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A GENERALIZED TRANSLATION THEOREM FOR FREE HOMEOMORPHISMS OF SURFACES

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ABSTRACT. Let h be a free homeomorphism with a finite set of fixed points on a compact surface. Then h is, on the complement of the fixed point set, everywhere 'semi-locally' conjugate to a translation. This generalizes the Brouwer plane translation theorem.

0. INTRODUCTION

Let S be a surface, that is, a metric 2-dimensional connected manifold without boundary, and h a homeomorphism of S. M. Brown introduced in [B2] the class of *free* homeomorphisms, that is such that if $h(D) \cap D = \varnothing$ for some 2-disc D in S, then $h^n(D) \cap D = \varnothing$ for any $n \in \mathbb{Z}\backslash\{0\}$. It is a famous result of Brouwer ([Br], see also [B1], [Fa], [G]) that every homeomorphism of $\mathbb{R}^2$ preserving the orientation and without fixed point is free. In fact, Brouwer proved the stronger ([Br], see also [Fr], [K], [G], [T]):

Plane Translation Theorem. *Let h be an orientation preserving fixed point free homeomorphism of $\mathbb{R}^2$. Then for every point p in $\mathbb{R}^2$ there exists an imbedding $\varphi\colon (\mathbb{R}^2, 0) \to (\mathbb{R}^2, p)$ such that*

(i) *$h\varphi = \varphi\tau$ where $\tau(x, y) = (x+1, y)$,*
(ii) *on each line $x \times \mathbb{R}$, φ restricts to a proper imbedding into $\mathbb{R}^2$, i.e., one with a closed image.*

The aim of the present paper is the following generalization of Brouwer's theorem:

Theorem. *Let S be a compact surface and h a free homeomorphism of S with a finite set F of fixed points. Then for every point p in $S\backslash F$ there exists an imbedding $\varphi\colon (\mathbb{R}^2, 0) \to (S\backslash F, p)$ such that*

(i) *$h\varphi = \varphi\tau$ where $\tau(x, y) = (x+1, y)$,*
(ii) *on each line $x \times \mathbb{R}$, φ restricts to a proper imbedding, i.e., $\varphi(x \times \mathbb{R})$ is closed in $S\backslash F$.*

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When S is the 2-sphere the theorem is due to E. Slaminka [S]. However his proof seems to use strongly Theorem 4.6 of M. Brown [B2], which has no proof (except for fixed point free orientation-preserving homeomorphisms of the plane, cf. [Br, Satz 6] or [G, Lemme 3.8]) and seems to be still an open problem. M. Brown's argument gives only a weaker result (see Proposition 1.5 below); the needed contortions to conclude with this weaker result are explained in §2 below which elaborates on the proof of Brouwer's theorem given in [G, §4]. (Nevertheless this paper does not depend on [G] and contains, once admitting that orientation preserving fixed point free homeomorphisms of $\mathbb{R}^2$ are free, still another (variant of) proof of Brouwer's theorem.) In §3 we prove the theorem, and in §4 we extend it to compact surfaces with boundary.

1. Free homeomorphisms

We will now recall some basic properties of free homeomorphisms; for more details, see M. Brown [B2].

Let S be a connected surface (in this section S may have a nonempty boundary) and h a free homeomorphism of S.

1.1 **Definition.** A subset E of S such that $h(E) \cap E = \varnothing$ will be called *free.*

1.2 **Proposition.** *If $E \subset S$ is arc connected and free, then $h^n(E) \cap E = \varnothing$ for $n \in \mathbb{Z}\backslash\{0\}$.*

Proof. See [B2, Corollary 3.3].

1.3 **Definition.** A closed simple arc α with endpoints p and $h(p)$ for some p in S is a *translation arc* if $\alpha \cap h(\alpha) = \{h(p)\}$. The invariant set $L = L(\alpha) = \bigcup_{n\in\mathbb{Z}} h^n(\alpha)$ is called a *translation line.* A translation line has a natural order where $h(\alpha)$ follows α.

1.4 **Proposition.** *A translation line L is a submanifold of S homeomorphic to $\mathbb{R}$ (generally not closed).*

Proof. That L has no double point follows easily from the freeness of h; see Lemma 4.2 of [B2]. To see that L is a submanifold, i.e. that $\overline{L}\backslash L$ is closed, it is enough to show that there exist no translation arc $\beta \subset L$ and no sequence (x_n) with $x_n \in L\backslash\beta$ and $x_n \to x \in \mathring{\beta}$. Suppose there is such a sequence and let $x_n = h^{p(n)}(y_n)$ with $y_n \in \beta$. One can ask that $y_n \to y \in \beta$ for some y. Then the subarc xy of β is free and we can find a disc D such that $xy \subset \mathring{D}$ and $h(D) \cap D = \varnothing$. But, for n big enough, $x_n \in D$ and $y_n \in D$, so that $x_n \in h^{p(n)}(D) \cap D$ which contradicts the freeness of h.

An example where L is not closed was already in Brouwer [Br, Beispiel p. 40]. The same example can also be see in [B2, Figure 7] or [G, Figure 3.4].

1.5 **Proposition.** *Let L be a translation line and α a translation arc generating L (so that $L = L(\alpha)$). Let C be an arc connected set such that:*

(i) *C touches α on at most one side,*
(ii) *C meets the two components of $L\backslash\alpha$,*
(iii) *there exists an n such that $C \cap L = C \cap (\bigcup_{|i|\leq n} h^n(\alpha))$.*

Then $C \cap h(C) \neq \varnothing$.

Proof. See the argument in the proof of Theorem 4.6 in [B2].

1.6 *Remark.* Theorem 4.6 of [B2] is the preceding proposition without hypothesis (iii), but in that case L may have accumulation points on C and not in L; such a case is overlooked in the given proof of Theorem 4.6. To help clarify a little of this problem, recall the following question of M. Brown:

Let D be a 2-disc such that $h(D) \cap D = \varnothing$ and f an homeomorphism supported in D. Is hf free?

We remark that a positive answer to this question implies the truth of Theorem 4.6 of [B2]. In fact, it is enough to prove the result when C is an arc λ not meeting α, and, restricting to a subarc of λ if necessary, we can suppose that the endpoints (p and q, say, with q following p in the natural order on L) of λ cut upon L a subarc pq such that $\lambda \cap pq = \{p, q\}$. Consider now a translation arc β for L such that $q \in \mathring{\beta}$ and $\lambda \cap (h^{-1}(\beta) \cup \cdots \cup h^{-m+1}(\beta)) = \varnothing$ where $m \geq 2$ is such that $p \in h^{-m}(\beta)$. Suppose then by contradiction that $h(\lambda) \cap \lambda = \varnothing$ and let D be a disc neighborhood of λ such that $h(D) \cap D = \varnothing$ and $D \cap (h^{-1}(\beta) \cup \cdots \cup h^{-m+1}(\beta)) = \varnothing$. Let also f be a homeomorphism supported in D such that $f(q) = p$. One has $h(D \cup \beta) \cap \beta = (h(D) \cap \beta) \cup (h(\beta) \cap \beta) = h(\beta) \cap \beta$ so that $hf(\beta) \cap \beta = h(\beta) \cap \beta$ and β is a translation arc for hf. But $f(q) = p$ and $h^i(p) \notin D$ for $1 \leq i \leq m-1$ so that $(hf)^m(q) = h^m(p) \in \beta$ and $(hf)^m(\beta) \cap \beta \neq \varnothing$ which contradicts the freeness of hf.

Finally we note that this remark gives another proof of the hypothetical Theorem 4.6 of [B2] for the case of orientation preserving and fixed point free homeomorphisms of the plane. In fact if h is fixed point free, then clearly hf is also fixed point free. Compare to [Br, Satz 6] or [G, Lemme 3.8].

1.7 **Proposition.** *If S is compact, then F is not empty.*

Proof. See [B2, Lemma 3.4].

1.8 **Proposition.** *There is no annulus A in S with boundary a simple closed curve α and its image $h(\alpha)$ such that $h(A) \cap A = h(\alpha)$.*

Proof. We argue by contradiction. If $h^2(A)$ does not contain all of α the proof of Lemma 5.2 in [B2] applies and leads to a contradiction with the freeness of h. If $h^2(A)$ contains α, then $S = A \cup h(A) \cup h^2(A)$ is a torus or a Klein bottle. Since h is fixed point free on A, it is fixed point free on S which is a contradiction to Proposition 1.7.

1.9 **Proposition.** *Any point of $S \backslash F$ is contained in a translation arc if F does not separate S.*

Proof. See [B2, Lemma 4.1].

2. An inductive construction

Let p be a point of $S \backslash F$ and α a translation arc containing p.

2.1 **Lemma.** *There is a triangulation K of $S \backslash F$ such that α is a subcomplex of K and the star of every vertex of K is free.*

Proof. We write down $S \backslash F$ as an ascending union of compact bordered surfaces $S_0 \subset S_1 \subset S_2 \subset \cdots$ with $\alpha \subset \mathring{S}_0$ and $S_i \subset \mathring{S}_{i+1}$. As is well known we can triangulate S_0 so that α is a subcomplex and the star of every vertex is free. Then we triangulate $S_1 \backslash \mathring{S}_0$ so that the star of every vertex is free and the triangulations induced on ∂S_0 by those of S_0 and $S_1\backslash\overset{\circ}{S}_0$ are coincident. To satisfy this last condition we may have to subdivide the triangulation of S_0 near ∂S_0. We then triangulate $S_2\backslash\overset{\circ}{S}_1$ so that the star of every vertex is free and the triangulations induced on ∂S_1 by those of S_1 and $S_2\backslash\overset{\circ}{S}_1$ are coincident. To do this we may have to subdivide the triangulation of S_1 near ∂S_1 but can leave it unchanged on S_0. Therefore, continuing this way, we conclude

2.2 **Lemma.** *Let K be a triangulation of $S\backslash F$ such that the star of every vertex is free and α is a subcomplex of K. Then, on each side of α, there is a triangle of K adjacent to α which meets neither $h(\alpha)$ nor $h^{-1}(\alpha)$.*

Proof. For each triangle T adjacent to α, let us consider the triangles included in T, of base a side of T in $\alpha \cap T$, and whose third vertex is located on the median of T which bisects (a chosen side of) $\alpha \cap T$. Let T' be the largest of those triangles such that $T' \cap (h(\alpha) \cup h^{-1}(\alpha)) \subset \partial T'$. We may have to allow $T' = \alpha \cap T$ when T contains an end point of α. Let us note that if $\overset{\circ}{T} \cap (h(\alpha) \cup h^{-1}(\alpha)) = \varnothing$, then $T' = T$. With these notations we have:

2.3 **Assertion.** *For every such T', $T' \cap h^i(\alpha) = \varnothing$ for $|i| \geq 2$.*

Proof. Let us suppose that $h^n(\alpha)$, $|n| \geq 2$, meets T' in a point x. We have $x = h^n(y)$ for some $y \in \alpha$. The arc made of $\alpha\backslash\alpha \cap T'$ and of an arc in T' joining the end points of $\alpha \cap T'$ without meeting $h^{-1}(\alpha) \cup h(\alpha)$ but going through x (and y if $y \in \alpha \cap T'$) is a translation arc β which contains x and y. Therefore $h^n(\beta) \cap \beta$ contains x which contradicts Proposition 1.4.

We now show that there exists a T' on each side of α such that $T' \cap (h(\alpha) \cup h^{-1}(\alpha)) = \varnothing$.

It follows from the preceding assertion and Proposition 1.5 that any T', being free, cannot meet both of $h(\alpha)$ and $h^{-1}(\alpha)$. On the other hand, because of the T' containing the end points of α, there is on each side of α a T' meeting $h(\alpha)$ and another one meeting $h^{-1}(\alpha)$. If all the T' on one side of α were meeting $h(\alpha)$ or $h^{-1}(\alpha)$, there would be two consecutive ones, one meeting $h(\alpha)$ and the other $h^{-1}(\alpha)$. But now we see, using the assertion above, that we have a contradiction to Proposition 1.5.

To conclude the proof of Lemma 2.2, it is enough to remark that a T' not meeting $h(\alpha)$ or $h^{-1}(\alpha)$ is equal to the T which contains it.

Let us keep with the notation of Lemma 2.2 and let $T_0 = T$ be a triangle not meeting $h(\alpha) \cup h^{-1}(\alpha)$ given by this lemma. Modifying $\alpha_0 = \alpha$ by T_0 we get a new translation arc α_1 which is still a subcomplex of the triangulation K. (α_1 is made of the arc in $\alpha_0 = p_0h(p_0)$ from p_0 to the first point r_0 of T_0, of the arc in α_0 from the last point t_0 in T_0 to $h(p_0)$, and of the arc from r_0 to t_0 in $\partial T_0\backslash\alpha_0$). We can then apply the same lemma again and get a triangle T_{-1} adjacent to α_0 on the side opposite to T_0 or a triangle T_1 adjacent to α_1 on the side opposite to T_0. This process can be iterated, and after n applications of Lemma 2.2 we get a sequence $T_p, \ldots, T_0, \ldots, T_q$ of triangles (a priori not necessarily distinct) with $-p+q+1 = n$, $p \leq 0$, $q \geq 0$, such that $\alpha \cup T_p \cup \cdots \cup T_0 \cup \cdots \cup T_q$ is a connected subcomplex of K. We also get a sequence $\alpha_p, \ldots, \alpha_0 = \alpha, \ldots, \alpha_{q+1}$ of translation arcs.

According to Assertion 2.3 we have $(\alpha \cup T_0) \cap h(\alpha \cup T_0) = \alpha \cap h(\alpha)$ and $(\alpha \cup T_0) \cap h^i(\alpha \cup T_0) = \varnothing$ for $|i| \geq 2$. I do not know how to show that this situation still holds (if it does!) when T_0 is replaced by $\mathfrak{T} = T_p \cup \cdots \cup T_0 \cup \cdots \cup T_q$ without using a hypothetical strong version of Proposition 1.5 (compare to Lemma 6 in [S]). The following lemma nevertheless shows that by exercising more care in the choice of the T_i's we can secure this situation.

2.4 **Definition.** We shall say that $\alpha \cup \mathfrak{T}$ is *critical* if $(\alpha \cup \mathfrak{T}) \cap h^i(\alpha \cup \mathfrak{T}) = \varnothing$ for $|i| \geq 2$ and if $(\alpha \cup \mathfrak{T}) \cap h(\alpha \cup \mathfrak{T}) = \alpha \cap h(\alpha)$. Note that this is the case if $n = 1$, in which case $\mathfrak{T} = T_0$.

2.5 **Lemma.** *Let us suppose that $\alpha \cup \mathfrak{T}$ is critical. Then there exist a triangle T of K adjacent to α_{q+1} on the side opposite to T_q and a triangle U of K adjacent to α_p on the side opposite to T_p such that $\alpha \cup \mathfrak{T} \cup T \cup U$ is still critical.*

Proof. Let $\widehat{\mathfrak{T}} = \alpha \cup \mathfrak{T}$. As in the proof of Lemma 2.2, to each triangle T adjacent to α_{q+1} on the side opposite to T_q we can associate a triangle $T' \subset T$ such that $T' \cap (h^{-1}(\widehat{\mathfrak{T}}) \cup h(\widehat{\mathfrak{T}})) \subset \partial T'$ and that if $\mathring{T} \cap (h^{-1}(\widehat{\mathfrak{T}}) \cup h(\widehat{\mathfrak{T}})) = \varnothing$, then $T' = T$. One shows as in Assertion 2.3 that such a triangle T' satisfies $T' \cap h^i(\widehat{\mathfrak{T}}) = \varnothing$ for $|i| \geq 2$, and one then deduces from Proposition 1.5 that it cannot meet both of $h(\widehat{\mathfrak{T}})$ and $h^{-1}(\widehat{\mathfrak{T}})$. On the other hand, because of the T' containing the endpoints of α_{q+1} (which are the same as those of α), there is a T' meeting $h(\widehat{\mathfrak{T}})$ and another one meeting $h^{-1}(\widehat{\mathfrak{T}})$. Using Proposition 1.5 again as in Lemma 2.2 one gets a T' not meeting either $h(\widehat{\mathfrak{T}})$ or $h^{-1}(\widehat{\mathfrak{T}})$. Such a T' is equal to the T which contains it so that $T \cap h^i(\widehat{\mathfrak{T}}) = \varnothing$ for all $i \neq 0$ and therefore $\widehat{\mathfrak{T}} \cup T = \alpha \cup \mathfrak{T} \cup T$ is critical. The same reasoning with α_p in place of α_{q+1} and $\mathfrak{T} \cup T$ in place of $\mathfrak{T}$ produces the desired U.

3. Proof of the theorem

We can iterate indefinitely the construction of Lemma 2.5 to get a translation arc α going through p and a doubly infinite sequence $\{T_n\}_{n\in\mathbb{Z}}$ of triangles such that $\alpha \cup (\bigcup_{n\in\mathbb{Z}} T_n)$ is critical. Since each T_i, $i \neq -1, 0$, meets a T_j, $|j| < |i|$, or α along an edge and since α consists of a finite number of edges, we can suppose that $T_i \cap T_{i+1}$ is always a common edge of T_i and T_{i+1} for $i \neq -1$. More precisely we find a subsequence of the T_i's satisfying these properties and reconstruct a sequence of α_i exactly as before but using only the T_i of the subsequence (and starting from $\alpha = \alpha_0$). Since $\alpha \cup \mathfrak{T}$ was critical these new α_i are certainly translation arcs. In what follows we consider only these new T_i's and α_i's.

3.1 **Lemma.** *The set $\alpha \cup (\bigcup_{n\in\mathbb{Z}} T_n)$ contains no simple closed curve crossing α (which means a curve isotopic, by an arbitrarily small isotopy, to a curve cutting α transversally in one point).*

Proof. Let c be such a curve. We first show that it does not bound a disc. If $c = \partial D$ for a 2-disc D, α has one of its end points in D and since $\alpha \cup c$ is critical $h(c)$ (or $h^{-1}(c)$ but we can replace h by h^{-1} if necessary in the following argument) is contained in $\mathring{D}$ and $h^{-1}(c)$ is outside D. So that, either $h(D) \subset \mathring{D}$, which contradicts Proposition 1.8, or $h(D) \cup D$ is a 2-sphere, and $S = S^2 = D \cup h(D)$ in which case the annulus A bounded by c and $h^{-1}(c)$ has to satisfy $A \cap h(A) = c$ which again contradicts Proposition 1.8.

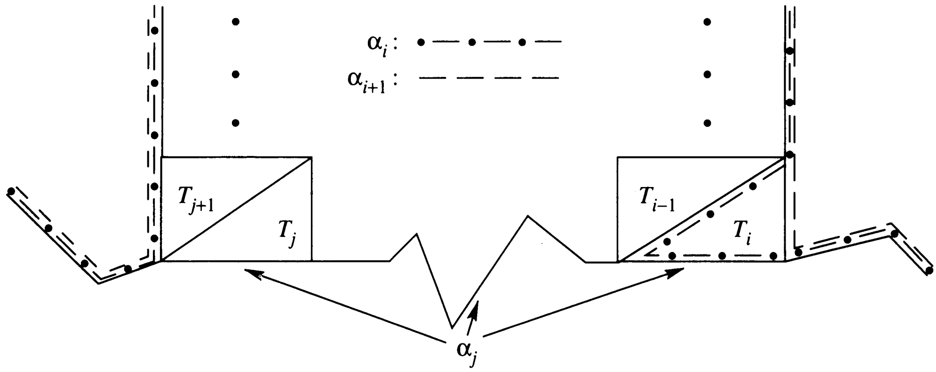

FIGURE 1

Second consider the family $\{h^n(c)\}_{n\in\mathbb{Z}}$ of disjoint simple closed curves. Since none bound a disc there must be a $k>0$ such that c and $h^k(c)$ bound an annulus A in S. Since $\alpha\cup c$ is critical, if $h(c)$ is contained in A we immediately have a contradiction to Proposition 1.8. If not, then $h(\alpha)\cap A=\varnothing$ so that $h^{-1}(\alpha)\subset A$ and $h^{-1}(c)\subset A$. Once again we arrive at a contradiction to Proposition 1.8.

3.2 **Lemma.** *The triangles of the family $\{T_n\}_{n\in\mathbb{Z}}$ are all distinct.*

Proof. Let us first show that given a translation arc α_j there is no $i\neq j-1$ such that T_i and T_{i+1} are on opposite sides of α_j, that is the sequences $\{T_i\}_{i>j}$ and $\{T_i\}_{i<j}$ may well come back to touch α_j but they cannot cross it. We will consider only the case $i>j$ the case $i<j$ being similar. So, suppose there exist such an α_j and such a T_i and choose them so that $i-j>0$ is minimal. Then $T_i\cap T_k=\varnothing$ for $j\le k<i-1$ and all T_k's are distinct for $j\le k\le i$. We now look at an imbedded closed band made of $T_j\cup T_{j+1}\cup\cdots\cup T_i$ and part of a neighborhood of α_j. It is free and so cannot be a Mœbius band since there are only a finite number of disjoint Mœbius bands in a compact surface. Also T_j and T_i have to be on the same side of α_j for if not we would have a contradiction to Lemma 3.1 which applies to α_j since $\alpha_j\cup(\bigcup T_n)_{n\in\mathbb{Z}}$ is critical. Therefore we are left with the situation depicted in Figure 1, which shows that $T_i\cap\alpha_{i+1}$ is not contained in α_j and so, that T_i and T_{i+1} cannot be on opposite sides of α_j.

As a consequence, all the T_i's, for $i\ge 0$ as well as for $i<0$, are distinct. And we cannot either have $T_i=T_j$ for some $i\ge 0$ and some $j<0$ since this would lead, using the first part of the proof, to a simple closed curve contradicting Lemma 3.1.

3.3 **Lemma.** *There is a fixed point q (resp. q') such that any neighborhood of q (resp. q') contains all but a finite number of the T_i's for $i>0$ (resp. $i<0$).*

Proof. Consider arbitrarily small distinct open balls, one around each fixed point, and call their union U. Then $S\backslash U$ is compact and meets only a finite number of T_n so there are $k>0$ and $l<0$ so that the connected sets $\bigcup_{n>k}T_n$ and $\bigcup_{n<l}T_n$ are each contained in a ball.

One can then choose a line K in $\alpha\cup(\bigcup_{n\in\mathbb{Z}}T_n)$ properly embedded in $S\backslash F$ whose ends converge to q and q' and crossing α.

3.4 **Lemma.** *The circle* $\overline{K} \cup h(\overline{K})$ *bounds a 2-disc* D *in* S *such that* $D \cap h(D) = h(\overline{K})$.

Proof. We cut S along $\overline{K}$ and think of the resulting bordered surface S' as a disc with a finite number of bands attached to it. K being free the arcs $h^n(\overline{K})$, $n \neq 0$, have the same endpoints and disjoint interiors, so clearly there are an s and an r, $s \neq r$, such that $h^s(\overline{K}) \cup h^r(\overline{K})$ bounds a disc in S' and therefore an m such that $\overline{K} \cup h^m(\overline{K})$ bounds a disc E in S. Since K crosses α and since $\alpha \cup K$ is critical, one of the two end points of α is in E and $h(K)$ or $h^{-1}(K)$ is also in E. Therefore $\overline{K} \cup h(\overline{K})$ bounds a disc D as required.

The proof of the theorem is now easily completed.

4. Further remarks

The theorem extends easily to the case of compact surfaces with boundary. Nevertheless we begin with a word of caution: the double of a free homeomorphism may not be free as shown by the following example.

4.1 **Example.** The homeomorphism of $S^2 = \mathbb{R}^2 \cup \{\infty\}$ given by $h(x, y) = (\lambda x, \lambda y)$ for $\lambda \neq 1$, $\lambda \in \mathbb{R}$, is not free (because of Proposition 1.7) but its restriction to the disc $\mathbb{R}^2_+ \cup \{\infty\}$ is free.

4.2 **Theorem.** *Let* S *be a compact surface, possibly with boundary, and* h *a free homeomorphism of* S *with a finite set* F *of fixed points. Then for every point* p *in* $S \backslash (\partial S \cup F)$ *there exists an imbedding* $\varphi : (\mathbb{R}^2, 0) \to (S \backslash (\partial S \cup F), p)$ *such that*

(i) $h\varphi = \varphi h$ *where* $\tau(x, y) = (x + 1, y)$,
(ii) *on each line* $x \times \mathbb{R}$, φ *restricts to a proper imbedding, i.e.,* $\varphi(x \times \mathbb{R})$ *is closed in* $S \backslash (\partial S \cup F)$.

And for every point $p \in \partial S \backslash F$ *there exists an imbedding*

$$\varphi : (\mathbb{R}^2_+, \mathbb{R}^2_+ \backslash \partial \mathbb{R}^2_+, \partial \mathbb{R}^2_+, 0) \to (S \backslash F, S \backslash (\partial S \cup F), \partial S \backslash F, p)$$

such that

(i) $h\varphi = \varphi h$ *where* $\tau(x, y) = (x + 1, y)$,
(ii) *on each half-line* $x \times [0, \infty)$, φ *restricts to a proper imbedding, i.e.,* $\varphi(x \times [0, +\infty))$ *is closed in* $S \backslash (\partial S \cup F)$

or an imbedding

$$\varphi : (\mathbb{R} \times [0, 1], \mathbb{R} \times (0, 1), \mathbb{R} \times \{0, 1\}, 0) \to (S \backslash F, S \backslash (\partial S \cup F), \partial S \backslash F, p)$$

such that

(i) $h\varphi = \varphi h$ *where* $\tau(x, y) = (x + 1, y)$.

Sketch of proof. We preserve the notation of §§2 and 3. If the processes explained in §2 can be continued indefinitely, then the proof goes on very similarly to the one given for the boundaryless case. If not, then some T_n for (say) $n \leq 0$ has an edge in common with ∂S and the construction can be continued indefinitely for $n > 0$, or there is also some T_m, $m > 0$, which has an edge in common with ∂S. In these cases, the line L of §3 is replaced by a half-line from ∂S to be a fixed point or a closed arc from ∂S to itself. Lemmas similar to Lemma 3.4 then lead to the desired imbeddings. The only new point occurs when $p \in S\backslash(\partial S \cup F)$ and L is a half-line from ∂S to a fixed point, in which case we obtain an imbedding $\psi\colon (\mathbb{R}^2_+, (0, 1)) \to (S\backslash F, p)$ sending interior into interior and boundary into boundary such that $\psi\tau = \tau\psi$ and that ψ restricted to each half-line $x \times [0, +\infty)$ is proper. We then consider the homeomorphism $\rho\colon \mathring{\mathbb{R}}^2_+ \to \mathring{\mathbb{R}}^2_+$ given by $\rho(x, y) = (y + x - 1, y^{-1})$; the map $\pi = \psi\rho$ from $\mathbb{R}^2 \cong \mathring{\mathbb{R}}^2_+$ to $S\backslash F$ is the imbedding we want.

4.3 *Remark.* The theorem does not extend as such to noncompact surfaces. In fact, if $S = \mathbb{R}^2\backslash\mathbb{Z} \times \{0\}$ and $h(x, y) = (x + 1, y)$, then no point of $\mathbb{R} \times \{0\}$ is contained in the image of an imbedding φ as in the theorem.

4.4 *Remark.* The homeomorphism of S^2 given in Example 4.1 is not free but is, outside of the fixed point set, everywhere 'semi-locally' conjugate to a translation (in the sense of the theorem). This raises the question: *Is there a dynamical characterisation of homeomorphisms of compact surfaces* (*with a finite number of fixed points*) *everywhere semi-locally conjugate to a translation* (*outside of the fixed point set*)?

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