ON THE EXISTENCE OF PERIODIC SOLUTIONS
FOR NONCONVEX-VALUED DIFFERENTIAL INCLUSIONS IN $\mathbb{R}^N$

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Abstract. In this paper we investigate the existence of periodic solutions for differential inclusions with nonconvex-valued orientor field. Using a tangential condition and directionally continuous selectors, we establish the existence of periodic trajectories.

1. Introduction

In this paper, we investigate the following multivalued periodic problem defined on $[0, b] \times \mathbb{R}^N$:

$$\begin{cases}
\dot{x}(t) \in F(t, x(t)) \text{ a.e.} \\
x(0) = x(b).
\end{cases}$$

(1)

Here $F : T \times \mathbb{R}^N \to 2^{\mathbb{R}^N} \setminus \{\emptyset\}$ is a set-valued vector field (orientor field), which has closed but not necessarily convex values. All earlier results on (1) assumed that the orientor field is convex-valued. We refer to the works of Haddad-Lasry [6] (Theorem B-II-1), Aubin-Cellina [1] (Theorem 4, p. 237), Macki-Nistri-Zecca [10] and Plaskacz [14] (Theorem 4.5).

In this paper using directionally continuous selectors for the orientor field $F(t, x)$ (cf. Bressan [3]) and a Nagumo type tangential condition, analogous to the one employed in [1], [6] and [14], we are able to establish the existence of solutions for (1). The approach of Macki-Nistri-Zecca [10] was based on degree-theoretic techniques.

2. Preliminaries

In what follows by $P_{k(c)}(\mathbb{R}^N)$ we will denote the collection of all nonempty, compact (and convex) subsets of $\mathbb{R}^N$. A multifunction (set-valued function)
F : T × ℝᴺ → Pk(ℝᴺ) is said to be measurable, if for all z ∈ ℝᴺ, (t, x) → d(z, F(t, x)) = \inf \{∥z - v∥ : v ∈ F(t, x)\} is measurable.

Let V, W be Hausdorff topological spaces and G : Y → 2Z\{∅}. We say that G(·) is lower semicontinuous (l.s.c.) (resp. upper semicontinuous (u.s.c.)), if for all U ⊆ W open the set G−(U) = {v ∈ V : G(v) ∩ U ≠ ∅} (resp. G+(U) = {v ∈ V : G(v) ⊆ U}) is open in V. We will say that G(·) has an open graph, if the set GrG = {(v, w) ∈ V × W : w ∈ G(v)} is open in V × W furnished with the product topology.

Let K be a nonempty and closed subset of ℝᴺ and let x ∈ K. The Bouligand tangent cone to K at x, TK(x), is defined by

$$TK(x) = \left\{ v ∈ ℝᴺ : \lim_{\lambda \to 0} \frac{d(x + \lambda v, K)}{\lambda} = 0 \right\}.$$ 

It is immediate from this definition that TK(x) is a closed, but not necessarily convex, cone in ℝᴺ. It is convex if K is convex or more generally locally convex at x.

Now let T = [0, b] and let K : T → Pk(ℝᴺ). Let (t, x) ∈ GrK = {(s, y) ∈ T × ℝᴺ : y ∈ K(s)} (the graph of K). The "contingent derivative" of K(·) at (t, x), DK(t, x), is the set-valued map whose graph is the Bouligand tangent cone to the graph of K(·) at (t, x); i.e. v ∈ DK(t, x)(τ) if and only if (τ, v) ∈ TGrK(t, x) (see Aubin-Cellina [1], Chapter 4).

Finally recall that on Pk(ℝᴺ) we can define a metric, known in the literature as the Hausdorff metric, by setting

$$h(A, B) = \max \left[ \sup_{a ∈ A} d(a, B), \sup_{b ∈ B} d(b, A) \right].$$

Recall that (Pk(ℝᴺ), h) is a Polish space (i.e. a complete, separable metric space) and Pk(ℝᴺ) is a closed subset of it. We will say that a multifunction K : T → Pk(ℝᴺ) is Hausdorff-Lipschitz (h-Lipschitz), if h(K(t'), K(t)) ≤ k|t' − t| for some k > 0 and all t, t' ∈ T.

3. Auxiliary results

In this section we prove some auxiliary results that we will need in the proof of our main theorem (Section 4).

We start with a Scorza-Dragoni type property for measurable orientor fields F(t, x), which are l.s.c. in x.

Proposition 3.1. If T = [0, b], X is a metric space, Z is a Polish space and F : T × X → 2Z\{∅} is a multifunction with closed values such that

1. (t, x) → F(t, x) is measurable,
2. x → F(t, x) is l.s.c.,

then for every ε > 0, there exists Tε ⊆ T compact with λ(Tε) < ε such that F|Tε×X is l.s.c. (here λ(·) denotes the Lebesgue measure on T).

Proof. Consider the function γ : T × X × Z → ℝ+ defined by γ(t, x, z) = d(z, F(t, x)). Because of the measurability hypothesis on F(t, x), (t, x) → γ(t, x, z) is measurable, while clearly z → γ(t, x, z) is continuous. So (t, x, z) → γ(t, x, z) is measurable. Furthermore, if (xn, zn) → (x, z) in
Then we have
\[ d(z_n, F(t, x_n)) = d(z_n, F(t, x_n)) + d(z, F(t, x_n)) \]
\[ \leq ||z_n - z|| + d(z, F(t, x_n)) \]
\[ \Rightarrow \lim d(z_n, F(t, x_n)) \leq \lim d(z, F(t, x_n)) \leq d(z, F(t, x)) , \]
the last inequality being a consequence of the hypothesis that \( x \to F(t, x) \) is l.s.c. (see for example Aubin-Cellina [1], Corollary 1, p. 52). So \( (t, x, z) \to d(z, F(t, x)) \) is a measurable integrand which is u.s.c. in \((x, z)\). Then define

\[ y_k(t, x, z) = \sup\{y(t, x, z) - kd_x(x, x') - kd_z(z, z') : x' \in X, z' \in Z\} \]
(here \( d_X(\cdot, \cdot) \) denotes the metric on \( X \) and \( d_Z(\cdot, \cdot) \) the metric on \( Z \)).

From Lemma 2.1 of Hiiai-Umegaki [7], we know that \( t \to y_k(t, x, z) \) is measurable, while it is easy to see that \((x, z) \to y_k(t, x, z)\) is \( k\)-Lipschitz and \( y_k \downarrow \gamma \) as \( k \to \infty \) (see for example Bertsekas-Shreve [2], Lemma 7.7, p. 125). Given \( \varepsilon > 0 \), apply the classical Scorza-Dragoni theorem to find \( T_\varepsilon \subseteq T \) compact with \( \lambda(T_\varepsilon) < \varepsilon \) such that \( y_k|_{T_\varepsilon \times X \times Z} \) is continuous for all \( k \geq 1 \). Since \( y_k \downarrow \gamma \), we get that \( \gamma|_{T_\varepsilon \times X \times Z} \) is u.s.c. (cf. Bertsekas-Shreve [2], Lemma 7.14, p. 147). So \( d(z, F(\cdot, \cdot))|_{T_\varepsilon \times X} \) is u.s.c. \( \Rightarrow F|_{T_\varepsilon \times X} \) is l.s.c. (see Klein-Thompson [8]).

In the next result \( B_\varepsilon(x) = \{y \in X : d_X(y, x) < \varepsilon\} \) (the open ball of radius \( \varepsilon \), centered at \( x \in X \)).

**Proposition 3.2.** If \( X \) is a metric space, \( h : X \to \mathbb{R}^N \) satisfies \( \|h(x)\| \leq M \) for all \( x \in X \) and we define \( G(x) = \bigcap_{\varepsilon > 0} \bar{\text{conv}}(h(B_\varepsilon(x))) \), then \( x \to G(x) \) is u.s.c. from \( X \) into \( P_{kc}(\mathbb{R}^N) \).

**Proof.** Since \( G(\cdot) \) is \( P_{kc}(\mathbb{R}^N) \)-valued, to establish its upper semicontinuity, it is enough to show that for every \( x^* \in \mathbb{R}^N \),
\[ x \to \sigma(x^*, G(x)) = \sup\{(x^*, v)_{\mathbb{R}^N} : v \in G(x)\} \]
is u.s.c. (cf. Klein-Thompson [8]; here \( \langle \cdot, \cdot \rangle_{\mathbb{R}^N} \) denotes the Euclidean inner product in \( \mathbb{R}^N \)). So fix \( x^* \in \mathbb{R}^N \). We will show that for every \( \lambda \in \mathbb{R} \), the level set \( U_\lambda = \{x \in X : \lambda \leq \sigma(x^*, G(x))\} \) is closed. To this end let \( \{x_n\}_{n \geq 1} \subseteq U_\lambda \) and assume that \( x_n \to x \) in \( X \). Find \( y_n \in G(x_n) \) such that \( \lambda \leq \langle x^*, y_n \rangle_{\mathbb{R}^N} \).

Let \( \varepsilon_n = 2d_X(x_n, x) \downarrow 0 \) as \( n \to \infty \). Then we have \( y_n \in G(B_{\varepsilon_n}(x)) \). Since \( \|y_n\| \leq M \), \( n \geq 1 \), by passing to a subsequence if necessary we may assume that \( y_n \to y \). Clearly then \( y \in G(x) \) and \( \langle x^*, y_n \rangle_{\mathbb{R}^N} \to \langle x^*, y \rangle_{\mathbb{R}^N} \Rightarrow \lambda \leq \langle x^*, y \rangle_{\mathbb{R}^N} \) and so \( \lambda \leq \sigma(x^*, G(x)) \). Therefore \( \lambda \in U_\lambda \), which implies that \( x \to \sigma(x^*, G(x)) \) is u.s.c. \( \Rightarrow x \to G(x) \) is u.s.c. \( \square \)

For the next auxiliary result we will need the following hypothesis.

**H(K) :** \( K : T \to P_{kc}(\mathbb{R}^N) \) is an \( h \)-Lipschitz multifunction.

**Proposition 3.3.** If hypothesis \( H(K) \) holds, \( G : T \times \mathbb{R}^N \to P_{kc}(\mathbb{R}^N) \) is a multifunction such that for all \( (t, x) \in \text{Gr} K \), \( t < b \), \( G(t, x) \subseteq DK(t, x)(1) \) and \( x : T \to \mathbb{R}^N \) is an absolutely continuous function such that \( \dot{x}(t) \in G(t, p(t, x(t))) \)
a.e., $x(0) \in K(0)$ (here $p(t, x) = \text{proj}(x; K(t))$, the metric projection on $K(t)$), then for all $t \in T$, $x(t) \in K(t)$.

Proof. Let $\varphi(t) = d(x(t), K(t))$. Using hypothesis $H(K)$, we can easily check that $\varphi(\cdot)$ is absolutely continuous on $T$, with $\varphi(0) = 0$. So to prove our proposition, we only need to show that $\varphi(t) \leq 0$ a.e.

To this end, let $t \in [0, b)$ be a point at which both $\dot{x}(\cdot)$ and $\varphi(\cdot)$ exist. Then we have

$$\varphi(t + h) - \varphi(t) = \frac{d(x(t + h), K(t + h)) - d(x(t), K(t))}{h}.$$

Observe that

$$d(x(t) + h\dot{x}(t), K(t + h)) - d(x(t), K(t)) \leq \frac{\|o(h)\|}{h} + \frac{d(x(t) + h\dot{x}(t), K(t + h)) - d(x(t), K(t))}{h}.$$

Since by hypothesis $\dot{x}(t) \in G(t, p(t, x(t))) \subseteq DK(t, p(t, x(t)))(1)$ a.e. on $[0, b)$, from Aubin-Cellina [1], pp. 190–191, we have that

$$\lim_{h \to 0} d \left( \frac{\dot{x}(t) - p(t, x(t))}{h}, \frac{K(t + h) - p(t, x(t))}{h} \right) = 0 \text{ a.e.} \Rightarrow \varphi(t) \leq 0 \text{ a.e.} \Rightarrow \varphi(t) = 0 \text{ for all } t \in T \text{ and so } x(t) \in K(t) \text{ for all } t \in T. \quad \square$$

The next auxiliary result establishes a useful property of lower semicontinuous multifunctions and will lead us to a new lower semicontinuity result concerning the intersection of two multifunctions.

**Proposition 3.4.** If $Z$ is a Hausdorff topological space, $F : Z \to 2^{R^N \setminus \emptyset}$ is l.s.c. with convex values and $B(x, \hat{r}) = \{y \in R^N : \|y - x\| < \hat{r}\} \subseteq F(z_0)$, then for all $r \in (0, \hat{r})$, there exists an open neighborhood $U$ of $z_0$ such that $B(x, r) \subseteq F(z)$ for all $z \in U$.

Proof. Let $r \in (0, \hat{r})$ and take $r', \varepsilon > 0$ such that $0 < r < r' < \hat{r}$ and $0 < \varepsilon < r' - r$. We have $B(x, r') \subseteq F(z_0)$. Let $\theta(z) = h^*(B(x, r'), F(z)) = \sup[d(v, F(z)) : v \in B(x, r')]$. We know (cf. Aubin-Cellina [1], Theorem 5, p. 52) that $\theta(\cdot)$ is u.s.c. and $\theta(z_0) = 0$. So we can find $U$ an open neighborhood of $z_0$ such that $\theta(z) < \varepsilon$ for all $z \in U$. Since by hypothesis $F(\cdot)$ is convex-
valued from Moreau [11], we know that
\[ d(x, \mathbb{R}^N \setminus \mathcal{B}(x, r')) - d(x, \mathbb{R}^N \setminus F(z)) \leq h^*(\mathcal{B}(x, r'), F(z)) = \theta(z) < \varepsilon \quad \text{for all } z \in U, \]
\[ \Rightarrow d(x, \mathbb{R}^N \setminus \mathcal{B}(x, r')) - \varepsilon \leq d(x, \mathbb{R}^N \setminus F(z)), \]
\[ \Rightarrow r < r' - \varepsilon \leq d(x, \mathbb{R}^N \setminus F(z)) \quad \text{for all } z \in U. \]

So we conclude that for all \( z \in U, \mathcal{B}(x, r) \subseteq F(z). \quad \square \)

Having this proposition, we can now prove a lower semicontinuity result for the intersection of two multifunctions.

**Proposition 3.5.** If \( Z \) is a Hausdorff topological space, \( H_1, H_2 : Z \to 2^{\mathbb{R}^N \setminus \{\emptyset\}} \) are l.s.c. multifunctions such that for all \( z \in Z, H_2(z) \) is open and convex and \( H_1(z) \cap H_2(z) \neq \emptyset \), then \( z \to H(z) = H_1(z) \cap H_2(z) \) is l.s.c.

**Proof.** We claim that \( \text{Gr} H_2 \) is open. Indeed let \((z, v) \in \text{Gr} H_2\). Then \( v \in H_2(z) \) and so we can find \( r > 0 \) such that \( \mathcal{B}(v, r) \subseteq H_2(z) \). Let \( r < \hat{r} \) and apply Proposition 3.4 to get \( U \) an open neighborhood of \( z \) such that \( \mathcal{B}(v, r) \subseteq H_2(z') \) for all \( z' \in U \). Set \( W = U \times \mathcal{B}(v, r) \). Clearly this is an open neighborhood of \((z, v) \in Z \times \mathbb{R}^N\) and for every \((z', v') \in W \) we have \( v' \in \mathcal{B}(v, r) \subseteq H_2(z') \). So \((z', v') \in \text{Gr} H_2\) for all \((z', v') \in W \Rightarrow \text{Gr} H_2\) is open and so by the lemma in Papageorgiou [13], we conclude that \( z \to H_1(z) \cap H_2(z) \) must be l.s.c. \quad \square

Let \( \Gamma \) be a cone in \( \mathbb{R}^m \) and \( X \) a metric space. Following Bressan [3], we say that the single-valued map \( f : \mathbb{R}^m \to X \) is \( \Gamma \)-continuous at \( v \in \mathbb{R}^m \), if and only if \( f(v_n) \to f(v) \) when \( v_n \to v \) in \( \mathbb{R}^m \) and \( v_n - v \in \Gamma \) for all \( n \geq 1 \) (equivalently for every \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that \( d_X(f(v), f(v')) < \varepsilon \) when \( \|v - v'\| < \delta, \ v' - v \in \Gamma \) ). We will say that \( f(\cdot) \) is \( \Gamma \)-continuous, if it is \( \Gamma \)-continuous at every \( v \in \mathbb{R}^m \). Let \( M > 0 \) and on \( \mathbb{R} \times \mathbb{R}^N \) consider the cone \( \Gamma^M = \{(t, x) \in \mathbb{R} \times \mathbb{R}^N : \|x\| \leq Mt\} \). A map \( h : T \times \mathbb{R}^N \to \mathbb{R}^N \) is said to be Scorza-Dragoni \( \Gamma^M \)-continuous, if for every \( \varepsilon > 0 \), we can find \( T_{\varepsilon} \subseteq T \) compact such that \( \lambda(T_{\varepsilon}) < \varepsilon \) and \( h|_{T_{\varepsilon} \times \mathbb{R}^N} \) is \( \Gamma^M \)-continuous.

Now let \( h : T \times \mathbb{R}^N \to \mathbb{R}^N \) be a Scorza-Dragoni \( \Gamma^M \)-continuous map such that \( \|h(t, x)\| \leq M \) for all \((t, x) \in T \times \mathbb{R}^N \). Let \( G(t, x) = \bigcap_{t \geq 0} \overline{\text{conv}} \mathcal{B}(t, x) \) (cf. Proposition 3.2). We consider the following two Cauchy problems:

(2) \[
\begin{align*}
&\mathcal{B}(t, x) \\
&x(0) = x_0
\end{align*}
\]

and

(3) \[
\begin{align*}
&\mathcal{B}(t, x) \\
&x(0) = x_0
\end{align*}
\]

We have the following result relating these two problems.
Proposition 3.6. If \( x : T \rightarrow \mathbb{R}^N \) is a solution of (2), then \( x(\cdot) \) also solves (3).

Proof. From Lusin’s theorem and the fact that \( h(\cdot, \cdot) \) is Scorza-Dragoni \( \Gamma^M \)-continuous, we can find \( T_n \subseteq T, \ n \geq 1, \) measurable sets such that \( \hat{x}|_{T_n} \) is continuous, \( h|_{T_n \times \mathbb{R}^N} \) is \( \Gamma^M \)-continuous, \( \hat{x}(t) \in G(t, x(t)) \) for all \( t \in T_n \) and \( \lambda(T \setminus \bigcup_{n \geq 1} T_n) = 0. \) Also invoking Lebesgue’s density theorem (see Oxtoby [12]), we can find sets \( N_n \subseteq T_n, \ with \ \lambda(N_n) = 0, \) such that every \( t \in T_n \setminus N_n \) is a density point for \( T_n. \) Next let \( t \in T_n \setminus N_n. \) We can find \( t_m \in T_n \setminus N_n, \ t_m > t, \ m \geq 1, \) and \( t_m \downarrow t \) as \( m \to \infty. \) So \( \hat{x}(t_m) \to \hat{x}(t). \) Also because by hypothesis \( \|h(t, x)\| \leq M, \) we have \( |G(t, x)| = \sup\{|v| : v \in G(t, x)\} \leq M \) and so \( \|x(t) - x(s)\| \leq M|t - s| \) for all \( t, s \in T. \) Let \( \varepsilon > 0. \) We have \( \hat{x}(t_m) \in G(t_m, x(t_m)) \subseteq h(t_m, x(t_m)) + (\varepsilon/2)B_1, \) where \( B_1 = \{y \in \mathbb{R}^N : |y| < 1\}. \) Since \( h|_{T_n \times \mathbb{R}^N} \) is \( \Gamma^M \)-continuous, we can find \( m_0(\varepsilon) \geq 1 \) such that for \( m \geq m_0 \) we have

\[
\|x(t_m) - h(t_m, x(t_m))\| < \frac{\varepsilon}{2}
\]

\[
\Rightarrow h(t_m, x(t_m)) \in h(t, x(t)) + \varepsilon B_1.
\]

Hence for \( m \geq m_0(\varepsilon) \) we have

\[
\hat{x}(t_m) \in h(t, x(t)) + \varepsilon B_1
\]

\[
\Rightarrow \hat{x}(t) \in h(t, x(t)) + \varepsilon B_1.
\]

Let \( \varepsilon \downarrow 0, \) to get that for all \( t \in \hat{T} = \bigcup_{n \geq 1}(T_n \setminus N_n), \ \lambda(T \setminus \hat{T}) = 0, \) we have \( \hat{x}(t) = h(t, x(t)), \ x(0) = x_0 \Rightarrow x(\cdot) \) is a solution of (3).  \( \square \)

4. Main theorem

In this section we state and prove our main result, concerning the existence of solutions for problem (1). For this we will need the following hypotheses:

- **H(F)** : \( F : T \times \mathbb{R}^N \to P_k(\mathbb{R}^N) \) is a multifunction such that
  1. \( (t, x) \to F(t, x) \) is measurable,
  2. \( x \to F(t, x) \) is l.s.c.,
  3. \( |F(t, x)| = \sup\{|v| : v \in F(t, x)\} \leq M \) for all \( (t, x) \in T \times \mathbb{R}^N. \)

- **H(K)'** : \( K : T \to P_{kc}(\mathbb{R}^N) \) is an h-Lipschitz multifunction such that \( K(0) \supseteq K(b), \) \( t \to DK(t, x(1)) \) is l.s.c. on \( \text{Gr} K \cap ([0, b] \times \mathbb{R}^N) \) and for every \( (t, x) \in \text{Gr} K \cap ([0, b] \times \mathbb{R}^N) \) we have \( \text{int} DK(t, x)(1) \neq \emptyset. \)

- **Remark.** Note that if \( K(t) = K \in P_{kc}(\mathbb{R}^N), \ t \in T, \) and \( \text{int} K \neq \emptyset, \) then hypothesis \( H(K)' \) is automatically satisfied. To see this, remark that \( DK(t, x)(1) = \text{proj}_{\mathbb{R}^N}(\{x\} \times \mathbb{R}^N \cap T_{\text{Gr} K}(t, x)) = \text{proj}_{\mathbb{R}^N}(\{y\} \times \mathbb{R}^N \cap T_{\text{Gr} K}(t, x)) = T_{\text{Gr} K}(x) \) and apply Theorem 1 and Proposition 4, pp. 220–221, of Aubin-Cellina [1]. Also note that our lower semicontinuity hypothesis on \( t \to DK(t, x)(1) \) implies that this multifunction is convex-valued (cf. Aubin-Cellina [1]).

- **H_{r}** : For all \( (t, x) \in \text{Gr} K, \ t < b, \) we have \( F(t, x) \cap \text{int} DK(t, x)(1) \neq \emptyset \) (tangential condition).

**Theorem 4.1.** If hypotheses \( H(F), H(K)' \) and \( H_{r} \) hold, then problem (1) admits a solution.
Proof. From Proposition 3.5, we have that \( x \to F(t, x) \cap \text{int} DK(t, x)(1) \) is l.s.c. \( \Rightarrow x \to F(t, x) \cap \text{int} DK(t, x)(1) \) is l.s.c. Also because of hypothesis \( H(F)(1) \), we have that \( (t, x) \to F(t, x) \cap \text{int} DK(t, x)(1) \) is measurable. Let \( H(t, x) = F(t, x) \cap \text{int} DK(t, x)(1) \). From Proposition 3.1 we know that given \( \varepsilon > 0 \), we can find \( T_\varepsilon \subseteq T \) compact with \( \lambda(T_\varepsilon) < \varepsilon \) such that \( H|_{T_\varepsilon \times \mathbb{R}^n} \) is l.s.c. So we can apply Theorem 1 of Bressan [3] and get a function \( h : T \times \mathbb{R}^n \to \mathbb{R}^n \) such that \( h(\cdot, \cdot) \) is Scorza-Dragoni \( \Gamma^M \)-continuous and for all \( (t, x) \in T \times \mathbb{R}^n \), \( h(t, x) \in H(t, x) \) (recall \( \Gamma^M \) is the cone \( \{(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n : ||x|| \leq Mt\} \)).

Then consider the multifunction \( G(t, x) = \bigcap_{\varepsilon > 0} \text{conv} h(B_\varepsilon(t, x)) \), where \( B_\varepsilon(t, x) = \{(t', x') \in T \times \mathbb{R}^n : |t - t'| < \varepsilon, ||x - x'|| < \varepsilon\} \). From Proposition 3.2 we know that \( (t, x) \to G(t, x) \) is u.s.c., and clearly from the above definition we have that \( G(t, x) \subseteq \text{conv} F(t, x) \subseteq \text{DK}(t, x)(1) \) for all \( (t, x) \in \text{Gr} K \), \( t < b \). Then consider the following Cauchy problem:

\[
\begin{align*}
\dot{x}(t) &\in G(t, p(t, x(t))) \text{ a.e.} \\
x(0) &= z \in K(0).
\end{align*}
\]

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\end{align*}
\]

It is well known (see for example, DeBlasi-Myjak [4] and Gorniewicz [5]) that the solution set \( S(z) \) of (4) is an \( R_3 \)-set in \( C(T, \mathbb{R}^n) \). In particular, then it is compact and acyclic (with respect to the Čech cohomology; cf. Proposition 3.1 of Gorniewicz [5]). Furthermore from Proposition 3.3, we know that \( S(z)(b) = \{x(b) : x \in S(z)\} \subseteq K(b) \subseteq K(0) \). Let \( e_b : C(T, \mathbb{R}^n) \to \mathbb{R}^n \) be the evaluation at \( b \) map; i.e. \( e_b(x) = x(b) \). Let \( R = e_b \circ S : K(0) \to \text{P}_b(K(0)) \). This is a pseudo-acyclic map in the sense of Lasry-Robert [9] (see Definition 5), and so by Theorem 7 of that paper we can find \( z \in R(z) \). Let \( x \in S(z) \) such that \( x(0) = x(b) = z \). Then \( x(t) \in G(t, p(t, x(t))) \) a.e., \( x(0) = x(b) \). Recalling that \( G(t, x) \subseteq \text{DK}(t, x)(1) \) for \( (t, x) \in \text{Gr} K \), \( t < b \), from Proposition 3.3 we get that \( x(t) \in K(t) \) for all \( t \in T \). So \( x(t) \in G(t, x(t)) \) a.e., \( x(0) = x(b) \). Invoking Proposition 3.6, we get \( x(t) = h(t, x(t)) \in F(t, x(t)) \) a.e., \( x(0) = x(b) \); i.e. \( x(\cdot) \) solves (1). \( \Box \)

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References


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