ON THE DETERMINANT
OF ELLIPTIC BOUNDARY VALUE PROBLEMS
ON A LINE SEGMENT

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Abstract. In this paper we present a formula for the determinant of a matrix-valued elliptic differential operator of even order on a line segment $[0, T]$ with boundary conditions.

1. Introduction and summary of the results

In this paper we present a formula for the determinant of a matrix-valued elliptic differential operator of even order on a line segment $[0, T]$ with boundary conditions. In order to state our results we introduce the following notation:

1. Denote by $A = \sum_{k=0}^{2n} a_k(x) D^k$ a differential operator, $D = D_x = -i \frac{d}{dx}$, where the coefficients are complex-valued $r \times r$ matrices depending smoothly on $x$, $0 < x < T$. The leading coefficient $a_{2n}(x)$ is assumed to be nonsingular and to have $\theta$ as a principal angle, i.e. $R_\theta \cap \text{Spec} a_{2n}(x) = \phi$ for $0 < x < T$, where $R_\theta := \{pe^{i\theta} \in \mathbb{C} \mid 0 < p < \infty\}$.

2. We impose boundary conditions of the form

$$\ell_j u(T) = 0, \quad m_j u(0) = 0 \quad (1 \leq j \leq n)$$

where $u \in C^\infty([0, T] ; \mathbb{C}^r)$ and $\ell_j, m_j$ are differential operators of the form

$$\ell_j := \sum_{k=0}^{\alpha_j} b_{jk} d_x^k, \quad m_j := \sum_{k=0}^{\beta_j} c_{jk} d_x^k \quad \left( d_x = \frac{d}{dx} \right)$$

such that $b_{jk}, c_{jk}$ are constant $r \times r$ matrices with $b_{j\alpha_j} = c_{j\beta_j} = \text{Id}$ and such that the integers $\alpha_j, \beta_j$ satisfy

$$0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_n \leq 2n - 1, \quad 0 \leq \beta_1 < \beta_2 < \cdots < \beta_n \leq 2n - 1.$$

Example 1. Dirichlet boundary conditions: $\alpha_D = \beta_D = (0, 1, \ldots, n - 1)$

$$b_{D, jk} = c_{D, jk} := \begin{cases} \text{Id} & \text{if } 1 \leq j \leq n, k = j - 1, \\ 0 & \text{otherwise.} \end{cases}$$

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Example 2. Neumann boundary conditions: \( \alpha_N = \beta_N = (n, n + 1, \ldots, 2n - 1) \)

\[
b_{N,j,k} = c_{N,j,k} := \begin{cases} 
\text{Id} & \text{if } 1 \leq j \leq n, k = n + j - 1, \\
0 & \text{otherwise.}
\end{cases}
\]

For convenience we write \( \alpha = (\alpha_1, \ldots, \alpha_n), \ |\alpha| = \sum_{j=1}^{n} \alpha_j \) and similarly \( \beta \) and \( |\beta| \). Boundary conditions of the above form are usually called separated.

Let \( B = (B_{jk}) \) and \( C = (C_{jk}) \), \( 1 \leq j \leq 2n, 0 \leq k \leq 2n - 1 \), be \( 2n \times 2n \) matrices whose entries are the following \( r \times r \) matrices

\[
B_{jk} := \begin{cases} 
\begin{cases} 
 b_{jk} & \text{if } 1 \leq j \leq n \text{ and } 0 \leq k \leq \alpha_j, \\
0 & \text{otherwise;}
\end{cases} \\
0 & \text{otherwise.}
\end{cases}
\]

\[
C_{jk} := \begin{cases} 
\begin{cases} 
 c_{j-n,k} & \text{if } n + 1 \leq j \leq 2n \text{ and } 0 \leq k \leq \beta_{j-n}, \\
0 & \text{otherwise.}
\end{cases} \\
0 & \text{otherwise.}
\end{cases}
\]

We denote by \( A = A_{B,C} \) the operator \( \mathcal{A} \) restricted to the space of smooth functions \( u : [0, T] \rightarrow \mathbb{C}^r \) satisfying the boundary conditions (1.1).

(3) \( \zeta \)-regularized determinant \( \text{Det}_\theta A \). In the case where \( A \) is not 1-1, define \( \text{Det}_\theta A = 0 \). In the case \( A \) is 1-1, one proceeds as follows. As the coefficient \( a_{2n}(x) \) has \( \theta \) as a principal angle, there exists \( \varepsilon > 0 \) so that \( L(\theta - \varepsilon, \theta + \varepsilon) \cap \text{Spec} a_{2n}(x) = \emptyset, 0 \leq x \leq T \), where \( L(\alpha, \beta) := \{ z \in \mathbb{C} | \alpha \leq \arg z \leq \beta \} \). Then the spectrum of \( A \), \( \text{Spec} A \), is discrete, \( \text{Spec} A = \{ \lambda_j, j \in \mathbb{N} \} \), \( |\lambda_j| \rightarrow \infty \), and \( \text{Spec} A \cap L(\theta - \varepsilon', \theta + \varepsilon') \) for any \( 0 < \varepsilon < \varepsilon' \) is finite.

If \( R_\theta \cap \text{Spec} A = \emptyset \), we define \( \zeta_{A, \theta}(s) = \sum_{j \geq 1} \lambda_j^{-s} = Tr A^{-s} \) where \( s \in \mathbb{C} \), \( \text{Re} s > 1/2n \) and where the complex powers are defined with respect to the angle \( \theta \). It is a well-known fact that \( \zeta_{A, \theta}(s) \) admits a meromorphic extension to \( \mathbb{C} \) with \( s = 0 \) being a regular point. According to Ray and Singer [RS] one defines \( \log \text{Det}_\theta A := -\sum_{s=0}^{\theta} \zeta_{A, \theta}(s) \). If \( R_\theta \cap \text{Spec} A = \emptyset \), then choose \( \theta' = (\theta - \varepsilon, \theta + \varepsilon) \) so that \( R_{\theta'} \cap \text{Spec} A = \emptyset \), and define \( \text{Det}_\theta A := \text{Det}_{\theta'} A \). It can be easily checked (cf. [BFK1]) that the definition is independent of the choice of \( \theta' \) in \( = (\theta - \varepsilon, \theta + \varepsilon) \).

(4) The fundamental matrix \( Y(x) = Y(x, \mathcal{A}) \). Denote by \( Y(x) = (y_{kt}(x)) \) \((x \in \mathbb{R})\) the fundamental matrix for \( \mathcal{A} \). Note that \( Y(x) \) is a \( 2n \times 2n \) matrix whose entries \( y_{kt}(x) (0 \leq k, \ell \leq 2n - 1) \) are \( r \times r \) matrices defined by

\[
y_{kt}(x) := d_x^k y(t)(x)
\]

where \( y_t(x) \) denotes the solution of the Cauchy problem \( \mathcal{A} y_t(x) = 0, y_{kt}(0) = \delta_{kt} \text{Id} \). Of particular interest is the \( 2n \times 2n \) matrix \( Y(T) \), the evaluation of the fundamental matrix at \( x = T \).

(5) Introduce the quantities

\[
g_{\alpha} := \frac{1}{2} \left( \frac{|\alpha|}{n} - n + \frac{1}{2} \right), \quad h_{\alpha} = \text{det} \left( \begin{array}{ccc} w_1^{\alpha_1} & \cdots & w_n^{\alpha_1} \\
\vdots & \ddots & \vdots \\
w_1^{\alpha_n} & \cdots & w_n^{\alpha_n} \end{array} \right)
\]

where \( w_1, \ldots, w_n \) denote the \( 2n \) th roots of \( (-1)^{n+1} \) with \( \text{Re} w > 0 \) given by \( w_k = \exp \left\{ \frac{2k - n - 1}{2n} \pi i \right\} \). For a \( r \times r \) matrix \( a \) with principal angle \( \theta \) and eigenvalues \( \lambda_1, \ldots, \lambda_r \), denote \( (\det a)^{\theta^*} = \prod_{j=1}^{r} |\lambda_j|^{\theta^*} \exp \{ ig_{\alpha} \arg \lambda_j \} \) where \( \theta - 2\pi < \arg \lambda_j < \theta \).
Example 1. Dirichlet boundary conditions:
\[ g_{\alpha D} = -\frac{n}{4}, \quad h_{\alpha D} = h_n := \prod_{i \geq j} (w_i - w_j). \]

Example 2. Neumann boundary conditions:
\[ g_{\alpha N} = \frac{n}{4}, \quad h_{\alpha N} = (-1)^n h_n. \]

The main result of this paper is

Theorem.
\[ \text{Det}_\theta A = K_\theta \exp \left\{ \frac{i}{2} \int_0^T \text{tr}(a_{2n}^{-1}(x)a_{2n-1}(x)) \, dx \right\} \text{det}(BY(T) - C) \]
where \( K_\theta \equiv K_\theta(\alpha, \beta) \) is given by
\[ K_\theta = ((-1)^{\beta/2}(2n)^{\alpha-1}h_\beta^{-1}) (\text{det} a_{2n}(0))^{\beta/\theta} (\text{det} a_{2n}(T))^{-\beta/\theta}. \]

Example 1. Dirichlet boundary conditions: 
\[ |\alpha_D| = \frac{n(n-1)}{2}, \]
\[ K_\theta = ((-1)^{|\alpha_D|}(2n)^{\alpha-2}h_\alpha^{-2}) (\text{det} a_{2n}(0))^{\theta/4} (\text{det} a_{2n}(T))^{-\theta/4}. \]

Example 2. Neumann boundary conditions: 
\[ |\alpha_N| = \frac{n(n-1)}{2}, \]
\[ K_\theta = ((-1)^{|\alpha_N|}(2n)^{\alpha-2}h_\alpha^{-2}) (\text{det} a_{2n}(0))^{\theta/4} (\text{det} a_{2n}(T))^{-\theta/4}. \]

Corollary. \( \text{Det}_\theta A \) is a complex number independent of \( \theta \) up to multiplication with a \( 2n \) th root of unity.

Remark 1. In the formula above all terms except the matrix \( Y(T) \) are easily computable from the coefficients of \( s/ \), \( e_\delta \) and \( m_i \). The matrix \( Y(T) \) requires the knowledge of the fundamental solutions. The matrix \( Y(T) \) and therefore \( \text{det}(BY(T) - C) \) can be calculated numerically within arbitrary accuracy by solving a finite difference equation approximating \( s/ \). So the determinant \( \text{Det}_\theta A \) can be calculated numerically within arbitrary accuracy.

Remark 2. Theorem is a companion of the corresponding result on the circle instead of the interval \([0, T]\) which was treated in an earlier paper [BFK1]. Again, the proof of Theorem relies on a deformation argument and explicit computations for certain special operators and special boundary conditions.

Remark 3. Introduce a spectral parameter \( \lambda \), and denote the fundamental matrix of \( s/ + \lambda \) by \( Y(x, \lambda) = Y(x, s/ + \lambda) \). One then verifies \( \text{det}(BY(T; \lambda) - C) = 0 \) iff \( \text{Det}_\theta (A + \lambda) = 0 \), i.e. \( -\lambda \) is an eigenvalue of \( A = A_B, C \).

Remark 4. First results of the type described in Theorem are due to Dreyfus and Dym [DD] and to Forman [Fo1] (cf. also [Fo2]). Forman proved by different methods that the quotient \( \text{Det}_\theta A / \text{Det}(BY(T) - C) \) only depends on the principal and subprincipal symbols of \( s/ \), and the principal symbol of the boundary operators \( \ell_j, m_j \ (1 \leq j \leq n) \). Our Theorem provides a formula for this quotient.
Remark 5. Analogous to results obtained in [BFK2], Theorem can be extended to the case where \( \mathcal{A} \) is a pseudodifferential operator. The determinant \( \text{Det}_\theta A \) can be written as a product of local invariants with a Fredholm determinant of a pseudodifferential operator of determinant class, canonically associated to \( A \). The Fredholm determinant corresponds to \( \det(BY(T) - C) \) in the case when \( \mathcal{A} \) is a differential operator.

2. Auxiliary results

In this section we collect some auxiliary results needed for the proof of Theorem. First we introduce some additional notation. Denote by \( EDO_{2n} \equiv EDO_{2n,T} \) the set of all elliptic differential operators \( \mathcal{A} \) of order \( 2n \) on \([0, T]\) as introduced in Section 1. We identify \( EDO_{2n} \) with the open set \( \{(a_2, \ldots, a_0) \in C^\infty([0, T]), \text{End} C' \} \) of the Frechet space \( C^\infty([0, T], \text{End} C') \). Further define \( EDO_{2n, \theta} := \{\mathcal{A} \in EDO_{2n} : \theta \text{ is principal angle for } a_2\} \). Clearly \( EDO_{2n, \theta} \) is an open connected subset in \( EDO_{2n} \).

In this section we introduce some additional notation. Denote by \( EDO_{2n, \theta} := \{A, B, C \in EDO_{2n}, B \in BDO_a, C \in BDO_\beta \} \) where \( A, B, C \) is the restriction of \( \mathcal{A} \) to the subspace of functions \( u \in C^\infty([0, T], C') \) satisfying the boundary conditions defined by \( B \) and \( C \). Similarly introduce \( EDO_{2n, \theta; \alpha; \beta} := \{A, B, C \in EDO_{2n; \theta; \alpha; \beta} : \mathcal{A} \in EDO_{2n; \theta} \} \). Observe that \( \{A, B, C \in EDO_{2n; \theta; \alpha; \beta} : \mathcal{A} \text{ is 1-1} \} \) is open.

Further, denote by \( EDO_{2n; \theta; \alpha; \beta} \) the open subset of \( EDO_{2n; \alpha; \beta} \) consisting of pairs \( (A, B, C, \theta) \) with \( A, B, C \in EDO_{2n; \theta; \alpha; \beta} \). As in [BFK1] we have the following

Proposition 2.1. (1) \( \text{Det}_\theta(A, B, C) \) is a smooth function on \( \widetilde{EDO}_{2n; \alpha; \beta} \) and is locally constant in \( \theta \).

(2) \( \text{Det}_\theta(A, B, C) \) is holomorphic when considered as a function on the open subset of injective operators in \( EDO_{2n; \theta; \alpha; \beta} \).

(3) \( \det(BY(T, \mathcal{A}) - C) \) is holomorphic on \( EDO_{2n} \times BDO_a \times BDO_\beta \).

Observe that a necessary and sufficient condition for \( A, B, C \) to have zero as an eigenvalue is that \( \det(BY(T) - C) = 0 \), which in view of Proposition 2.1 (3) implies that the subsets of \( EDO_{2n; \theta; \alpha; \beta} \) and \( EDO_{2n; \alpha, \beta} \) consisting of injective operators are open (as we already noticed) and connected, and therefore, \( \widetilde{EDO}_{2n; \alpha; \beta} \) is open and connected as well.

Let \( s : [0, T] \to GL(C') \) be a smooth map. Given \( \mathcal{A} \in EDO_{2n} \) and boundary operators \( \ell_j, m_j (1 \leq j \leq n) \) introduce \( \mathcal{A}_1 := s(x)^{-1} \mathcal{A} s(x), \ell_{ij} := s(T)^{-1} \ell_j s(x) |_{x=T}, \text{ and } m_{ij} := s(0)^{-1} m_j s(x) |_{x=0} \). Denote by \( (B_{ijk}) \) and \( (C_{ijk}) \) the matrices introduced in Section 1 corresponding to the boundary operators \( (\ell_{ij}, m_{ij})_{1 \leq i, j \leq n} \) and write \( Y_1(x) = Y(x, \mathcal{A}_1) \) for short.

Proposition 2.2. \( \det(B_1 Y_1(T) - C_1) = (\det s(0)s(T)^{-1})^n \det(BY(T) - C) \).
Proof. Let $L = L(x)$ be a $2n \times 2n$ matrix with entries $L_{k\ell}$ which are the following $r \times r$ matrices ($0 \leq k, \ell \leq 2n - 1$)

$$L_{k\ell} := \binom{k}{\ell} d_s^{k-\ell} s(x) \quad \text{if } k \geq \ell; \quad L_{k\ell} = 0 \quad \text{if } k < \ell.$$ 

Thus we obtain

$$B_1 = \text{diag} \left( s(T)^{-1}, \ldots, s(T)^{-1} \right) BL(T)$$

where $\text{diag} \left( s(T)^{-1}, \ldots, s(T)^{-1} \right)$ is a $2n \times 2n$ diagonal matrix whose entries on the diagonal are all equal to the $r \times r$ matrix $s(T)^{-1}$. Similarly, one obtains

$$C_1 = \text{diag} \left( s(0)^{-1}, \ldots, s(0)^{-1} \right) CL(0).$$

Further, by a straightforward computation, $Y_1$ is given by

$$Y_1(x) = L(x)^{-1} Y(x) L(0).$$

Thus

$$B_1 Y_1(T) - C_1 = \text{diag} \left( s(T)^{-1}, \ldots, s(T)^{-1}, s(0)^{-1}, \ldots, s(0)^{-1} \right) \cdot [B Y(T) - C] L(0).$$

Now observe that $\det L(0) = (\det s(0))^{2n}$ as $L(0)$ is lower triangular with diagonal entries all equal to the $r \times r$ matrix $s(0)$. This implies that

$$\det (B_1 Y_1(T) - C_1) = (\det s(0)s(T)^{-1})^n \det (B Y(T) - C).$$

Next consider for $A = A_B, C$ in $EDO_{2n}; \theta; \alpha; \beta$ and $\Phi \in C^\infty([0, T], GL_r(\mathbb{C}))$ the generalized $\zeta$-function

$$\zeta_{\Phi, A; \theta}(s) := \text{tr}_A \Phi^{s-1}.$$ 

Again this is a function which is holomorphic in $\text{Re } s > \frac{1}{2n}$ and has a meromorphic extension to the whole complex plane. Moreover $s = 0$ is a regular point. Recall that we have introduced $g_\alpha := \frac{1}{2}(\frac{|\alpha|}{n} - n + \frac{1}{2})$, and similarly $g_\beta$.

**Proposition 2.3.**

(2.1) $\zeta_{\Phi, A; \theta}(0) = g_\beta \text{tr} \Phi(0) + g_\alpha \text{tr} \Phi(T)$

As an immediate consequence we obtain

**Corollary 2.4.** $\zeta_{A; \theta}(0) = r(g_\alpha + g_\beta) = r\left(\frac{|\alpha| + |\beta|}{2n}\right) - n + 1$.

**Proof (Proposition 2.3).** We first prove that there are numbers $\tilde{g}_\alpha, \tilde{g}_\beta \in \mathbb{C}$ which only depend on $\alpha$ and $\beta$ respectively such that (2.1) holds. The actual values of $\tilde{g}_\alpha, \tilde{g}_\beta$ are computed at the end of section 3 by considering the case $\Phi(x) \equiv K$ with $K > 1$, $\mathcal{A} = D^n + \lambda$, $\theta = \pi$. In the course of the proof we use a number of results due to Seeley [Se 1, 2]. For the convenience of the reader we partly keep Seeley’s notation. For simplicity, we write $\zeta(s) = \zeta_{\Phi, A; \theta}(s)$. According to [Se2], the value $\zeta(0)$ consists of a sum of two terms, $\zeta(0) = I + II$ where $I$ represents the contribution to $\zeta(0)$ of the resolvent of $\mathcal{A} - \lambda$ and $II$ represents a correction term due to the boundary conditions. According to [BFK1, p. 8],

$$I = -\frac{e^{i\theta}}{4\pi n} \sum_{\tau = \pm 1} \int_0^T dx \int_0^\infty dr \text{tr} \{\Phi(x)c_{-2n-1}(x, \tau, re^{i\theta})\}$$

where $c_{-2n-1}(x, \tau, \lambda)$ comes from the expansion of the symbol

$$r(x, \tau, \lambda) = c_{-2n}(x, \tau, \lambda) + c_{-2n-1}(x, \tau, \lambda) + \cdots$$
of the parametrix for \( \mathcal{A} - \lambda = (a_{2n}(x)D^{2n} - \lambda) + \sum_{j=0}^{2n-1} a_j(x)D^j \) and is given by

\[
c_{-2n-1}(x, \tau, \lambda) = -\tau^{2n-1}c_{-2n}a_{2n-1}c_{-2n} - i2n\tau^{4n-1}c_{-2n}a_{2n}c_{-2n} \left( \frac{d}{dx}a_{2n} \right) c_{-2n},
\]

where \( c_{-2n} \equiv c_{-2n}(x, \tau, \lambda) = (a_{2n}(x)\tau^{2n} - \lambda)^{-1} \).

As in [BFK1], Proposition 2.8, in view of the fact that \( c_{-2n-1} \) is odd in \( \tau \), we conclude \( I = 0 \). From [Se2], p. 968, it follows that \( II \) is of the form

\[
II = \text{tr} \{ \Delta'_0(0)\Phi(0) + \Delta'_T(0)\Phi(T) \}
\]

where \( \Delta'_0(s) \) and \( \Delta'_T(s) \) are smooth functions described below. Let us first consider the scalar case, \( r = 1 \). In first approximation the kernel \( r(x, y, \lambda) \) of \( (A_B, c - \lambda)^{-1} \) is given by

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} (a_{2n}(x)\tau^{2n} - \lambda)^{-1}e^{(x-y)\tau} d\tau + r_0(x, y, \lambda) + r_T(x, y, \lambda)
\]

where \( r_0(x, y, \lambda) \) and \( r_T(x, y, \lambda) \) are correction terms so that in first approximation \( r(x, y, \lambda) \) satisfies the boundary conditions at \( x = 0 \) and \( x = T \). Let us explain how to obtain \( r_0(x, y, \lambda) \); for \( r_T(x, y, \lambda) \) one proceeds in a similar fashion. Consider the boundary value problem

\[
(2.2) (aD^{2n} - \lambda)u = 0
\]

with the boundary condition

\[
(2.3) \lim_{x \to \infty} u(x) = 0; \quad D^\beta_i u(0) = -(a\tau^{2n} - \lambda)^{-1}\tau^\beta_i e^{-iy\tau}
\]

where \( a = a_{2n}(0) \) and \( D = \frac{1}{i} \frac{d}{dx} \). The solution \( u(x) = u(x, \tau, y, \lambda) \) of the boundary value problem (2.2)-(2.3) is given by \( u(x) = \sum_{\nu=1}^{n} u_\nu e^{ix(-\lambda/a)^{1/2n}w_\nu} \) where \( w_\nu \) (\( 1 \leq \nu \leq n \)) are the \( 2n \)th roots of \(-1\) with strictly positive imaginary part and where \((-\lambda/a)^{1/2n} = (|\lambda/a|^{1/2n} e^{i(\theta-\pi-\arg a)/2n} \) with \( \lambda = |\lambda|e^{i\theta} \) and \( \theta - 2\pi < \arg a < \theta \). The coefficients \( u_\nu = u_\nu(\tau, y, \lambda) \) are then determined by (2.3)

\[
\sum_{\nu=1}^{n} u_\nu \left( -\frac{\lambda}{a} \right)^{\beta_j/2n} w_\nu^{\beta_j} = -\tau^\beta_i (a\tau^{2n} - \lambda)^{-1} e^{-iy\tau}.
\]

Thus

\[
\sum_{\nu=1}^{n} u_\nu = -\sum_{j=1}^{n} \mathcal{H}_{\beta_j}(\frac{\lambda}{a})^{-\beta_j/2n}\tau^\beta_j(a\tau^{2n} - \lambda)^{-1} e^{-iy\tau}
\]

with \( \mathcal{H}_{\beta_j} \) defined by

\[
(2.4) \sum_{j=1}^{n} \mathcal{H}_{\beta_j} w_k^{\beta_j} = \delta_{\nu k}.
\]

The term \( r_0(x, y, \lambda) \) is then given by

\[
r_0(x, y, \lambda) = \sum_{\nu=1}^{n} e^{ix(-\lambda/a)^{1/2n}w_\nu} \sum_{j=1}^{n} \mathcal{H}_{\beta_j} \frac{1}{i}(\frac{\lambda}{a})^{-\beta_j/2n} f
\]
where \( \mathcal{J} \) is the sum of residues

\[
\mathcal{J} = \sum_{k=1}^{n} \text{Res}_{z=\lambda} \left\{ \tau^{\lambda} (a \tau^{2n} - \lambda)^{-1} e^{-iy\tau} \right\}
\]

of \( \tau^{\lambda} (a \tau^{2n} - \lambda)^{-1} e^{-iy\tau} \) in the lower half plane. One obtains

\[
\mathcal{J} = \sum_{k=1}^{n} ((-\lambda/a) \tau^{2n} \bar{w}_k)^{\beta_j} \left( (-\lambda/a)^{2n} - 1 \right) \frac{1}{2na} \exp\left\{-iy(-\lambda/a)^{2n} \bar{w}_k \right\}.
\]

Summarizing one obtains

\[
r_0(x, y, \lambda) = \frac{i}{2na} (-\lambda/a)^{-(2n-1)/2n} \sum_{\nu, j, k} \mathcal{H}_{\nu j} \bar{w}_k^{\beta_j+1} \exp\left\{i(-\lambda/a)^{1/2n} (x \nu - y \bar{w}_k) \right\}.
\]

Following Seeley, we now define for \( \text{Re} s > 0 \)

\[
\Delta_0(s) := \int_{0}^{T/2} dx \frac{1}{2\pi i} \int_{\Gamma_0} d\lambda \lambda^{-s} r_0(x, x, \lambda)
\]

where \( \Gamma_0 \) is the contour that goes from \( \infty \) to \( 0 \) along the lower side of ray \( \{re^{i\theta} : r > 0\} \), goes around the origin and then returns to \( \infty \) along the upper side of the ray \( \{re^{i\theta} : r > 0\} \). By a standard computation,

\[
\frac{1}{2\pi i} \int_{\Gamma_0} d\lambda \lambda^{-s} r_0(x, x, \lambda) = a^{-s} e^{-\pi ns} \frac{\sin \pi s}{\pi} \Gamma(1 - 2ns) \sum_{\nu, j, k} \mathcal{H}_{\nu j} \bar{w}_k^{\beta_j+1} ((x \nu - y \bar{w}_k) x)^{-1+2ns}
\]

and therefore

\[
\Delta_0(0) = \frac{1}{2n} \sum_{\nu, j, k} \mathcal{H}_{\nu j} \frac{\bar{w}_k^{\beta_j+1}}{w_\nu - \bar{w}_k}.
\]

In the case \( r \geq 2 \), we first treat the case where all eigenvalues of \( a_{2n}(0) \) are different which can be easily reduced to scalar case \( r = 1 \). By a continuity argument we then conclude that

\[
\tilde{g}_\beta = \frac{1}{2n} \sum_{\nu, j, k} \mathcal{H}_{\nu j}(\beta) \bar{w}_k^{\beta_j+1} (w_\nu - \bar{w}_k)^{-1}
\]

where \( \mathcal{H}_{\nu j} = \mathcal{H}_{\nu j}(\beta) \) are determined by (2.4). Similarly one obtains

\[
\tilde{g}_\alpha = \frac{1}{2n} \sum_{\nu, j, k} \mathcal{H}_{\nu j}(\alpha) \bar{w}_k^{\alpha_j+1} (w_\nu - \bar{w}_k)^{-1}.
\]

3. Proof of Theorem 1

For the proof of Theorem we need two deformation results. The first one is the analogue of Proposition 3.1 in [BFK1] and proved in a similar way (cf. also [DD] and [Fo1]).
Proposition 3.1. Suppose \( \mathcal{A} = \sum_{k=0}^{2n} a_k(x) D^k \) and \( \mathcal{A}' = \sum_{k=0}^{2n} a_k'(x) D^k \) are in \( EDO_{2n; \theta} \) with \( a_{2n} = a'_{2n} \) and \( a_{2n-1} = a'_{2n-1} \). Then, for \( B \in BDO_\alpha \) and \( C \in BDO_\beta \)

\[
\text{Det}_\theta(AB, C) \det(B Y(T, \mathcal{A}')) - C) = \text{Det}_\theta(A_B', C) \det(B Y(T, \mathcal{A}) - C).
\]

The second result concerns a deformation of the boundary conditions. Consider boundary operators \( 1 \leq j \leq n, d_x = \frac{d}{d_x} \)

\[
\ell_j = \sum_{k=0}^{\alpha_j} b_{jk} d_x^k, \quad m_j = \sum_{k=0}^{\beta_j} c_{jk} d_x^k; \quad b_{j\alpha_j} = c_{j\beta_j} = \text{Id}
\]

and

\[
\ell'_j = d_x^{\alpha_j}, \quad m'_j = d_x^{\beta_j}.
\]

Form the matrices \( B, C \) and \( B', C' \) as in Section 1.

Proposition 3.2. Fix \( \mathcal{A} \in EDO_{2n; \theta} \). Then

\[
\text{Det}_\theta(A_B', C') \det(B Y(T) - C) = \text{Det}_\theta(A_B, C) \det(B' Y(T) - C').
\]

Proof. Without loss of generality we may assume that both \( A_B, C \) and \( A_B', C' \) are injective. Note that \( \{ A_B, C : A_B \rightarrow 1 \}, \tilde{B} \in BDO_\alpha, \tilde{C} \in BDO_\beta \} \)

is arcwise connected in \( BDO_\alpha \times BDO_\beta \). Define, for \( 0 \leq t \leq 1 \),

\[
\ell_{ij} = d_x^{\alpha_i} + t \sum_{k=0}^{\alpha_i-1} b_{jk} d_x^k, \quad c_{ij} = d_x^{\beta_i} + t \sum_{k=0}^{\beta_i-1} c_{jk} d_x^k
\]

such that, with \( B_t \) and \( C_t \) the corresponding matrices in \( BDO_\alpha \) and \( BDO_\beta \),

\[
(3.1) \quad A_{B_t}, C_t \text{ is 1-1 for } 0 \leq t \leq 1;
\]

\[
(3.2) \quad (B_0, C_0) = (B', C'), \quad (B_1, C_1) = (B, C).
\]

Introduce

\[
w(t) := \frac{d}{dt} \text{Det}_\theta(A_B, C_t) \quad \delta(t) := \frac{d}{dt} \frac{\text{Det}(B_t Y(T) - C_t)}{\text{Det}(B Y(T) - C)}.
\]

The claimed result follows once we show that \( w(t) = \delta(t) \) \((0 \leq t \leq 1)\). Let us first consider \( \delta(t) \). Denote by \( P_t \) the Poisson operator corresponding to the boundary value problem defined by \( (B_t, C_t) \). Then \( P_t \) is given by \( P_t = Y(x)(B_t Y(T) - C_t)^{-1} \) and

\[
\delta(t) = \text{tr} \{ (B_t Y(T) - C_t)(B_t Y(T) - C_t)^{-1} \}
\]

\[
= \text{tr} \{ (\ell_{ij}, \ell_{ij})_{1 \leq i \leq n} P_t \}
\]

when \( \ell_{ij} = \frac{d}{dt} \) and \( (\ell_{ij}, \ell_{ij})_{1 \leq i \leq n} \) is the operator associating to a section \( u \) the boundary values \( (\ell_{ij} u(T), \ell_{ij} u(0))_{1 \leq j \leq n} \).

Next we consider \( w(t) \); with the notation \( A_t = A_{B_t}, C_t \),

\[
w(t) = F.p.s=0 \text{tr} (A_t^{-1} A_t^{-1-s})
\]

where \( F.p.s=0 \) denotes the finite part at \( s = 0 \). In order to evaluate \( A_t^{-1} A_t' = -(A_t^{-1})' A_t \), consider for a fixed section \( u : [0, T] \rightarrow C' \) the section \( v_t := A_t^{-1} u \), i.e. \( v_t \) satisfies

\[
\mathcal{A} v_t = u, \quad B_t v_t(T) = 0, \quad C_t v_t(0) = 0.
\]
Taking derivatives with respect to $t$ we obtain
\[ \mathcal{A} v^*_i = 0, \quad \ell^*_{ij} v^*_i(T) = -\ell^*_{ij} v_i(T), \quad m^*_{ij} v^*_i(0) = -m^*_{ij} v_i(0) \quad (1 \leq j \leq n). \]
Thus
\[ v^*_i = -P_i(\ell^*_{ij} v_i(T), m^*_{ij} v_i(0))_{1 \leq j \leq n} \]
where $P_i$ again denotes the Poisson operator. Thus we have proved that $(A_t^{-1})^* = -P_i(\ell^*_{ij}, m^*_{ij})_{1 \leq j \leq n}$. Note that $(A_t^{-1}) A_t = -P_i(\ell^*_{ij}, m^*_{ij})_{1 \leq j \leq n}$ is a singular Green's operator of order $\leq -2$ and then of trace class. Thus
\[ w(t)^* = \text{tr} P_i(\ell^*_{kj}, m^*_{ij})_{1 \leq j \leq n}. \]
\[ \Box \]

**Proof of Theorem.** We have to prove that
\[ f_\theta(A_B, c) := \text{Det}_\theta(A_B, c) - K_\theta \exp \left\{ \frac{i}{2} \int_0^T \text{tr} (a_{2n}(x)^{-1} a_{2n-1}(x)) dx \right\} \]
\[ \cdot \det(BY(T) - C) \]
vanishes identically on $\{A_B, c \in EDO_{2n}; \theta : A_B, c \text{ is 1-1}\}$. First observe that it suffices to consider the case $\theta = \pi$: For $\mathcal{A}$ in $EDO_{2n}; \theta \in EDO_{2n}; \pi$ we have $\log \text{Det}_\pi(e^{i(\pi-\theta)} A_B, c) = \log \text{Det}_\theta(A_B, c) + \zeta_{A_B, \theta}(0) \log e^{i(\pi-\theta)}$ and $\log K_\theta(e^{i(\pi-\theta)} \mathcal{A}) = \log K_\pi(t) + r(g_\theta + g_\theta) i(\pi - \theta)$; thus Corollary 2.4 allows to conclude the result as soon as we check it for $\theta = \pi$.

To make writing easier, let $f = f_\pi$, $K = K_\pi$, $\theta \equiv \pi$.

**Deformation 1.** Consider the factorization $\mathcal{A} = a_{2n}(D^{2n} + \mathcal{H})$ where $\mathcal{H}$ is a differential operator with ord $\mathcal{H} \leq 2n - 1$. Consider the 1-parameter family $(0 \leq t \leq 1)$
\[ \mathcal{A}_t := \alpha_t(D^{2n} + \mathcal{H}), \quad A_t := A_{t, B, c} \]
when $\alpha_t(x) = ta_{2n}(x) + (1 - t)$. Clearly $\theta = \pi$ is a principal angle for $\alpha_t$ and $A_t$ is 1-1 for $0 \leq t \leq 1$. Moreover $A_t^* = (a_{2n}(x) - 1)(ta_{2n}(x) + (1 - t))^{-1} A_t$. Thus, with $w(t) = \log \text{Det}_\pi A_t$ and Proposition 2.3
\[ w(t)^* = F_p s_0 = \text{tr}(a_{2n}(x) - 1)(ta_{2n}(x) + (1 - t))^{-1} A(t)^{-t} \]
\[ = g_\beta \text{tr} [(a_{2n}(0) - 1)(ta_{2n}(0) + (1 - t))^{-1}] \]
\[ + g_\alpha \text{tr} [(a_{2n}(T) - 1)(ta_{2n}(T) + (1 - t))^{-1}] \]
\[ = \frac{d}{dt} \{ g_\beta \log \det[t a_{2n}(0) + (1 - t)] \]
\[ + g_\alpha \log \det[t a_{2n}(0) + (1 - t)] \}. \]

Thus
\[ \log \text{Det}_\pi A_1 - \log \text{Det}_\pi A_0 = \int_0^1 w(t)^* dt = g_\beta \log \det(a_{2n}(0)) + g_\alpha \log \det(a_{2n}(T)). \]
Hence we may and will assume that $a_{2n}(x) \equiv \text{Id}$. 

**Deformation 2.** Define $s \in C^\infty([0, T]; \text{End } \mathbb{C}^r)$ by
\[ \frac{d}{dx} s(x) = \frac{i}{2n} a_{2n-1}(x)s(x) \quad (0 \leq x \leq T); \quad s(0) = \text{Id}. \]
Observe that $\det(s(x)) = \exp\{\frac{1}{2n} \int_0^x \text{tr} (a_{2n-1}(y)) dy\} \neq 0$ for $0 \leq x \leq T$ and therefore $s(x) \in GL_\pi(\mathbb{C})$. Now consider $s_1 := s(x)^{-1}s_0(x)$ and boundary conditions defined by $B_1, C_1$ (cf. Proposition 2.2). Then $\text{Det}_\pi(A_1) = \text{Det}_\pi(A)$ as the spectrum of $A$ and the operator $A_1$, defined by $s_1$ and boundary conditions $(B_1, C_1)$ do coincide. By Proposition 2.2,

$$\det(B_1 Y(T) - C_1) = (\det s(T))^{-n} \det(BY(T) - C).$$

As we have noted above, $\det s(T) = \exp\{\frac{1}{2n} \int_0^T \text{tr} (a_{2n-1}(y)) dy\}$. Finally note that $s_1$ is of the form

$$s_1 = D^{2n} + \sum_{k=0}^{2n-2} a_{1k}(x) D^k$$

and then we may and will assume that for $s_1$, $a_{2n}(x) \equiv \text{Id}$ and $a_{2n-1}(x) \equiv 0$.

**Deformation 3.** Applying Proposition 3.1 and Proposition 3.2 we conclude that it remains to prove that $f(AB_1, C) = 0$ for $s_1 = D^{2n} + \lambda$ and $B, C$ given by

$$\ell_j = d_{\alpha j}, \quad m_j = d_{\beta j} \quad (1 \leq j \leq n)$$

where $\lambda$ is chosen positive and sufficiently large so that $A_{B, C}$ is $1-1$. This is verified by an explicit computation. To make writing easier we restrict ourselves to that case $r = 1$. However, to obtain the explicit formulas for $g_\alpha$ and $g_\beta$ we consider $s_1 = \rho D^{2n} + \lambda$ with $\rho > 1$. Denote by $Y(x, \lambda)$ the fundamental matrix for $\rho D^{2n} + \lambda$. For $\lambda > 0$, let $\mu = (\lambda^2)\frac{1}{2n}$. Then, with $w_k := \exp(\frac{i 2k-n-1}{2n} \pi)$, $Y(x, \lambda)$ is equal to

$$Y(x, \lambda) = \left( \begin{array}{cccccc} e^{\mu w_1 x} & \cdots & e^{\mu w_{2n} x} \\ \mu w_1 e^{\mu w_1 x} & \cdots & \mu w_{2n} e^{\mu w_{2n} x} \\ \vdots & \ddots & \vdots \\ (\mu w_1)^{2n-1} e^{\mu w_1 x} & \cdots & (\mu w_{2n})^{2n-1} e^{\mu w_{2n} x} \end{array} \right)^{-1} \left( \begin{array}{cccc} 1 & \cdots & 1 \\ \mu w_1 & \cdots & \mu w_{2n} \\ \vdots & \ddots & \vdots \\ (\mu w_1)^{n-1} & \cdots & (\mu w_{2n})^{2n-1} \end{array} \right).$$

Further define $B = (B_{jk})$, $C = (C_{jk})$ by

$$B_{jk} = \begin{cases} 1 & \text{if } 1 \leq j \leq n \text{ and } k = \alpha_j, \\ 0 & \text{otherwise}; \end{cases}$$

$$C_{jk} = \begin{cases} 1 & \text{if } n + 1 \leq j \leq 2n \text{ and } k = \beta_{j-n}, \\ 0 & \text{otherwise}. \end{cases}$$

We have to show that

$$\text{Det}_\pi((\rho D^{2n} + \lambda)_{B, C}) = (-1)^{|B|} (2n)^n (h_\alpha h_\beta)^{-1} \rho^{s_\alpha + s_\beta} \det(BY(T, \lambda) - C).$$

For that purpose we introduce

$$w(\lambda) := \log \text{Det}_\pi((\rho D^{2n} + \lambda)_{B, C}),$$

$$\delta(\lambda) := \log \det(BY(T; \lambda) - C).$$

As $n \geq 1$, we know from Proposition 3.1 that $\frac{d}{d\lambda} w(\lambda) = \frac{d}{d\lambda} \delta(\lambda)$. Therefore it suffices to consider the asymptotics of $w(\lambda)$ and $\delta(\lambda)$ as $\lambda \to +\infty$.

First recall from [Fr] (cf. also [Vo]) that $w(\lambda)$ admits an asymptotic expansion of the form $\sum_{k=-1}^\infty p_k \lambda^{-k/n} + \sum_{j=0}^\infty q_j \lambda^{-j} \log \lambda$ with the property that $p_0 = 0$. To find the asymptotics of $\delta(\lambda)$ as $\lambda \to +\infty$, write $Y(T, \lambda)$ in the form

$$Y(T; \lambda) = LWE(LW)^{-1}.$$
where $L = \text{diag}(1, \mu, \mu^2, \ldots, \mu^{2n-1})$, $E := \text{diag}(e^{\mu w_1 T}, \ldots, e^{\mu w_{2n} T})$ and

$$W = \begin{pmatrix}
1 & \cdots & 1 \\
w_1 & \cdots & w_{2n} \\
\vdots & \vdots & \vdots \\
w_{2n-1} & \cdots & w_{2n-1}
\end{pmatrix}.$$ 

Thus $\delta(\lambda) = \log(\det W^{-1} L^{-1}) + \log \det(\text{BLWE} - \text{CLW})$. Observe that the $(j,k)$th coefficient of the matrix $\text{BLWE} - \text{CLW}$ is of the form $e^{w_j T} f_{jk}(\mu) + g_{jk}(\mu)$ where $f_{jk}(\mu)$ and $g_{jk}(\mu)$ are rational functions of $\mu$. We conclude that, with $\Omega = \sum_{j=1}^n w_j = \sum_{j=1}^n \text{Re } w_j$,

$$\log \det(\text{BLWE} - \text{CLW}) = \mu\Omega T + \log \det[\text{BLW} \begin{pmatrix} \text{Id}_n & 0 \\ 0 & 0 \end{pmatrix} - \text{CLW} \begin{pmatrix} 0 & 0 \\ 0 & \text{Id}_n \end{pmatrix} ] + e(\lambda)$$

where $\lim_{\lambda \to \infty} e(\lambda) = 0$. The matrix $\text{BLW} \begin{pmatrix} \text{Id}_n & 0 \\ 0 & 0 \end{pmatrix} - \text{CLW} \begin{pmatrix} 0 & 0 \\ 0 & \text{Id}_n \end{pmatrix}$ is of the form

$$F^{(1)} F^{(2)}$$

where $F^{(i)}$ are $n \times n$ matrices given by $(1 \leq j, k \leq n)$

$$F^{(1)} := \mu^{a_j} w^{a_j}_k, \quad F^{(2)} := -\mu^{b_j} w^{b_j}_k = (-1)^{b_j+1} \mu^{b_j} w^{b_j}_k$$

where we used that $w_{n+k} = -w_k$. Therefore, with $|\alpha| = \sum_{j=1}^n \alpha_j$, $|\beta| = \sum_{j=1}^n \beta_j$

$$\det(\text{BLW} \begin{pmatrix} \text{Id}_n & 0 \\ 0 & 0 \end{pmatrix} - \text{CLW} \begin{pmatrix} 0 & 0 \\ 0 & \text{Id}_n \end{pmatrix} ] = \mu^{\alpha} \det(w^{a_j}_k) \mu^{\beta} (-1)^{|\beta|+n} \det(w^{b_j}_k).$$

In view of the fact that $\det L^{-1} |_{\lambda=1} = \prod_{j=0}^{2n-1} (1 - \mu^{-j})^{-2n} = \rho^{2n-1}$, this implies that the 0th order coefficient of the asymptotic expansion of $\delta(\lambda)$ for $\lambda \to \infty$ is of the form

$$\delta_{+\infty} := \det L^{-1} |_{\lambda=1} + \log(\det(W^{-1}) \det(w^{a_j}_k)(-1)^{|\beta|+n} \det(w^{b_j}_k) \rho^{-|\alpha|+|\beta|}/2n})$$

$$= \log \rho^{2n-1} - \log \rho^{(|\alpha|+|\beta|)/}\log((-1)^{|\beta|+n} \det(W^{-1})h_{\alpha} h_{\beta})$$

where $h_{\alpha} = \det(w^{a_j}_k)$, $h_{\beta} \equiv \det(w^{b_j}_k)$.

By a straightforward computation we have $\det W = (-1)^n (2n)^n$ and therefore

$$w(\lambda) = \delta(\lambda) - \delta_{+\infty} = \delta(\lambda) + \log((-1)^{|\beta|+(2n)^n}h_{\alpha}^{-1} \rho^{(\frac{|\alpha|}{2n} - \frac{3}{4}) + \frac{|\beta|}{2n} - \frac{1}{4}}).$$

The claim (3.4) then follows from the following.

Lemma 3.3. $\bar{g}_{\alpha} = \frac{1}{2} \left( \frac{|\alpha|}{n} - n + \frac{1}{2} \right)$.

Proof. In view of Proposition 2.3 we obtain from (3.5) in the case $\alpha = \beta$

$$2 \bar{g}_{\alpha} = 2 \left( \frac{|\alpha|}{2n} - \frac{n}{2} + \frac{1}{4} \right) \quad \text{or} \quad \bar{g}_{\alpha} = \frac{1}{2} \left( \frac{|\alpha|}{n} - n + \frac{1}{2} \right).$$

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