\text{rank}(\lambda_0 C - H) = n - m$.

The following lemma can be deduced from [4, Lemma 4.1].
Lemma 3. Let \( H \) be an \( n \times n \) Hermitian matrix with eigenvalues \( \lambda_1 \leq \cdots \leq \lambda_n \). If \( H(1, \ldots, r) \) has eigenvalues \( \lambda_{r+1}, \ldots, \lambda_n \), then \( H = H_1 \oplus H_2 \) where \( H_1 \in \mathcal{M}_r \) has eigenvalues \( \lambda_1, \ldots, \lambda_r \), and \( H_2 \in \mathcal{M}_{n-r} \) has eigenvalues \( \lambda_{r+1}, \ldots, \lambda_n \).

Let \( x_j \) denote the \( j \)-th column of \( X^{-1} \), i.e. \( X^{-1} = [x_1: \cdots : x_n] \).

Lemma 4. Let \( r_1 + r_2 \leq n \). If \( \lambda_1 \leq \cdots \leq \lambda_n \) are real numbers with \( \lambda_1 < \lambda_n \) such that for \( i = 1, 2, \) and for any choice of \( \eta_1^{(i)} \leq \cdots \leq \eta_{n-r_i}^{(i)} \) satisfying

\[
\lambda_j \leq \eta_j^{(i)} \leq \lambda_{j+r_i}, \quad j = 1, \ldots, n - r_i,
\]

there is an \( n \times n \) Hermitian matrix \( H \) such that (2) has roots \( \lambda_1, \ldots, \lambda_n \) and

\[
\det((\lambda C - H)(r_i - 1, \ldots, r_i - 1 + r_i)) = 0
\]

has roots \( \eta_1^{(i)}, \ldots, \eta_{n-r_i}^{(i)} \), then

(a) \( \lambda_1 = \cdots = \lambda_{r_1 + r_2} \),
(b) \( \lambda_{n-r_1 - r_2 + 1} = \cdots = \lambda_n \),
(c) \( (x_p, x_q) = 0 \) for \( p = 1, \ldots, r_1 \), and \( q = r_1 + 1, \ldots, r_1 + r_2 \).

Proof. We first derive some preliminary results. Without loss of generality, we assume \( r_1 \leq r_2 \). Let \( T \) be an upper triangular matrix such that \( X^{-1} T = W^* \) where \( W \) is unitary. Write

\[
T = \begin{bmatrix}
T_{11} & T_{12} & T_{13} \\
0 & T_{22} & T_{23} \\
0 & 0 & T_{33}
\end{bmatrix}
\]

where \( T_{11} \in \mathcal{M}_{r_1}, T_{22} \in \mathcal{M}_{r_2}, \) and the \( T_{ii} \)'s are invertible. By Lemma 1, for any Hermitian \( H \) such that (2) has roots \( \lambda_1, \ldots, \lambda_n \),

\[
\lambda C - H = T(\lambda I - WU\text{diag}(\lambda_1, \ldots, \lambda_n)U^*W^*)T^*
\]

for some unitary \( U \). Let \( A = WU\text{diag}(\lambda_1, \ldots, \lambda_n)U^*W^* \) so that it is a Hermitian matrix with eigenvalues \( \lambda_1, \ldots, \lambda_n \), and

\[
\lambda C - H = T(\lambda I - A)T^*.
\]

Now suppose \( H \) is chosen such that \( \eta_1^{(1)} = \lambda_{r_1 + 1}, \ldots, \eta_{n-r_1}^{(1)} = \lambda_n \). From (7), and as in the proof of Theorem 1, we deduce that \( \det(\lambda I - A(1, \ldots, r_1)) = 0 \) has roots \( \lambda_{r_1 + 1}, \ldots, \lambda_n \), i.e. \( A(1, \ldots, r_1) \) has eigenvalues \( \lambda_{r_1 + 1}, \ldots, \lambda_n \). By Lemma 3, let

\[
A = \begin{bmatrix}
A_{11} & O & O \\
O & A_{22} & A_{23} \\
O & A_{32} & A_{33}
\end{bmatrix}
\]

where \( A_{11} \in \mathcal{M}_{r_1} \) has eigenvalues \( \lambda_1, \ldots, \lambda_{r_1} \), \( A_{22} \in \mathcal{M}_{r_2} \), \( \begin{bmatrix} A_{32} & A_{33} \end{bmatrix} \) has eigenvalues \( \lambda_{r_1 + 1}, \ldots, \lambda_n \). This is the case irrespective of the choice of \( \eta_1^{(2)}, \ldots, \eta_{n-r_2}^{(2)} \). With the above \( H \) (and hence \( A \)),

\[
\det((\lambda C - H)(r_1 + 1, \ldots, r_1 + r_2)) = \det \left( \begin{bmatrix}
T_{11}(\lambda I - A_{11})T_{11} & O \\
O & O
\end{bmatrix}
\right)
\]

\[
+ \begin{bmatrix}
T_{12} & T_{13} \\
O & T_{33}
\end{bmatrix}(\lambda I - \begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}) \begin{bmatrix}
T_{12}^* & O \\
T_{13}^* & T_{33}^*
\end{bmatrix}
\right).
\]
Let
\[
\begin{align*}
    f_1(\lambda) &= \begin{bmatrix} T_{11}(\lambda I - A_{11})T_{11}^* & 0 \\ 0 & O \end{bmatrix}, \\
    f_2(\lambda) &= \begin{bmatrix} T_{12} & T_{13} \\ O & T_{33} \end{bmatrix} \begin{bmatrix} \lambda I & [A_{22} \ A_{23}] \\ [A_{32} \ A_{33}] & [T_{12}^* \ O] \end{bmatrix}.
\end{align*}
\]

(a) Suppose $H$ is chosen such that $\eta_1^{(2)} = \lambda_{r_2+1}, \ldots, \eta_{n-r_2}^{(2)} = \lambda_n$. Then $\lambda_n$ is a root of $\det(f_1(\lambda) + f_2(\lambda)) = 0$. If $\lambda_{r_1} = \lambda_n$, then (as $r_1 \leq r_2$) $\eta_1^{(2)} = \ldots = \eta_{n-r_2}^{(2)} = \lambda_{r_1}$ and so by Lemma 2 $f_1(\lambda_{r_1}) + f_2(\lambda_{r_1}) = O$. Since $[A_{22} \ A_{32} \ A_{33}]$ has eigenvalues $\lambda_{r_1+1}, \ldots, \lambda_n$, we see that $f_2(\lambda_{r_1}) = O$ and hence $f_1(\lambda_{r_1}) = O$. Consequently, as $T_{11}$ is invertible, $A_{11} = \lambda_{r_1}I$ and so $\lambda_1 = \ldots = \lambda_n$. This contradicts $\lambda_1 < \lambda_n$.

Now suppose $\lambda_{r_1} < \lambda_n$. Let $z = (z_1, z_2)$, where $z_1 \in \mathbb{C}^{r_1}$ and $z_2 \in \mathbb{C}^{n-r_1-r_2}$, be a nonzero vector such that
\[
z(f_1(\lambda_n) + f_2(\lambda_n))z^* = 0.
\]
As $T_{11}(\lambda_n I - A_{11})T_{11}^*$ is positive definite and $f_2(\lambda_n)$ is positive semidefinite, we deduce that $z_1 = 0$ and
\[
z_2T_{33}(\lambda_n I - A_{33})T_{33}^*z_2^* = 0.
\]
If $r_1 + r_2 = n$ so that no $A_{33}$ appears, then $z = z_1$ and thus $z_1 = 0$ gives a contradiction. We now assume $r_1 + r_2 < n$. As $\lambda_n I - A_{33}$ is positive semidefinite and $T_{33}$ is invertible, we conclude that $\lambda_n$ is an eigenvalue of $A_{33}$. If $\lambda_{r_1} < \lambda_{n-1}$, let $V$ be a unitary matrix such that $V A_{33} V^* = \text{diag}(\alpha_1, \ldots, \alpha_{n-r_1-r_2-1}, \lambda_n)$. Then by Lemma 3
\[
(I \oplus V) \begin{bmatrix} A_{22} & A_{23} \\
A_{32} & A_{33} \end{bmatrix} (I \oplus V)^* = \begin{bmatrix} A_{22} & \tilde{A}_{23} & O \\
A_{32} & D & O \\
O & O & \lambda_n \end{bmatrix}
\]
where
\[
D = \text{diag}(\alpha_1, \ldots, \alpha_{n-r_1-r_2-1}).
\]
By multiplying on the left a suitable matrix of the form $[I \ 1]$ to the matrix $(I \oplus V T_{33}^{-1})(f_1(\lambda) + f_2(\lambda))(I \oplus V T_{33}^{-1})^*$, we see that
\[
\det \left\{ \begin{bmatrix} T_{11}(\lambda I - A_{11})T_{11}^* & 0 \\ 0 & O \end{bmatrix} + \begin{bmatrix} T_{12} & \tilde{T}_{13} \\ O & I \end{bmatrix} \begin{bmatrix} \lambda I & [A_{22} \ A_{23}] \\ [A_{32} \ D] & [T_{12}^* \ I] \end{bmatrix} \right\} = 0
\]
has root $\lambda_{r_1+1}, \ldots, \lambda_{n-1}$. Here, $\tilde{T}_{13}$ is the matrix obtained by deleting the last column of $T_{13} V^*$. Repeating the same argument as above, we see that $\lambda_{n-1} = \alpha_j$ for some $j$. Inductively, we conclude that if $\lambda_{r_1} < \lambda_{\ell}$, then $\lambda_{\ell}$ is an eigenvalue of $A_{33}$ counting multiplicity. Consequently, we deduce that $\lambda_{r_1} = \ldots = \lambda_{r_1+r_2}$. Furthermore, by Theorem A, the eigenvalues of $A_{33}$ are the $n - r_1 - r_2$ largest eigenvalues of $[A_{22} \ A_{23} \ A_{32} \ A_{33}]$. By Lemma 3, $A_{23} = O$, $A_{32} = O$, and $A_{22} = \lambda_{r_1}I$. Now suppose $\lambda_{r_1} = \ldots = \lambda_{r_1+r_2} = \ldots = \lambda_s < \lambda_{s+1}$ where
s \geq r_1 + r_2$. Then, by Lemma 2, \( \text{rank}(f_1(\lambda r_1) + f_2(\lambda r_2)) = n - s \). Now, as

\[ f_1(\lambda r_1) + f_2(\lambda r_2) = T_{11}(\lambda r_1 I - A_{11})T_{11}^* + T_{33}(\lambda r_1 I - A_{33})T_{33}^* , \]

we conclude that \( A_{11} = \lambda r_1 I \). Hence \( \lambda_1 = \cdots = \lambda_{r_1+r_2} \).

(b) If \( \lambda_1, \ldots, \lambda_n \) satisfy the condition given in the hypothesis of the lemma, then so are \(-\lambda_n, \ldots, -\lambda_1\) because one can take \(-H\) instead of \(H\) in the consideration of roots of (2) and (5). So, by part (a), we have \( \lambda_{n-r_1-r_2+1} = \cdots = \lambda_n \).

(c) It suffices to show \( T_{12} \) given in (6) is the zero matrix. By (a) and (b), we now have \( \lambda_1 = \cdots = \lambda_s < \lambda_{s+1} \leq \cdots \leq \lambda_n \) where \( r_1 + r_2 \leq s \leq n - r_1 - r_2 \). By Lemma 2, if we choose \( H \) such that \( f_1^{(2)}(\lambda_1, \ldots, \lambda_{n-r_2}) = \lambda_{n-r_2} \), then \( \text{rank}(f_1(\lambda_1) + f_2(\lambda_1)) = n - r_2 - s \). Notice that now we have \( f_1(\lambda_1) = O \). If \( T_{12} \neq O \), then \( \text{rank} \left( \begin{bmatrix} T_{12} & T_{13}^* \end{bmatrix} \right) > n - r_1 - r_2 \) and so \( \text{rank}(f_2(\lambda_1)) > n - r_1 - r_2 - (s - r_1) = n - s - r_2 \). Thus we have a contradiction. Hence \( T_{12} = O \). \( \square \)

**Theorem 2.** Let \( C \) be positive definite, \( \lambda_1 \leq \cdots \leq \lambda_n \) be fixed real numbers with \( \lambda_1 < \lambda_n \), and \( r_1, \ldots, r_k \) \((k \geq 2)\) be positive integers such that \( r_1 + \cdots + r_k \leq n \). Then the following conditions (I) and (II) are equivalent.

(I) For any choice of \( \eta_1^{(i)} \leq \cdots \leq \eta_n^{(i)} \), \( i = 1, \ldots, k \), satisfying (4) there is an \( n \times n \) Hermitian \( H \) such that the roots of (2) are \( \lambda_1, \ldots, \lambda_n \) and, for \( i = 1, \ldots, k \), the roots of \( \eta_1^{(i)} - \eta_{n-r_1}^{(i)} \).

(II) (i) Each distinct number among \( \lambda_1, \ldots, \lambda_n \) has multiplicity at least \( r_1 + \cdots + r_k \), and

(ii) if \( C^{-1} = (M_{ij})_{i,j=1,\ldots,k+1} \) with \( M_{ii} \in \mathcal{M}_r \), \( i = 1, \ldots, k \), then \( M_{ij} = O \) for \( 1 \leq i \neq j \leq k \).

**Proof.** Let \( N_r = \{1, \ldots, r_1\}, \ldots, N_s = \{r_1 + \cdots + r_{k-1} + 1, \ldots, r_1 + \cdots + r_k\} \). If we consider pairwise \( N_r, N_s \) and \( P \) such that \( P \) satisfies (II)(ii). This means that \( X^{-1} \in [x_1; \ldots; x_n] \), \( x_p \perp x_q \), \( p \in N_r \), \( q \in N_s \), \( 1 \leq r \neq s \leq k \). As \( C^{-1} = X^{-1*} X^{-1} \), this is equivalent to saying that \( C^{-1} \) satisfies (II)(ii). Now suppose this condition holds. Then we can find an upper triangular

\[
T = \begin{bmatrix}
T_{11} & O & \cdots & O \\
O & T_{22} & \cdots & O \\
\vdots & \vdots & \ddots & \vdots \\
O & O & \cdots & T_{kk} \\
O & O & \cdots & O
\end{bmatrix}
\]

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such that $X = TW$ where $W$ is unitary. Then for any

$$H = XU\text{diag}(\lambda_1, \ldots, \lambda_n)U^*X^*$$

with $U$ unitary, and $1 \leq i \leq k$,

$$\det((\lambda C - H)(r_1 + \cdots + r_{i-1} + 1, \ldots, r_1 + \cdots + r_i))$$

$$= \left(\prod_{j=1, j \neq i}^{k+1} \det T_{jj} \det T_{jj}^*\right)$$

$$\times \det(\lambda I - WU\text{diag}(\lambda_1, \ldots, \lambda_n)$$

$$\cdot U^*W^*(r_1 + \cdots + r_{i-1} + 1, \ldots, r_1 + \cdots + r_i)).$$

That is, the roots of (8) are exactly the eigenvalues of

$$WU\text{diag}(\lambda_1, \ldots, \lambda_n)U^*W^*(r_1 + \cdots + r_{i-1} + 1, \ldots, r_1 + \cdots + r_i).$$

The result follows from Lemma 1 and Theorem B. □

Finally, we state without proof the following theorem for completeness.

**Theorem 3.** The roots of (2) satisfy $\lambda_1 = \cdots = \lambda_n = \alpha$ if and only if $H = \alpha C$. If this is so,

$$\det((\lambda C - H)(r_1 + \cdots + r_{i-1} + 1, \ldots, r_1 + \cdots + r_i))$$

$$= \det(C(r_1 + \cdots + r_{i-1} + 1, \ldots, r_1 + \cdots + r_i)) (\lambda - \alpha)^{n-r_i}.$$  

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**References**


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