A FULL EXTENSION OF THE ROGERS-RAMANUJAN CONTINUED FRACTION

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Abstract. In this paper, we present the natural extension of the Rogers-Ramanujan continued fraction to the nonterminating very well-poised basic hypergeometric function \( \phi_7 \). In a letter to Hardy, Ramanujan indicated that he possessed a four variable generalization. Our generalization has seven variables and is, perhaps, all one can expect from this method.

1. Introduction

One of the most intriguing results from classical \( q \)-series is the Rogers-Ramanujan continued fraction [1, p. 440]:

\[
\prod_{n=0}^{\infty} \frac{(1 - q^{5n+2})(1 - q^{5n+3})}{(1 - q^{5n+1})(1 - q^{5n+4})} = 1 + \frac{q}{1 + \frac{q^2}{1 + \frac{q^3}{1 + \cdots}}},
\]

where \(|q| < 1\) throughout this paper.

The proof of (1.1) relies fundamentally on the following. Let

\[
(1.2) \quad G(z, q) = G(z) = 1 + \sum_{n=1}^{\infty} \frac{z^n q^n}{(1 - q)(1 - q^2) \cdots (1 - q^n)}.
\]

Then

\[
(1.3) \quad G(z) = \prod_{n=1}^{\infty} (1 - zq^n)^{-1}
\]

\[
\times \left(1 + \sum_{n=1}^{\infty} \frac{(1 - zq)(1 - zq^2) \cdots (1 - zq^{n-1})(-1)^n z^{2n} q^{n(5n-1)/2}(1 - zq^{2n})}{(1 - q)(1 - q^2) \cdots (1 - q^n)}\right),
\]

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and
\[
\frac{G(z)}{G(zq)} = 1 + \frac{zq}{1 + \frac{zq^2}{1 + \frac{zq^3}{1 + \cdots}}.}
\]

G. N. Watson [7] gave the following five-parameter generalization of (1.3) which is perhaps the most natural nonmultiple series generalization:
\[
\phi_7\left(z, q^{\sqrt{z}}, -q^{\sqrt{z}}, a_1, a_2, a_3, a_4, a_5, q, \frac{z^2q^5}{a_1a_2a_3a_4a_5}\right)
\]
\[
= (zq)_N^{-1} \frac{(zq)_N}{(zq)_N^{(a_1a_2a_3a_4a_5)}} \phi_3\left(z, a_1, a_2, a_3, a_4, a_5; q, q\right)
\]
where \(a_5 = q^{-N}\) with \(N\) a nonnegative integer, and
\[
(a; q)_N = (a)_N = (1 - a)(1 - aq) \cdots (1 - aq^{N-1})
\]
\[
r+1 \phi_r\left(a_0, a_1, \ldots, a_r; q, t\right) = \frac{r!}{b_1! \cdots b_r!} \sum_{n=0}^{\infty} \frac{(a_0)_n(a_1)_n \cdots (a_r)_n t^n}{(q)_n(b_1)_n \cdots (b_r)_n}.
\]

We remark that the expressions in (1.5) are most naturally viewed as functions of \(a_1, a_2, a_3, a_4, a_5\) and are, in fact, continuous in each \(a_i\) around zero, i.e. \(a_i = \infty\). Consequently when we set any of \(a_1, \ldots, a_5 = \infty\) throughout this paper we shall be doing so in light of the above comments. We could, of course, have begun with each \(a_i\) replaced by its reciprocal; however this would not be consistent with standard notation [5, pp. 4, 35].

When \(a_1 = a_2 = a_3 = a_4 = a_5 = \infty\) (i.e. \(a_5 = \infty\) means \(\lim_{N \to \infty} q^{-N}\)), (1.5) reduces term-by-term to (1.3). In [1, p. 434], it is noted that the case \(a_3 = a_4 = a_5 = \infty\) of (1.5) is quite possibly the function used in a general continued fraction identity alluded to by Ramanujan [6, p. xxviii]. One is naturally led to ask: Can one extend (1.4) wherein \(G(z)\) is replaced by the very well-poised \(8\phi_7\) of (1.5)?

We answer this question affirmatively in Theorem 1. It is to be emphasized that Theorem 1 does not require any of \(a_1, a_2, \ldots, a_5\) to be of the form \(q^{-N}\). Consequently our result holds for the nonterminating very well-poised \(8\phi_7\) first considered by W. N. Bailey in [4] (cf. [5, p. 42, eq. (2.10.10)]).

2. \textit{q-Hypergeometric Background}

Our work is based on the \(q\)-difference equations for very well-poised basic hypergeometric series studied in [1] (cf. [2]). Thus we require many of the auxiliary functions defined there.

\[
C_k,i(a_1, a_2, \ldots, a_k; x; q) = C_k,i((a); x; q)k
\]
\[
= \sum_{n \geq 0} (-1)^{n(\lambda+1)} x^{kn}(a_1a_2 \cdots a_k)^{-n} q^{\left(2k-\lambda+1\right)n^2+(\lambda+1)n-2n^2}/2
\]
\[
(1 - x^i q^{2ni}) \frac{(x)_n(a_1)_n(a_2)_n \cdots (a_k)_n}{(1 - x)q_{a_1}_n(a_k)_n \cdots (a_k)_n}.
\]
A FULL EXTENSION OF THE ROGERS-RAMANUJAN CONTINUED FRACTION

\[ H_{k,i}(a_1, a_2, \ldots, a_k; x, q) = H_{k,i}((a); x; q)_\lambda \]

\[ = \frac{(\frac{xq}{a_1})_\infty (\frac{xq}{a_2})_\infty \cdots (\frac{xq}{a_k})_\infty}{(xq)_\infty} C_{k,i}((a); x; q)_\lambda . \]

We note that [1, p. 434]

\[ C_{k,1}((a); x; q)_{2k+1} \]

\[ = 2k+4\phi_{2k+3} \left( \frac{x, q\sqrt{x}, -q\sqrt{x}, a_1, a_2, \ldots, a_{2k+1}; q, \frac{xq^k}{a_1a_2\cdots a_{2k+1}}}{\sqrt{x}, -\sqrt{x}, \frac{xq}{a_1}, \frac{xq}{a_2}, \ldots, \frac{xq}{a_{2k+1}}} \right) . \]

The crucial relations among these functions are the following [1, p. 435, Theorem 1]:

\[ H_{k,i}((a); x; q)_\lambda - H_{k,-i}((a); x; q)_\lambda \]

\[ = \sum_{j=0}^{i-1} (-1)^j \sigma_j \left( \frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \frac{1}{a_4}, \frac{1}{a_5} \right) x^j q^j H_{k,k+1-i-j}((a); xq; q)_\lambda , \]

where \( \sigma_j(y_1, \ldots, y_\lambda) \) is the \( j \)-th elementary symmetric function of \( y_1, \ldots, y_\lambda \);

\[ H_{k,-i}((a); x; q)_\lambda = -x^{-i} H_{k,i}((a); x; q)_\lambda ; \]

\[ H_{k,0}((a); x; q)_\lambda \equiv 0 . \]

Henceforth we are only interested in the case \( k = 2, \lambda = 5 \). Given these values of \( k \) and \( \lambda \), we abbreviate

\[ H_i(x) = H_{2,i}(a_1, a_2, a_3, a_4, a_5; x; q) \]

and

\[ \sigma_j = \sigma_j \left( \frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \frac{1}{a_4}, \frac{1}{a_5} \right) . \]

The four instances of (2.4) with \( i = 2, 1, 0, -1 \) reduce to the following once we use (2.5) and (2.6) for simplification:

\[ H_2(x) - H_1(x) - \sigma_3 x^2 q H_2(xq) + x^2 q \sigma_4 H_3(xq) - x^2 q \sigma_5 H_4(xq) \]

\[ = (x - x^2 q \sigma_2) H_1(xq) ; \]

\[ H_1(x) - (1 - x^2 q^2 \sigma_4) H_2(xq) - x^2 q^2 \sigma_5 H_3(xq) \]

\[ = (-xq \sigma_1 + x^2 q^2 \sigma_3) H_1(xq) ; \]

\[ H_1(x) + (xq \sigma_1 - x^3 q^3 \sigma_5) H_2(xq) - H_3(xq) \]

\[ = (x^2 q^2 \sigma_2 - x^3 q^3 \sigma_4) H_1(xq) ; \]

\[ H_2(x) - x H_1(x) - x^2 q^2 \sigma_2 H_2(xq) + xq \sigma_1 H_3(xq) - H_4(xq) \]

\[ = (-x^3 q^3 \sigma_3 + x^4 q^4 \sigma_5) H_1(xq) . \]

These four functional equations will be used in Section 3 to obtain the relevant second order \( q \)-difference equation for \( H_{2,1}(a_1, a_2, a_3, a_4, a_5; x; q) \).

3. THE MAIN RESULTS

Our next step consists of replacing \( x \) by \( xq \) in each of (2.9)--(2.12). We rewrite the resulting equations so that \( H_1(xq) \) and \( H_1(xq^2) \) appear on the right
of each equation.

\[
H_2(xq) - \sigma_3 x^2 q^3 H_2(xq^2) + x^2 q^3 \sigma_4 H_3(xq^2) - x^2 q^3 \sigma_3 H_4(xq^2) = H_1(xq) + (xq - x^2 q^3 \sigma_2) H_1(xq^2);
\]

\[
- (1 - x^2 q^4 \sigma_4) H_2(xq^2) - x^2 q^4 \sigma_5 H_3(xq^2)
= (-x q^2 \sigma_1 + x^2 q^4 \sigma_3) H_1(xq^2) - H_1(xq);
\]

\[
(x q^2 \sigma_1 - x^3 q^6 \sigma_5) H_2(xq^2) - H_3(xq^2)
= (x^2 q^4 \sigma_2 - x^3 q^6 \sigma_4) H_1(xq^2) - H_1(xq);
\]

\[
H_2(xq) - x^2 q^4 \sigma_2 H_2(xq^2) + x q^2 H_3(xq^2) - H_4(xq^2)
= (-x^3 q^6 \sigma_3 + x^4 q^8 \sigma_5) H_1(xq^2).
\]

Equations (2.9)-(2.12), (3.1)-(3.4) constitute a system of eight linear equations in the eight unknowns: \( H_1(x), H_2(x), H_2(xq), H_2(xq^2), H_3(xq), H_3(xq^2), H_4(xq), H_4(xq^2) \). The system turns out to be nonsingular, and consequently Cramer's Rule assisted by MACSYMA allows us to solve for \( H_1(x) \) which upon inspection reveals a linear relation among \( H_1(x), H_2(x), \) and \( H_1(xq^2) \). To make this relation most succinct we introduce

\[
p(x) = p(x; \sigma_1, \sigma_4, \sigma_5, q) = 1 - x^2 q^2 \sigma_2 + x^3 q^3 \sigma_1 \sigma_5 - x^5 q^5 \sigma_2^5.
\]

Using this notation, we find that after simplification and division by \((-1 + x^2 q^5\sigma_5)\)

\[
Q(x) H_1(x) = P(x) H_1(xq) + R(x) H_1(xq^2),
\]

where

\[
Q(x) = (1 - x^2 q^4 \sigma_5)(1 - x^2 q^3 \sigma_5)p(xq),
\]

\[
P(x) = -xq(1 - x^2 q^3 \sigma_5)p(xq)(\sigma_1 - xq \sigma_3 + x^3 q^3 \sigma_2 \sigma_5 - x^4 q^4 \sigma_4 \sigma_5) - p(x) \left\{ (-1 - xq \sigma_1 + x^3 q^4 \sigma_5)p(xq) - x^6 q^{11} \sigma_2 \sigma_5 + x^5 q^9 \sigma_1 \sigma_4 \sigma_5 - x^5 q^9 \sigma_4 \sigma_5 + x^4 q^7 \sigma_3 \sigma_5 \right.

+ x^4 q^7 \sigma_2 \sigma_5 - x^4 q^7 \sigma_2^3 - x^3 q^5 \sigma_1 \sigma_5 + x^2 q^3 \sigma_4 - x^2 q^3 \sigma_3 + xq \sigma_1 \left\} ,
\]

\[
R(x) = xq p(x) \prod_{1 \leq i < j \leq 5} \left( 1 - \frac{xq^2}{a_i a_j} \right).
\]

**Theorem 1.**

\[
\frac{H_1(x)}{H_1(xq)} = \frac{P(x)}{Q(x)} + \frac{R(x)/Q(x)}{P(xq) + R(xq)/Q(xq) + \frac{R(xq^2)/Q(xq^2)}{P(xq^2) + R(xq^2)/Q(xq^2) + \ldots}}.
\]
Proof. This assertion is merely the iteration of a restatement of (3.6) written
in the form
\[
\frac{H_1(x)}{H_1(xq)} = \frac{P(x)}{Q(x)} + \frac{R(x)/Q(x)}{H_1(xq)}.
\]
Convergence is guaranteed by the fact that \(P(x), Q(x)\) and \(R(x)\) are polynomi-
als in \(x\), \(Q(0) = 1\), and \(R(x)\) has no constant term. □

Related contiguous relations and continued fractions may be derived from
(3.6). For example, if we put \(a_5 = q^{-N}\) and let \(N \to \infty\) in (1.5) we find
\[
H_1(a_1, a_2, a_3, a_4; x; q)
\]
\[
(3.12) = \left(\frac{xq}{a_1}\right) \left(\frac{xq}{a_2}\right) \left(\frac{xq}{a_3a_4}\right) \phi_2 \left(\frac{xq}{a_1a_2}, \frac{xq}{a_3}, \frac{xq}{a_4}; xq\right).
\]
Making the change of variables
\[
x \mapsto \frac{de}{aq}, \ a_1 \mapsto \frac{e}{a}, \ a_2 \mapsto \frac{d}{a}, \ a_3 \mapsto b, \ a_4 \mapsto c,
\]
cancelling infinite products and simplifying the resulting polynomials give the
following contiguous relation for \(\phi_2\):
\[
S_3\phi_2 \left(\begin{array}{ccc}
a & b & c \\
d & e & \frac{de}{abc}
\end{array} \right) = T_3\phi_2 \left(\begin{array}{ccc}
aq & b & c \\
dq & eq & \frac{de}{abcq}
\end{array} \right)
+ U_3\phi_2 \left(\begin{array}{ccc}
aq^2 & b & c \\
dq^2 & eq^2 & \frac{de}{abcq^2}
\end{array} \right),
\]
where the polynomials \(S, T\) and \(U\) are
\[
S = bc(d)^2(e)^2(abc - de)(bc - deq^2),
\]
\[
T = (1 - dq)(1 - eq)(bc - deq)[bcde(a(b + c) + d + e)(1 + q) - (b + c)de
+ abc(d + e))(bc + deq) + a(bc - de)(bc - deq^2)],
\]
\[
\]
Applying the transformation (3.2.7) of [5] term-by-term to the contiguous
relation and simplifying the products yield
\[
S_3'\phi_2 \left(\begin{array}{ccc}
a & b & c \\
d & e & \frac{de}{abc}
\end{array} \right) = T_3'\phi_2 \left(\begin{array}{ccc}
aq & bq & cq \\
dq & eq & \frac{de}{abc}
\end{array} \right)
+ U_3'\phi_2 \left(\begin{array}{ccc}
aq^2 & bq^2 & cq^2 \\
dq^2 & eq^2 & \frac{de}{abc}
\end{array} \right),
\]
where now
\[
S' = a^2b^2c^2(d)^2(e)^2(1 - eq^2),
\]
\[
T' = abc(1 - dq)(eq)^3(de(ab + ac + bc + e)(1 + q)
- d(abc + (a + b + c)e)(1 + eq)
+ abc(1 - e)(1 - eq^2)),
\]
\[
\]
Iterating these two contiguous relations gives
\begin{equation}
3\phi_2\left(\frac{a, b, c}{d, e}\; ; \; \frac{de}{abc}\right) = \frac{T(a, d, e)}{S(a, d, e)} + \frac{U(a, d, e)}{T(aq, dq, eq)/S(aq, dq, eq)} + \ldots
\end{equation}
and
\begin{equation}
3\phi_2\left(\frac{aq, b, c}{dq, eq}\; ; \; \frac{de}{abc}\right) = \frac{T'(a, b, c, d, e)}{S'(a, b, c, d, e)}
\end{equation}
\begin{equation}
+ \frac{U'(a, b, c, d, e)}{T'(aq, bq, cq, dq, eq^2)/S'(aq, bq, cq, dq, eq^2)} + \ldots.
\end{equation}

For both continued fractions convergence follows from the fact that after canceling common factors of powers of $q$ from the partial numerators and denominators, the partial numerators tend to zero, while the partial denominators do not.

Obviously, although Theorem 1 is explicit, it nonetheless lacks the elegance of (1.1) or (1.4). If we let $a_4$ and $a_5$ tend to infinity (so that $a_4 = a_5 = 0$) and denote the resulting $P(x)$, $Q(x)$ and $R(x)$ by $P(x)$, $Q(x)$ and $R(x)$ respectively, then we see that
\begin{equation}
Q(x) = 1,
\end{equation}
\begin{equation}
R(x) = xq \left(1 - \frac{xq^2}{a_1a_2}\right) \left(1 - \frac{xq^2}{a_1a_3}\right) \left(1 - \frac{xq^2}{a_2a_3}\right),
\end{equation}
\begin{equation}
P(x) = 1 - \frac{xq}{a_1} - \frac{xq}{a_2} - \frac{xq}{a_3} + \frac{x^2q^2}{a_1a_2a_3} (1 + q).
\end{equation}

Thus we obtain

Corollary 1.
\begin{equation}
\frac{H_{2,1}(a_1, a_2, a_3; x; q)}{H_{2,1}(a_1, a_2, a_3; q; q)} = P(x) + \frac{R(x)}{P(xq) + \frac{R(xq)}{P(xq^2) + \ldots}}.
\end{equation}

Several remarks are now in order. First it is clear from (3.17)-(3.20) that (1.4) is the case $a_1 = a_2 = a_3 = a_4 = a_5 = +\infty$. Also the form for the $H_{k,i}$ given in (2.2) will yield the right-hand side of (1.3) and not the right-hand side of (1.2). However, for completeness, we note we may deduce from (2.2), (2.3) and (1.5) that
\begin{equation}
H_{2,1}(a_1, a_2, a_3; x; q) = \left(\frac{xq}{a_1a_3}\right) \left(\frac{xq}{a_2}\right) \sum_{n=0}^{\infty} (a_1)_n (a_3)_n \left(\frac{xq}{a_1a_3}\right)_n,
\end{equation}
and this last expression reduces to the right-hand side of (1.2) as $a_1$, $a_2$ and $a_3 \to \infty$.

To conclude we examine the simplest case involving three finite parameters. We replace $q$ by $q^4$ and then set $a_1 = q$, $a_2 = q^2$, $a_3 = q^3$ in Corollary 1. By (3.21) we see that

$$\lim_{x \to 1^-} H_{2,1}(q, q^2, q^3; x; q^4) = \lim_{x \to 1^-} (x; q^4)_{\infty} (xq^2; q^4)_{\infty} \sum_{n=0}^{\infty} \frac{(q, q^2)_{2n} x^n}{(q^4; q^4)_n (xq^2; q^4)_n} = (q; q^2)_{\infty},$$

and if we define

$$f(q) = H_{2,1}(q, q^2, q^3; q^4, q^4),$$

then by (3.21) after simplification

$$f(q) = \frac{(q^2; q^4)^2_{\infty}}{q(1-q)} \sum_{n=0}^{\infty} \frac{(q; q^2)_{2n+1} q^{2n+1}}{(q^2; q^2)_{2n+1}}$$

$$= \frac{(q^2; q^4)^2_{\infty}}{q(1-q)} \sum_{n=0}^{\infty} \frac{(q; q^2)_n q^n (1 - (-1)^n)}{2 (q^2; q^2)_n}$$

$$= \frac{(q^2; q^4)^2_{\infty}}{2q(1-q)} \left( \frac{(q^2; q^2)_{\infty}}{q; q^2_{\infty}} - \frac{(-q^2; q^2)_{\infty}}{-q; q^2_{\infty}} \right)$$

$$= \frac{(q^2; q^4)^2_{\infty}}{2q(1-q)} \left( (-q; -q)_{\infty} - (q; -q)_{\infty} \right).$$

Hence by [3, p. 23, eq. (2.2.12)]

$$f(-q) = \frac{-(q^2; q^4)_{\infty}}{2q(1+q)} (-q)_{\infty} 2 \sum_{n=1}^{\infty} (-1)^n q^{n^2}$$

$$= (-q^3; q^2_{\infty}) \sum_{n=1}^{\infty} (-1)^{n-1} q^{n^2-1},$$

and so

$$H_{2,1}(q, q^2, q^3; q^4; q^4) = (q^3; q^2_{\infty}) \sum_{n=0}^{\infty} q^{n^2+2n}.$$

Consequently Corollary 1 reduces to

$$1 - q \frac{\sum_{n=0}^{\infty} q^{n^2+2n}}{1 - q^6 + q^6 + \frac{q^4(1-q^3)(1-q^4)(1-q^5)}{1-q^5 - q^6 - q^{10} + q^{14} + \frac{q^8(1-q^7)(1-q^8)(1-q^9)}{1-q^7 - q^{10} - q^{11} + q^{14} + \frac{q^{13}(1-q^9)(1-q^{10})(1-q^{11})}{1-q^9 - q^{10} - q^{11} + q^{14} + \ldots}}}}.$$
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