ON THE BETTI NUMBER
OF THE IMAGE OF A CODIMENSION-\(k\) IMMERSION
WITH NORMAL CROSSINGS

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Abstract. Let \(f: M \to N\) be a codimension-\(k\) immersion with normal crossings of a closed \(m\)-dimensional manifold. We show that \(f\) is an embedding if and only if the \((m-k+1)\)-th Betti numbers of \(M\) and \(f(M)\) coincide, under a certain condition on the normal bundle of \(f\).

1. Introduction

Let \(f: M \to N\) be a codimension-\(k\) \(C^1\)-immersion with normal crossings, where \(M\) is a closed \(m\)-dimensional manifold and \(N\) is an \((m+k)\)-dimensional manifold \((k \geq 1)\). In [BR, BMS1, BMS2], it is shown that when \(k = 1\), \(H^1(N; \mathbb{Z}_2) = 0\), \(M\) is orientable, and \(f\) is not an embedding, then

\[ \beta_0(N - f(M)) \geq 3, \]

where \(\beta_i\) denotes the \(i\)-th Betti number in \(\mathbb{Z}_2\)-coefficient (see also [S]). This is equivalent to showing that \(f\) is an embedding if and only if \(\beta_m(f(M)) = \beta_m(M)\) (see [BMS2, Lemma 2.2]). In this paper we generalize this result, showing the following.

Theorem 1.1. Let \(f: M \to N\) be a codimension-\(k\) \(C^1\)-immersion with normal crossings, where \(M\) is a closed \(m\)-dimensional manifold. Then \(f\) is an embedding if and only if

\[ \beta_{m-k+1}(f(M)) = \beta_{m-k+1}(M) \quad \text{and} \quad v_k(f) = w_k(v_f), \]

where \(v_k(f) = f^* \circ \beta(\Omega_1) \in H^k(M; \mathbb{Z}_2)\), \(\Omega_1 \in H^k(N, N - f(M); \mathbb{Z}_2)\) is the transverse class defined in [He], \(\beta: H^k(N, N - f(M); \mathbb{Z}_2) \to H^k(N; \mathbb{Z}_2)\) is the homomorphism induced by the inclusion, and \(w_k(v_f) \in H^k(M; \mathbb{Z}_2)\) is the top Stiefel-Whitney class of the normal bundle \(v_f\) of the immersion \(f\).

Note that, when \(k = 1\), \(H^1(N; \mathbb{Z}_2) = 0\), and \(M\) is orientable, we always have \(v_1(f) = w_1(v_f) = 0\).

In particular, we have the following.

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Corollary 1.2. Let \( f: M \to N \) be a codimension-\( k \) \( C^1 \)-immersion with normal crossings, where \( M \) is a closed \( m \)-dimensional manifold. Suppose that either \( f^*: H^k(N; \mathbb{Z}_2) \to H^k(M; \mathbb{Z}_2) \) or \( f_*: H_m(M; \mathbb{Z}) \to H_m(N; \mathbb{Z}_2) \) is the zero map. Then \( f \) is an embedding if and only if
\[
\beta_{m-k+1}(f(M)) = \beta_{m-k+1}(M) \quad \text{and} \quad w_k(\nu_f) = 0.
\]

Note that the top Stiefel-Whitney class \( \omega_k(\nu_f) \) of the normal bundle \( \nu_f \) is the modulo 2 reduction of the Euler class, which is the obstruction to the existence of a nowhere zero cross section. Note also that \( w_k(\nu_f) \) depends only on the homotopy class of \( f \).

As to the Betti number of the complement of the image of an immersion, we have the following.

Corollary 1.3. Let \( f: M \to N \) be a codimension-\( k \) \( C^1 \)-immersion with normal crossings, where \( M \) is a closed \( m \)-dimensional manifold, \( \beta_{2k}(N) = \beta_{2k-2}(N) = \beta_{2k-2}(M) = 0 \), and \( w_k(\nu_f) = 0 \). Here \( \tilde{\beta}_{2k-2} \) denotes the dimension of the reduced \( (2k-2) \)-th homology group in \( \mathbb{Z}_2 \)-coefficient. Then \( f \) is an embedding if and only if \( \tilde{\beta}_{2k-2}(N - f(M)) = \beta_{k-1}(M) \).

Note that Corollary 1.3 is a generalization of the results in [BR, BMS1, BMS2] concerning the case \( k = 1 \) (see also [S]).

In the following, all the homology and cohomology groups are with \( \mathbb{Z}_2 \)-coefficients.

2. Proof of Theorem 1.1

Proof of Theorem 1.1. Set \( A = \{ x \in M : f^{-1}(f(x)) \neq \{ x \} \} \) and \( B = f(A) \). Note that \( A \) and \( B \) are ANR. We suppose that \( f \) is not an embedding; i.e., \( A \neq \emptyset \). Consider the following diagram of homologies with exact rows:

\[
\cdots \to H_i(A) \to H_i(M) \to H_i(M, A) \to H_{i-1}(A) \to \cdots
\]

\[
\cdots \to H_i(B) \to H_i(f(M)) \to H_i(f(M), B) \to H_{i-1}(B) \to \cdots
\]

where the vertical homomorphisms are induced by \( f \). Note that the homomorphism \( f_*: H_i(M, A) \to H_i(f(M), B) \) is an isomorphism by excision. Then it is not difficult to extract the following exact sequence:

\[
\cdots \to H_{m-k+1}(A) \to H_{m-k+1}(B) \oplus H_{m-k+1}(M) \to H_{m-k+1}(f(M))
\]

\[
\to H_{m-k}(A) \to H_{m-k}(B) \oplus H_{m-k}(M) \to \cdots.
\]

Since \( A \) and \( B \) are of dimension \( m - k \), we have the exact sequence

\[
0 \to H_{m-k+1}(M) \to H_{m-k+1}(f(M)) \to H_{m-k}(A) \xrightarrow{\alpha} H_{m-k}(B) \oplus H_{m-k}(M) \to \cdots,
\]

where \( \alpha = (f|A)_* \oplus j_* \) and \( j: A \to M \) is the inclusion map. Then we have

\[
\beta_{m-k+1}(f(M)) = \beta_{m-k+1}(M) + \dim \ker \alpha.
\]

Now consider the fundamental class \( [A] \in H_{m-k}(A) \), which is known to exist and to be non-zero ([He]). Then we have \( (f|A)_*[A] = 0 \), since \( f|A \) is a double cover away from the codimension-\( k \) set \( \{ x \in A : \#(f^{-1}(f(x))) \geq 3 \} \), where \( \# \)
denotes the cardinality. On the other hand, by Herbert [He], we have a formula for calculating $j_\ast[A]$, which is
\[ j_\ast[A] = D_M \circ f^* \circ \beta(\Omega_1) - D_M(w_k(\nu_f)) \]
\[ = D_M(v_k(f) - w_k(\nu_f)), \]
where $D_M : H^k(M) \to H_{m-k}(M)$ is the Poincaré dual, $\beta : H^k(N, N - f(M)) \to H^k(N)$ is the homomorphism induced by the inclusion $(N, \emptyset) \to (N, N - f(M))$, and $\Omega_1 \in H^k(N, N - f(M))$ is the transverse class defined in [He]. (For this formula, see also [W, (18.5)].) Now suppose that $v_k(f) = w_k(\nu_f)$. Then we have $j_\ast[A] = 0$. This implies that
\[ \beta_{m-k+1}(f(M)) = \beta_{m-k+1}(M) + \dim \ker \alpha \]
\[ > \beta_{m-k+1}(M), \]
since $[A]$ is a non-zero element of $\ker \alpha$. Thus, if $f$ is not an embedding, we have
\[ \beta_{m-k+1}(f(M)) > \beta_{m-k+1}(M) \]
or
\[ v_k(f) \neq w_k(\nu_f). \]
On the other hand, if $f$ is an embedding, we clearly have
\[ \beta_{m-k+1}(f(M)) = \beta_{m-k+1}(M). \]
Furthermore, the formula of Herbert [He] cited above shows $v_k(f) = w_k(\nu_f)$. This completes the proof of Theorem 1.1.

**Proof of Corollary 1.2.** If $f^\ast : H^k(N) \to H^k(M)$ is the zero map, it is easy to see that $v_k(f) = 0$, by the definition of $v_k(f)$. Thus, for the proof of Corollary 1.2, we have only to show that, if $f_* : H_m(M) \to H_m(N)$ is the zero map, then $v_k(f) = 0$. Let
\[ D_1 : H^k(N, N - f(M)) \to H_m(f(M)) \]
and
\[ D_2 : H_m(N, N - f(M)) \to H^k(f(M)) \]
be the duality isomorphisms. Furthermore, we denote by $i : f(M) \to N$ and $l : (N, \emptyset) \to (N, N - f(M))$ the inclusion maps. First note that $D_1(\Omega_1) = f_*[M] \in H_m(f(M))$ by [He, Proposition 4.1], where $[M] \in H_m(M)$ is the fundamental class of $M$. Then we have
\[ v_k(f) = f^\ast \circ \beta(\Omega_1) = f^\ast \circ (l \circ i)^\ast(\Omega_1) \]
\[ = f^\ast \circ D_2 \circ (l \circ i)_* \circ D_1(\Omega) = f^\ast \circ D_2 \circ (l \circ i)_*(f_*[M]) \]
\[ = f^\ast \circ D_2 \circ i_*(f_*[M]) = 0, \]
since $f_*[M] = 0 \in H_m(N)$ by our hypothesis. This completes the proof.

**Remark 2.1.** We can interpret the cohomology classes $v_k(f)$ and $w_k(\nu_f) \in H^k(M)$ geometrically as follows. Let $\gamma \in H_k(M)$ be an arbitrary homology class and $C (\subset M)$ a singular cycle representing $\gamma$. Then we see that $\langle v_k(f), \gamma \rangle$ is equal to the modulo 2 intersection number of $f(M)$ and $f(C)$ in $N$, where we move $f(C)$ slightly so that it intersects $f(M)$ transversely. On the other hand, $\langle w_k(\nu_f), \gamma \rangle$ is equal to the modulo 2 self-intersection number of $C$ in...
the total space of $i^*\nu_f$, where $i: C \to M$ is the inclusion map. In other words, $\langle w_k(\nu_f), \gamma \rangle$ is equal to the modulo 2 intersection number of $f(M)$ and $f(C)$ in $N$ off the self-intersection of $f$.

Remark 2.2. The condition about the top Stiefel-Whitney class of the normal bundle $\nu_f$ of $f$ is necessary in Theorem 1.1 and Corollaries 1.2 and 1.3. For example, consider the immersion with normal crossings $f: K \to \mathbb{R}^3$ as in Figure 1, where $K$ is the Klein bottle. We see that the immersion $f$ is not an embedding, but that

$$\beta_2(f(K)) = \beta_1(K) = 1.$$ 

Note that, in this example, we have $0 = v_1(f) \neq w_1(\nu_f)$. Furthermore, Theorem 1.1 implies that, for any immersion $g: K \to N$ with normal crossings into a 3-manifold $N$ with $g(K)$ homeomorphic to $f(K)$, $v_1(g)$ never coincides with $w_1(\nu_g)$.

Remark 2.3. Suppose that there exist integers $p$ and $q$ such that $q \leq p + 1$, $p + q = k$, $f^*(w_i(N)) = 0$ ($0 < i < q$), and $w_j(M) = 0$ ($0 < j \leq p$), where $w_i$ denotes the $i$-th Stiefel-Whitney class. Then we have $f^*(w_k(N)) = w_k(M) + w_k(\nu_f)$. This can be proved as follows. By the definition of the normal bundle of an immersion, we have

$$f^*(w(N)) = w(M) \cup w(\nu_f),$$

where $w$ denotes the total Stiefel-Whitney class. Then we have

$$w(\nu_f) = f^*(w(N)) \cup (w(M))^{-1},$$

which implies that $w_i(\nu_f) = 0$ for $0 < i < q$. Then we have

$$f^*(w_k(N)) = w_k(M) + w_k(\nu_f).$$

Thus, if in addition we have $w_k(M) = f^*(w_k(N)) = 0$, then we have $w_k(\nu_f) = 0$. For example, if $k = 1$ and $M$ and $N$ are orientable, $w_1(\nu_f)$ always vanishes. If $k = 2$ and $M$ and $N$ are spin manifolds, $w_2(\nu_f)$ always vanishes.

Proof of Corollary 1.3. First note that, since $H_k(N) = 0$, the hypotheses of Corollary 1.2 are satisfied for $f$. Now consider the following exact sequence of homology:

$$\tilde{H}_{2k-1}(N) \to H_{2k-1}(N, N - f(M)) \to \tilde{H}_{2k-2}(N - f(M)) \to \tilde{H}_{2k-2}(N).$$
Note that
\[ H_{2k-1}(N) = 0, \quad H_{2k-1}(N, N - f(M)) \cong H^{m-k+1}(f(M)), \]
and
\[ H_{2k-2}(N) = 0. \]
Thus we have
\[ \tilde{\beta}_{2k-2}(N - f(M)) = \beta_{m-k+1}(f(M)). \]
Note also that \( \beta_{m-k+1}(M) = \beta_{k-1}(M) \) by Poincaré duality. Then, combining this with Corollary 1.2, we obtain the conclusion. This completes the proof. \( \square \)

Remark 2.4. Corollary 1.3 seems a little bit tedious. However, when \( N = \mathbb{R}^{m+k} \), it takes a simpler form as follows: a codimension- \( k \) \( C^1 \)-immersion with normal crossings \( f: M \to \mathbb{R}^{m+k} \) of a closed \( m \)-dimensional manifold \( M \) with vanishing \( k \)-th dual Stiefel-Whitney class \( w_k(M) \) \(( \in H^k(M)) \) is an embedding if and only if \( \tilde{\beta}_{2k-2}(\mathbb{R}^{m+k} - f(M)) = \beta_{k-1}(M) \).

Remark 2.5. In [Hi], Hirsch has shown that, if \( f: M \to N \) is a codimension- \( k \) proper \( C^2 \)-immersion and \( H_k(N) = 0 \), then \( H_{k-1}(N - f(M)) \) is non-trivial. Using the techniques used in the proof of Theorem 1.1, we can prove a refinement of Hirsch’s result for immersions with normal crossings as follows. Let \( f: M \to N \) be a codimension- \( k \) \( C^1 \)-immersion with normal crossings, where \( M \) is a closed \( m \)-dimensional manifold, \( \dim H_{k-1}(N) \) is finite, and \( H_k(N) = 0 \). Then we have the following.

(1) We always have
\[ \beta_{k-1}(N - f(M)) ( = \beta_{k-1}(N) + \beta_m(f(M))) \geq \beta_{k-1}(N) + \beta_0(M). \]

(2) When \( k = 1 \) and \( w_1(M) = 0 \), the equality holds in (1) if and only if \( f \) is an embedding.

(3) When \( k \geq 2 \), the equality in (1) always holds.

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References


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