A COUNTEREXAMPLE
ON THE SEMICONTINUITY OF MINIMA

FERNANDO LUQUE-VÁSQUEZ AND ONÉSIMO HERNÁNDEZ-LERMA

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Abstract. Let $X$ and $Y$ be metric spaces, $\Phi$ a multifunction from $X$ to $Y$, and $v$ a real-valued function on $X \times Y$. We give an example in which $\Phi$ is continuous, and $v$ is continuous, inf-compact and bounded below, but the minimum function $v^*(x) := \inf_{y \in \Phi(x)} v(x, y)$ on $X$ is not lower semicontinuous.

Introduction

An important problem in optimization, control theory and related fields is the following. Let $X$ and $Y$ be metric spaces, and $\Phi$ a multifunction (or set-valued map) from $X$ to $Y$ with $\Phi(x) \neq \emptyset$ for all $x$ in $X$. Given a real-valued function $v$ on the graph $\text{Gr}(\Phi)$ of $\Phi$, which is defined as

$$\text{Gr}(\Phi) := \{(x, y)|x \in X, \ y \in \Phi(x)\},$$

the problem is to give conditions under which the minimum function

$$v^*(x) := \inf_{y \in \Phi(x)} v(x, y), \quad x \in X,$$

is lower semicontinuous (l.s.c.).

For example, a well-known result (see, e.g., [1], p. 14, Proposition 1.7; [3], p. 148, Proposition 7.32) states that $v^*$ is l.s.c. if $\Phi(x) = Y$ is compact and $v$ is l.s.c. Other known results include Theorems 1 and 2 below.

Theorem 1 (cf. [2], p. 116, Theorem 2; [5], Theorem 10.2). Suppose that $\Phi$ is compact-valued and upper semicontinuous. If $v$ is l.s.c. and bounded below, then so is $v^*$.

Theorem 2 (cf. [4], Lemma 3.2(f)). Suppose that $\text{Gr}(\Phi)$ is a Borel subset of $X \times Y$, and that $v$ is l.s.c. and inf-compact on $\text{Gr}(\Phi)$—where inf-compactness means that for every real number $r$ and $x$ in $X$, the set $\{y \in \Phi(x)|v(x, y) \leq r\}$

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is compact. If, in addition, the multifunction
\[ x \mapsto \Phi^*(x) := \{ y \in \Phi(x) | v^*(x) = v(x, y) \} \]
is l.s.c., then \( v^* \) is l.s.c.

The essential difference between Theorem 1 and Theorem 2 is that the compactness and upper semicontinuity of \( \Phi \) in the former is replaced by inf-compactness of \( v \) and lower semicontinuity of \( \Phi^* \) in the latter. We thank the referee for pointing out that Theorem 1 also holds even if the boundedness assumption on \( v \) is dropped, which can be deduced, for instance, from Theorem 1.4.16 in J.-P. Aubin and H. Frankowska, *Set-valued analysis*, Birkhäuser, 1990.

The following example shows that \( v^* \) fails to be l.s.c. if the assumptions in the above theorems are weakened.

**The example**

Suppose that \( X = \mathbb{R} \) and \( Y = \Phi(x) = [0, \infty) \) for all \( x \in X \), and let \( v \) be defined as

\[
\begin{align*}
v(x, y) &= 1 + y & \text{if } (x \leq 0) \text{ or } (x > 0, 0 \leq y \leq 1/2x), \\
       &= (2 + 1/x) - (2x + 1)y & \text{if } x > 0, 1/2x < y \leq 1/x, \\
       &= y - 1/x & \text{if } x > 0, y > 1/x.
\end{align*}
\]

Then the multifunction \( x \mapsto \Phi(x) \) is continuous, and \( v \) is continuous, inf-compact on \( \text{Gr}(\Phi) = \mathbb{R} \times [0, \infty) \) and bounded below (nonnegative), but

\[ v^*(x) = \begin{cases} 1 & \text{if } x \leq 0, \\ 0 & \text{if } x > 0 \end{cases} \]

is not l.s.c.

**REFERENCES**


**DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD DE SONORA, ROSALES Y BOULEVARD LUIS ENCINAS, 83000 HERMOSILLO, SONORA, MEXICO**

**DEPARTAMENTO DE MATEMÁTICAS, CINVESTAV-IPN, A. POSTAL 14-740, 07000 MÉXICO, D.F., MEXICO**

*E-mail address: ohernand@math.cinvestav.mx*