SYMmetric ORTHOGONAL POLYNOMIALS
and the Associated Orthogonal L-Polynomials

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ABSTRACT. We show how symmetric orthogonal polynomials can be linked to polynomials associated with certain orthogonal L-polynomials. We provide some examples to illustrate the results obtained. Finally as an application, we derive information regarding the orthogonal polynomials associated with the weight function \((1 + kx^2)(1 - x^2)^{-1/2}, k > 0\).

1. INTRODUCTION

Let \(Y(n)\) be the space of all real monic polynomials \(P_n(x)\) of degree \(n\) that satisfy the symmetric property \(P_n(x) = (-1)^nP_n(-x)\). Let \(Z(n, \beta)\) be the space of all real monic polynomials \(Q_n(t)\) of degree \(n\) that satisfy the rather different symmetric property \(Q_n(t) = t^nQ_n(\beta^2/t)/(-\beta)^n\).

It is well known (see \([1, 13]\)) that if \(w(x)\) is a weight function on \((-d, d)\), where \(0 < d \leq \infty\), such that \(w(x) = w(-x)\), then the associated monic orthogonal polynomials \(B_n(w; x)\), \(n \geq 0\), are such that \(B_n(w; x) \in Y(n)\). Furthermore, they satisfy

\[
B_{n+1}(w; x) = xB_n(w; x) - \alpha_{n+1}(w)B_{n-1}(w; x), \quad n \geq 1,
\]

where

\[
\alpha_{n+1}(w) = \frac{\int_{-d}^d x^n B_n(w; x)w(x) \, dx}{\int_{-d}^d x^{n-1} B_{n-1}(w; x)w(x) \, dx} > 0, \quad n \geq 1.
\]

Now, some lesser known results are the following. Let \(\nu(t)\) be a strong weight function defined on \((\beta^2/b, b)\), where \(0 < \beta < b \leq \infty\), such that

\[
\sqrt{t} \nu(t) = \sqrt{\beta^2/t} \nu(\beta^2/t).
\]

Then the monic polynomials \(\widetilde{B}_n(\nu; t)\), \(n \geq 0\), defined uniquely by

\[
\int_{\beta^2/b}^b t^{-n+s}\widetilde{B}_n(\nu; t) \nu(t) \, dt = 0, \quad 0 \leq s \leq n - 1,
\]

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are such that $\widetilde{B}_n(\nu ; t) \in Z(n, \beta)$ and satisfy

$$
\widetilde{B}_{n+1}(\nu ; t) = (t - \beta)\widetilde{B}_n(\nu ; t) - \alpha_{n+1}(\nu)\widetilde{B}_{n-1}(\nu ; t), \quad n \geq 1,
$$

where

$$
\alpha_{n+1}(\nu) = \frac{\int_{\beta^2/b}^{b} \widetilde{B}_n(\nu ; t)\nu(t) \, dt}{\int_{\beta^2/b}^{b} \widetilde{B}_{n-1}(\nu ; t)\nu(t) \, dt} > 0, \quad n \geq 1.
$$

For proofs, and other information on these results, see [8, 12]. In [9, 10, 11, 12] some results can be found regarding the polynomials defined by (1.3) for weight functions satisfying a property different from (1.2). For some studies of polynomials given by a variation of (1.3), see [5, 6, 7].

It is easily verified from (1.3) that the Laurent polynomials or L-polynomials given by

$$
R_{2m}(t) = t^{-m}\widetilde{B}_{2m}(\nu ; t), \quad R_{2m+1}(t) = t^{-m-1}\widetilde{B}_{2m+1}(\nu ; t), \quad m \geq 0,
$$

form a sequence of orthogonal L-polynomials on $(\beta^2/b, b)$ in relation to the strong weight function $\nu(t)$. For information on orthogonal L-polynomials and polynomials similar to $\widetilde{B}_n(\nu ; t)$, but where $\nu(t)$ is any strong weight function, we refer to [2, 3, 4].

In this article we show the relation between the polynomials $B_n(w ; x)$ and $\widetilde{B}_n(\nu ; t)$, when the associated weight functions are linked in a certain way. We also provide some examples to illustrate the results obtained. Finally as an application, we derive information regarding the orthogonal polynomials $B_n(W ; x)$, $n \geq 0$, where $d = 1$ and $W(x) = (1 + kx^2)(1 - x^2)^{-1/2}$, for $k > 0$.

2. Preliminary results

Throughout the rest of this article we assume $\alpha > 0$ and $\beta > 0$.

**Theorem 2.1.** Let the sequence of polynomials $\{P_n(x)\}$ be such that $P_n(x) \in Y(n)$. Then

$$
P_n(x) = B_n(w ; x), \quad n \geq 0,
$$

if and only if

$$
\int_{-d}^{d} \left\{ \sqrt{\alpha x^2 + \beta} + \sqrt{\alpha x} \right\}^{-(n-1)+2s} P_n(x) \frac{w(x)}{\sqrt{\alpha x^2 + \beta}} \, dx = 0, \quad 0 \leq s \leq n - 1.
$$

**Proof.** Since $\left\{ \sqrt{\alpha x^2 + \beta} + \sqrt{\alpha x} \right\}^{-1} = \{ \sqrt{\alpha x^2 + \beta} - \sqrt{\alpha x} \}/\beta$, (2.1) is equivalent to

$$
\int_{-d}^{d} \left\{ \sqrt{\alpha x^2 + \beta} \pm \sqrt{\alpha x} \right\}^{2l} P_{2m+1}(x) \frac{w(x)}{\sqrt{\alpha x^2 + \beta}} \, dx = 0
$$

and

$$
\int_{-d}^{d} \left\{ \sqrt{\alpha x^2 + \beta} \pm \sqrt{\alpha x} \right\}^{2l+1} P_{2m+2}(x) \frac{w(x)}{\sqrt{\alpha x^2 + \beta}} \, dx = 0,
$$

for $l = 0, 1, \ldots, m$ and $m \geq 0$. 

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Using the symmetric property of $P_n(x)$, this can also be given in the equivalent form

\[(2.2)\]

\[
\int_{-d}^{d} \left[ \sum_{r=0}^{l-1} \binom{2l}{2r+1} (\alpha x^2 + \beta)^{l-r-1} \{\pm \sqrt{\alpha x}\}^{2r+1} \right] P_{2m+1}(x)w(x) \, dx = 0,
\]

\[
\int_{-d}^{d} \left[ \sum_{r=0}^{l} \binom{2l+1}{2r} (\alpha x^2 + \beta)^{l-r} \{\pm \sqrt{\alpha x}\}^{2r} \right] P_{2m+2}(x)w(x) \, dx = 0,
\]

for $l = 0, 1, \ldots, m$ and $m \geq 0$.

Now, if $P_n(x) = B_n(w ; x)$ for $n \geq 0$, then (2.2) must hold as $B_n(w ; x)$ is orthogonal to all polynomials of degree $\leq n$.

Conversely, if (2.2) holds, then, for example, $P_{2m+1}(x)$ is orthogonal to the odd polynomials

\[
\sum_{r=0}^{l-1} \binom{2l}{2r+1} (\alpha x^2 + \beta)^{l-r-1} \{\pm \sqrt{\alpha x}\}^{2r+1},
\]

of "precise" degree $2l - 1$, for $l = 0, 1, \ldots, m$. Since these polynomials must form a basis for all odd polynomials of degree $\leq 2m - 1$, $P_{2m+1}(x)$ must be orthogonal to all odd polynomials of degree $\leq 2m - 1$. Hence, as $P_{2m+1}(x)$ is monic and is also odd, we obtain $P_{2m+1}(x) = B_{2m+1}(w ; x)$. In a similar way we also show that $P_{2m}(x) = B_{2m}(w ; x)$. This completes the proof of the theorem.

We now consider the transformation

\[ t(x) = \left\{ \sqrt{\alpha x^2 + \beta + \sqrt{\alpha x}} \right\}^2, \quad x \in (-\infty, \infty). \]

The transformation $t(x)$ represents a one-to-one correspondence between $(-\infty, \infty)$ and $(0, \infty)$. The inverse of $t(x)$ is

\[(2.3)\]

\[ x(t) = \frac{1}{2\sqrt{\alpha}} \left( \sqrt{t - \beta} / \sqrt{t} \right), \quad t \in (0, \infty). \]

If $x = d$ corresponds to $t = b$, that is, if $\sqrt{b} = \sqrt{\alpha d^2 + \beta + \sqrt{\alpha d}}$, then it is easily seen that $x = -d$ corresponds to $t = \beta^2 / b$.

**Theorem 2.2.** With $x(t)$ given by (2.3), let $Q(t) = (2\sqrt{\alpha t})^n P(x(t))$. Then

$P(x) \in \mathcal{Y}(n)$ iff $Q(t) \in \mathcal{Z}(n, \beta)$.

**Proof.** Since $x(\beta^2 / t) = -x(t)$, it follows that

\[ \frac{t^n Q(\beta^2 / t)}{(-\beta)^n} = (-1)^n (2\sqrt{\alpha t})^n P(-x(t)). \]

Hence, $P(x) = (-1)^nP(-x)$ iff $Q(t) = t^n Q(\beta^2 / t)/(-\beta)^n$. Thus, all we need to show is that if $P(x)$ is a monic polynomial of degree $n$, then so is $Q(t)$ and vice versa. This can be established using the following observations.

If $Q(t) \in \mathcal{Z}(n, \beta)$ and if $Q(t) = \sum_{r=0}^{n} q_r t^r$, where $q_n = 1$, then $q_{n-r} = (-\beta)^{2r-n} q_r$. Hence, for $r = [(n+1)/2], \ldots, n$

\[ \left\{ 2\sqrt{\alpha t(x)} \right\}^{-n} [q_r \{t(x)\}^n + q_{n-r} \{t(x)\}^{n-r}] \]
is a polynomial of the form

$$(4\alpha)^{-n}q_rx^{2r-n} + \text{lower degrees}.$$  

On the other hand, if $P(x) \in Y(n)$, then for $n = 2m$ we can write $P(x) = \sum_{r=0}^{m} p_{2r}x^{2r}$, where $p_{2m} = 1$. Hence, $(2\sqrt{\alpha t})^{2m}p_{2r}(x(t))^{2r}$ is the polynomial

$$(4\alpha)^{m-r}p_{2r}t^{m-r}(t-\beta)^{2r},$$

for $r = 0, 1, \ldots, m$. Similarly for $n = 2m + 1$, we can write $P(x) = \sum_{r=0}^{m} p_{2r+1}x^{2r+1}$, where $p_{2m+1} = 1$. Hence, $(2\sqrt{\alpha t})^{2m+1}p_{2r+1}(x(t))^{2r+1}$ is the polynomial

$$(4\alpha)^{m-r}p_{2r+1}t^{m-r}(t-\beta)^{2r}, \quad r = 0, 1, \ldots, m.$$  

3. The main results

We first consider a result regarding the weight functions.

**Theorem 3.1.** Let $b$ and $d$ be such that $\sqrt{b} = \sqrt{\alpha d^2 + \beta} + \sqrt{\alpha d}$ and let $V(t) = At^{-1/2}W(x(t))$. Then $W(x)$ is a weight function on $(-d, d)$ such that $W(x) = W(-x)$ if and only if $V(t)$ is a strong weight function on $(\beta^2/b, b)$ such that $\sqrt{t}V(t) = \sqrt{\beta^2/t}V(\beta^2/t)$. Here, $A$ is any positive number.

**Proof.** It is easily verified that if $W(x)$ is positive and satisfies $W(x) = W(-x)$ in $(-d, d)$, then $V(t)$ is positive and satisfies $\sqrt{t}V(t) = \sqrt{\beta^2/t}V(\beta^2/t)$ in $(\beta^2/b, b)$ and vice versa. Therefore we need only to prove that when the moments (including the negative ones for $V(t)$) exist and are finite for one of the functions, the same is also true for the other function. We have

$$\int_{\beta^2/b}^{b} t^r V(t) \, dt = A 2\sqrt{\alpha} \int_{-d}^{d} \left\{ \sqrt{\alpha x^2 + \beta} + \sqrt{\alpha x} \right\}^{2r+1} \frac{W(x)}{\sqrt{\alpha x^2 + \beta}} \, dx.$$  

Since $W(x) = W(-x)$, this can be written as

$$\int_{\beta^2/b}^{b} t^r V(t) \, dt = \int_{-d}^{d} S_r(x)W(x) \, dx,$$

where $S_r(x)$ is an even polynomial of exact degree $2r$ for $r \geq 0$ and is an even polynomial of exact degree $|2r| - 2$ for $r \leq -1$. As the monomials $x^{2r}$ and the polynomials $S_r(x)$ can be expressed as linear combinations of each other, the proof of the theorem follows.

**Theorem 3.2.** Let $W(x)$ and $V(t)$ be a pair of weight functions given by Theorem 3.1. Then for $n \geq 0$

$$(3.1) \quad \tilde{B}_n(V; t) = (2\sqrt{\alpha t})^nB_n(W; x(t)).$$

**Proof.** From Theorems 2.1 and 3.1 it follows that

$$\int_{\beta^2/b}^{b} t^{-n+s}(2\sqrt{\alpha t})^nB_n(W; x(t))V(t) \, dt = 0, \quad 0 \leq s \leq n - 1.$$  

Hence from (1.3) and Theorem 2.2, the result of the theorem is immediate.
As a consequence of (3.1) we obtain from (1.1) and (1.4) that

\[ \tilde{\alpha}_{n+1}(V) = 4\alpha\alpha_{n+1}(W), \quad n \geq 1. \]

Furthermore, if \( G(x, u) \) is a generating function for the polynomials \( B_n(W; x) \), then

\[ F(t, u) = G(x(t), 2\sqrt{\alpha}tu) \]

is a generating function for \( \tilde{B}_n(V; t) \).

### 4. Illustrative examples

**Example 1. The Tchebyshev case.** If \( W(x) = 1/\sqrt{1 - x^2} \) in \((-1, 1)\), then from Theorem 3.1

\[ V(t) = \frac{1}{\sqrt{b} - t\sqrt{t - a}} \quad \text{in } (a, b), \]

where \( \sqrt{b} = \sqrt{\alpha + \beta} + \sqrt{\alpha} \) and \( \sqrt{a} = \sqrt{\alpha + \beta} - \sqrt{\alpha} \).

The associated polynomials \( \tilde{B}_n(V; t) \), \( n \geq 1 \), and their interpolatory quadrature rules have already been studied in [8]. From [8], we can also confirm that (3.2) holds.

Now from (3.3) we get

\[ F(t, u) = \left(1 - (t - \beta)u / 1 - 2(t - \beta)u + 4\alpha tu^2\right)^{-1}, \]

as a generating function for \( \tilde{B}_n(V; t) \).

**Example 2. The Gegenbauer case.** For \( \lambda > 0 \), if \( W(x) = (1 - x^2)^{\lambda-1/2} \) in \((-1, 1)\), then

\[ V(t) = t^{-\lambda}(b - t)^{\lambda-1/2}(t - a)^{-\lambda-1/2} \quad \text{in } (a, b), \]

where \( \sqrt{b} = \sqrt{\alpha + \beta} + \sqrt{\alpha} \) and \( \sqrt{a} = \sqrt{\alpha + \beta} - \sqrt{\alpha} \). From the recurrence relation for the monic Gegenbauer polynomials it then follows that

\[ \tilde{\alpha}_{n+1}(V) = n(n + 2\lambda - 1) / (n + \lambda)(n + \lambda - 1)\alpha, \quad n \geq 1. \]

For \( \lambda = 1/2 \), that is, for the Legendre case, this result has already been given in [8].

From (3.3), we obtain as a generating function for \( \tilde{B}_n(V; t) \),

\[ F(t, u) = (1 - 2(t - \beta)u + 4\alpha tu^2)^{-\lambda}. \]

**Example 3. The Hermite case.** If \( W(x) = \exp(-x^2) \) in \((0, \infty)\), then

\[ V(t) = t^{-1/2}\exp\left(-\frac{(t + \beta^2/t)}{4\alpha}\right) \quad \text{in } (0, \infty). \]

From the recurrence relation for the monic Hermite polynomials we find

\[ \tilde{\alpha}_{n+1}(V) = 2\alpha n, \quad n \geq 1. \]

This result for \( \alpha = 1/2 \) has also been given in [8].

Using (3.3) as a generating function for \( \tilde{B}_n(V; t) \) we have

\[ F(t, u) = \exp(2(t - \beta)u - 4\alpha tu^2). \]
5. AN APPLICATION

Let \( W(x) = (1 + kx^2)(1 - x^2)^{-1/2} \) in \((-1, 1)\), where \( k > 0 \). Choosing \( \alpha \) and \( \beta \) such that \( k = \alpha/\beta \), we obtain

\[
V(t) = \frac{t^{-1}(t + \beta)^2}{\sqrt{b - t}\sqrt{1 - t}} = \frac{t(1 + \beta/t)^2}{\sqrt{b - t}\sqrt{1 - t}} \quad \text{in} \ (a, b),
\]

where \( \sqrt{b} = \sqrt{\alpha + \beta + \sqrt{\alpha}} \) and \( \sqrt{a} = \sqrt{\alpha + \beta - \sqrt{\alpha}} \). Equivalently, we can also write \( \beta = \sqrt{ab} \) and \( \alpha = (\sqrt{b} - \sqrt{a})/2 \).

It has been shown in [12] that the polynomials \( \tilde{B}_n(V; t) \) satisfy the recurrence relation (1.4) with

\[
\tilde{\alpha}_{n+1}(V) = \frac{K_{n-1}K_{n+2}}{K_nK_{n+1}} \alpha, \quad n \geq 1,
\]

with \( K_n = (1 + l)^n + (1 - l)^n, \ n \geq 0, \) and \( l = \sqrt{1 + \alpha/\beta} \). Hence from (3.2) we obtain the following result.

**Theorem 5.1.** The monic orthogonal polynomials \( B_n(W; x), \ n \geq 0, \) associated with the weight function

\[
W(x) = \frac{1+kx^2}{\sqrt{1-x^2}} \quad \text{in} \ (-1, 1),
\]

where \( k > 0 \), satisfy

(5.1) \( B_{n+1}(W; x) = xB_n(W; x) - \frac{1}{4} \frac{K_{n-1}K_{n+2}}{K_nK_{n+1}} B_{n-1}(W; x), \quad n \geq 1, \)

with \( B_0(W; x) = 1, \ B_1(W; x) = x \). Here \( K_n = (1 + l)^n + (1 - l)^n, \ n \geq 0, \) and \( l = \sqrt{1 + k} \).

Since \( \int_{-1}^{1} W(x) \, dx = (2 + k)\pi/2 \), we obtain from (5.1) the orthogonality property

\[
\int_{-1}^{1} B_m(W; x)B_n(W; x) \frac{1+kx^2}{\sqrt{1-x^2}} \, dx = \delta_{mn}2^{-2n} \frac{K_{n+2}}{K_n} \pi/2.
\]

Finally, if we let \( P_n(W; x) = 2^{n-1}K_nB_n(W; x) \) for \( n \geq 0 \), then

\[
\int_{-1}^{1} P_m(W; x)P_n(W; x) \frac{1+kx^2}{\sqrt{1-x^2}} \, dx = \delta_{mn}K_nK_{n+2}\pi/8
\]

and

\[
K_nP_{n+1}(W; x) = 2K_{n+1}xP_n(W; x) - K_{n+2}P_{n-1}(W; x), \quad n \geq 1,
\]

with \( P_0(W; x) = 1 \) and \( P_1(W; x) = 2x \).

**References**


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