A FULL EXTENSION OF THE ROGERS-RAMANUJAN CONTINUED FRACTION

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ABSTRACT. In this paper, we present the natural extension of the Rogers-Ramanujan continued fraction to the nonterminating very well-poised basic hypergeometric function $g^7$. In a letter to Hardy, Ramanujan indicated that he possessed a four variable generalization. Our generalization has seven variables and is, perhaps, all one can expect from this method.

1. Introduction

One of the most intriguing results from classical $q$-series is the Rogers-Ramanujan continued fraction [1, p. 440]:

\[
\prod_{n=0}^{\infty} \frac{(1 - q^{5n+2})(1 - q^{5n+3})}{(1 - q^{5n+1})(1 - q^{5n+4})} = 1 + \frac{q}{1 + \frac{q^2}{1 + \frac{q^3}{1 + \cdots}}},
\]

where $|q| < 1$ throughout this paper.

The proof of (1.1) relies fundamentally on the following. Let

\[
G(z, q) = G(z) = 1 + \sum_{n=1}^{\infty} \frac{z^n q^{n^2}}{(1 - q)(1 - q^2) \cdots (1 - q^n)}.
\]

Then

\[
G(z) = \prod_{n=1}^{\infty} (1 - z q^n)^{-1}
\]

\[
\times \left(1 + \sum_{n=1}^{\infty} \frac{(1 - z q)(1 - z q^2) \cdots (1 - z q^{n-1})(-1)^n z^{2n} q^{n(5n-1)/2} (1 - z q^{2n})}{(1 - q)(1 - q^2) \cdots (1 - q^n)}\right),
\]

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and

\[
\frac{G(z)}{G(zq)} = 1 + \frac{zq}{1 + \frac{zq^2}{1 + \frac{zq^3}{\ddots}}}.
\]

G. N. Watson [7] gave the following five-parameter generalization of (1.3) which is perhaps the most natural nonmultiple series generalization:

\[
\begin{align*}
&\phi(q,qz^2,qz^4,qz^6,qz^8,qz^{10},qz^{12},qz^{14},qz^{16},qz^{18},qz^{20}); \left(\frac{zq^n}{a_1a_2\cdots a_5}\right)_{n}
\end{align*}
\]

\[
(1.5)
\]

where \(a_5 = q^{-N}\) with \(N\) a nonnegative integer, and

\[
(a;q)_N = (a)_N = (1-a)(1-aq)\cdots(1-aq^{N-1}),
\]

(1.6)

\[
l_{r+1}(a_0, a_1, \cdots, a_r; q, t; b_1, \cdots, b_r) = \sum_{n=0}^{\infty} \left(\frac{a_0}{a_1}\right)_{n} \cdots \left(\frac{a_r}{a_{r+1}}\right)_{n} \frac{t^n}{(q)_n(b_1)_n \cdots (b_r)_n}.
\]

(1.7)

We remark that the expressions in (1.5) are most naturally viewed as functions of \(\alpha_1', \alpha_2', \alpha_3', \alpha_4', \alpha_5'\) and are, in fact, continuous in each \(\frac{1}{\alpha_i}\) around zero, i.e. \(\beta_i = \infty\). Consequently, when we set any of \(\alpha_1, \ldots, \alpha_5 = \infty\) throughout this paper we shall be doing so in light of the above comments. We could, of course, have begun with each \(\alpha_i\) replaced by its reciprocal; however this would not be consistent with standard notation [5, pp. 4, 35].

When \(\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \infty\) (i.e. \(\alpha_5 = \infty\) means \(\lim_{N \to \infty} q^{-N}\)), (1.5) reduces term-by-term to (1.3). In [1, p. 434], it is noted that the case \(\alpha_3 = \alpha_4 = \alpha_5 = \infty\) of (1.5) is quite possibly the function used in a general continued fraction identity alluded to by Ramanujan [6, pp. xxviii]. One is naturally led to ask: Can one extend (1.4) wherein \(G(z)\) is replaced by the very well-poised \(\phi\) of (1.5)?

We answer this question affirmatively in Theorem 1. It is to be emphasized that Theorem 1 does not require any of \(\alpha_1, \alpha_2, \ldots, \alpha_5\) to be of the form \(q^{-N}\). Consequently, our result holds for the nonterminating very well-poised \(\phi\) first considered by W. N. Bailey in [4] (cf. [5, p. 42, eq. (2.10.10)]).

2. \(q\)-HYPERGEOMETRIC BACKGROUND

Our work is based on the \(q\)-difference equations for very well-poised basic hypergeometric series studied in [1] (cf. [2]). Thus we require many of the auxiliary functions defined there.

\[
C_{k,i}(a_1, a_2, \ldots, a_k; x; q) = C_{k,i}(a_1; x; q)_{k}\]

\[
= \sum_{n \geq 0} (-1)^{n(\lambda+1)} x^{kn} (a_1a_2\cdots a_k)^{-n} q^{(2k-\lambda+1)n^2+(\lambda+1)n-2in)/2}
\]

\[
(1 - x+q^{2n}) \frac{(x)_n(a_1)_n(a_2)_n\cdots(a_k)_n}{(1-x)q_{a_1}_n(a_2)_n\cdots(a_k)_n};
\]

(2.1)
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\[ H_{k,i}(a_1, a_2, \ldots, a_k; x, q) = H_{k,0}(a_1; x, q) \cdot C_{k,i}(a_1; x, q). \]

(2.2)

We note that [1, p. 434]

\[ C_{k,0}(a_1; x, q) = 2^{k+1} \phi_{2k+3} \left( \frac{x}{a_1}, \frac{x}{a_2}, \ldots, \frac{x}{a_{2k+1}} \right). \]

(2.3)

The crucial relations among these functions are the following [1, p. 435, Theorem 1]:

\[ H_{k,i}(a_1; x, q) - H_{k,i-1}(a_1; x, q) = (-1)^i \sigma_i \left( \frac{1}{a_1}, \ldots, \frac{1}{a_k} \right) x^{i-1} H_{k,0}(a_1; x, q), \]

where \( \sigma_i(y_1, \ldots, y_k) \) is the \( i \)-th elementary symmetric function of \( y_1, \ldots, y_k \);

\[ H_{k,0}(a_1; x, q) = -x^{-i} H_{k,i}(a_1; x, q); \]

(2.5)

\[ H_{k,0}(a_1; x, q) = 0. \]

(2.6)

Henceforward we are only interested in the case \( k = 2, \lambda = 5 \). Given these values of \( k \) and \( \lambda \), we abbreviate

\[ H_i(x) = H_{2,i}(a_1, a_2, a_3, a_4, a_5; x, q) \]

(2.7)

and

\[ \sigma_i = \sigma_i \left( \frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \frac{1}{a_4}, \frac{1}{a_5} \right). \]

(2.8)

The four instances of (2.4) with \( i = 2, 1, 0, -1 \) reduce to the following once we use (2.5) and (2.6) for simplification:

\[ H_2(x) - H_1(x) - \sigma_3 x^2 q H_2(xq) + x^2 q \sigma_4 H_3(xq) - x^2 q \sigma_5 H_4(xq) = (x - x^2 q \sigma_2) H_1(xq); \]

(2.9)

\[ H_1(x) - (1 - x^2 q^2 \sigma_4) H_2(xq) - x^2 q^2 \sigma_5 H_3(xq) = (-x q \sigma_1 + x^2 q^2 \sigma_3) H_1(xq); \]

(2.10)

\[ H_1(x) + (x q \sigma_1 - x^3 q^3 \sigma_5) H_2(xq) - H_3(xq) = (x^2 q^2 \sigma_2 - x^3 q^3 \sigma_4) H_1(xq); \]

(2.11)

\[ H_2(x) - x H_1(xq) - x^2 q \sigma_2 H_2(xq) + x q \sigma_1 H_3(xq) - H_4(xq) = (-x^3 q^3 \sigma_3 + x^4 q^4 \sigma_5) H_1(xq). \]

(2.12)

These four functional equations will be used in Section 3 to obtain the relevant second order \( q \)-difference equation for \( H_{2,1}(a_1, a_2, a_3, a_4, a_5; x, q) \).

3. The main results

Our next step consists of replacing \( x \) by \( xq \) in each of (2.9)–(2.12). We rewrite the resulting equations so that \( H_1(xq) \) and \( H_1(xq^2) \) appear on the right.
of each equation.

\begin{align}
    (3.1) \quad & H_2(xq) - \sigma_3 x^2 q^3 H_2(xq^2) + x^2 q^3 \sigma_4 H_3(xq^2) - x^2 q^3 \sigma_5 H_4(xq^2) \\
    & = H_1(xq) + (xq - x^2 q^3 \sigma_2) H_1(xq^2) ; \\
    (3.2) \quad & -(1 - x^2 q^4 \sigma_4) H_2(xq^2) - x^2 q^4 \sigma_5 H_3(xq^2) \\
    & = (-x^2 q^2 \sigma_1 + x^2 q^4 \sigma_3) H_1(xq^2) - H_1(xq) ; \\
    (3.3) \quad & (xq^2 \sigma_1 - x^3 q^6 \sigma_5) H_2(xq^2) - H_3(xq^2) \\
    & = (x^2 q^4 \sigma_2 - x^3 q^6 \sigma_4) H_1(xq^2) - H_1(xq) ; \\
    (3.4) \quad & H_2(xq) - x^2 q^4 \sigma_2 H_2(xq^2) + xq^2 H_3(xq^2) - H_4(xq^2) \\
    & = (-x^3 q^6 \sigma_3 + x^4 q^8 \sigma_5) H_1(xq^2).
\end{align}

Equations (2.9)-(2.12), (3.1)-(3.4) constitute a system of eight linear equations in the eight unknowns: \( H_1(x) \), \( H_2(x) \), \( H_2(xq) \), \( H_2(xq^2) \), \( H_3(xq) \), \( H_3(xq^2) \), \( H_4(xq) \), \( H_4(xq^2) \). The system turns out to be nonsingular, and consequently Cramer's Rule assisted by MACSYMA allows us to solve for \( H_1(x) \) which upon inspection reveals a linear relation among \( H_1(x) \), \( H_1(xq) \) and \( H_1(xq^2) \). To make this relation most succinct we introduce

\begin{align}
    p(x) &= p(x; \sigma_1, \sigma_4, \sigma_5, q) \\
    &= 1 - x^2 q^2 \sigma_4 + x^3 q^3 \sigma_1 \sigma_5 - x^5 q^5 \sigma_2.
\end{align}

Using this notation, we find that after simplification and division by \((-1 + x^2 q \sigma_5)\)

\begin{align}
    (3.6) \quad & Q(x) H_1(x) = P(x) H_1(xq) + R(x) H_1(xq^2), \\
    \text{where} \quad & Q(x) = (1 - x^2 q^2 \sigma_5)(1 - x^2 q^3 \sigma_5)p(xq), \\
    (3.7) \quad & P(x) = -xq(1 - x^2 q^3 \sigma_5)p(xq)(\sigma_1 - xq \sigma_3 + x^3 q^3 \sigma_2 \sigma_5 - x^4 q^4 \sigma_4 \sigma_5) \\
    & - p(x) \left\{ (-1 - xq \sigma_1 + x^3 q^4 \sigma_5)p(xq) \\
    & - x^6 q^{11} \sigma_2 \sigma_5 + x^5 q^9 \sigma_1 \sigma_4 \sigma_5 - x^5 q^9 \sigma_4 \sigma_5 + x^4 q^7 \sigma_3 \sigma_5 \\
    & + x^4 q^7 \sigma_2 \sigma_5 - x^4 q^7 \sigma_2^2 - x^3 q^5 \sigma_1 \sigma_5 + x^2 q^3 \sigma_4 - x^3 q^3 \sigma_3 + xq \sigma_1 \right\}, \\
    (3.8) \quad & R(x) = xq p(x) \prod_{1 \leq i < j \leq 5} \left( 1 - \frac{xq^2}{a_i a_j} \right).
\end{align}

\textbf{Theorem 1.}

\begin{align}
    (3.10) \quad & \frac{H_1(x)}{H_1(xq)} = \frac{P(x)}{Q(x)} + \frac{R(x)/Q(x)}{P(xq)/Q(xq)} + \frac{R(xq)/Q(xq)}{P(xq^2)/Q(xq^2)} + \frac{R(xq^2)/Q(xq^2)}{P(xq^3)/Q(xq^3)} + \ldots.
\end{align}
Proof. This assertion is merely the iteration of a restatement of (3.6) written in the form

\[
\frac{H_1(x)}{H_1(xq)} = \frac{P(x)}{Q(x)} + \frac{R(x)/Q(x)}{H_1(xq)/Q(xq^2)}.
\]

Convergence is guaranteed by the fact that \(P(x), Q(x)\) and \(R(x)\) are polynomials in \(x\), \(Q(0) = 1\), and \(R(x)\) has no constant term. \(\square\)

Related contiguous relations and continued fractions may be derived from (3.6). For example, if we put \(a_5 = q^{-N}\) and let \(N \to \infty\) in (1.5) we find

\[
H_1(a_1, a_2, a_3, a_4; x; q)
\]

(3.12)

\[
= \left( \frac{xq}{a_1} \right)_\infty \left( \frac{xq}{a_2} \right)_\infty \left( \frac{xq}{a_3a_4} \right)_\infty \phi_2\left( \frac{xq}{a_1}, \frac{xq}{a_2}, \frac{xq}{a_3a_4}; a_3, a_4, xq \right).
\]

Making the change of variables

\[
x \mapsto \frac{de}{aq}, \quad a_1 \mapsto \frac{e}{a}, \quad a_2 \mapsto \frac{d}{a}, \quad a_3 \mapsto b, \quad a_4 \mapsto c,
\]

cancelling infinite products and simplifying the resulting polynomials give the following contiguous relation for a \(\phi_2\):

\[
S \phi_2\left( \frac{a}{d}, \frac{b}{e}, \frac{c}{f}; \frac{de}{abc} \right) = T \phi_2\left( \frac{aq}{dq}, \frac{bq}{eq}, \frac{cq}{eq}; \frac{de}{abc} \right)
\]

(3.13)

\[
+ U \phi_2\left( \frac{aq^2}{dq^2}, \frac{bq^2}{eq^2}, \frac{cq^2}{eq^2}; \frac{de}{abc} \right),
\]

where the polynomials \(S, T\) and \(U\) are

\[
S = bc(d)(e)(abc - de)(bc - deq^2),
\]

\[
T = (1 - dq)(1 - eq)(bc - deq)[bcde(a(b + c) + d + e)(1 + q) - ((b + c)de
\]

\[
+ abc(d + e))(bc + deq) + a(bc - de)(bc - deq^2)],
\]

\[
\]

Applying the transformation (3.2.7) of [5] term-by-term to the contiguous relation and simplifying the products yield

\[
S' \phi_2\left( \frac{a}{d}, \frac{b}{e}, \frac{c}{f}; \frac{de}{abc} \right) = T' \phi_2\left( \frac{aq}{dq}, \frac{bq}{eq}, \frac{cq}{eq}; \frac{de}{abc} \right)
\]

(3.14)

\[
= U' \phi_2\left( \frac{aq^2}{dq^2}, \frac{bq^2}{eq^2}, \frac{cq^2}{eq^2}; \frac{de}{abc} \right),
\]

where now

\[
S' = a^2b^2c^2(d)(e)(1 - eq^2),
\]

\[
T' = abc(1 - dq)(eq)(de(ab + ac + bc + e)(1 + q)
\]

\[
- d(abc + (a + b + c)e)(1 + eq)
\]

\[
+ abc(1 - e)(1 - eq^2)),
\]

\[
\]
Iterating these two contiguous relations gives

\[
3\phi_2 \left( \frac{a, b, c}{d, e}; \frac{de}{abc} \right) = \frac{T(a, d, e)}{S(a, d, e)} + \frac{U(a, d, e)}{T(aq, dq, eq)/S(aq, dq, eq) + \ldots}
\]

and

\[
3\phi_2 \left( \frac{a, b, c}{d, q}; \frac{de}{abc} \right) = \frac{T'(a, b, c, d, e)}{S'(a, b, c, d, e)} + \frac{U'(a, b, c, d, e)}{S'(aq, bq, cq, dq, eq^2)/S'(aq, bq, cq, dq, eq^2) + \ldots}
\]

For both continued fractions convergence follows from the fact that after canceling common factors of powers of \( q \) from the partial numerators and denominators, the partial numerators tend to zero, while the partial denominators do not.

Obviously, although Theorem 1 is explicit, it nonetheless lacks the elegance of (1.1) or (1.4). If we let \( a_4 \) and \( a_5 \) tend to infinity (so that \( a_4 = a_5 = 0 \)) and denote the resulting \( P(x) \), \( Q(x) \) and \( R(x) \) by \( P(x) \), \( Q(x) \) and \( R(x) \) respectively, then we see that

\[
Q(x) = 1,
\]
\[
R(x) = xq \left( 1 - \frac{xq^2}{a_1a_2} \right) \left( 1 - \frac{xq^2}{a_1a_3} \right) \left( 1 - \frac{xq^2}{a_2a_3} \right),
\]
\[
P(x) = 1 - \frac{xq}{a_1} - \frac{xq}{a_2} - \frac{xq}{a_3} + \frac{x^2q^2}{a_1a_2a_3} (1 + q).
\]

Thus we obtain

**Corollary 1.**

\[
\frac{H_{2,1}(a_1, a_2, a_3; x; q)}{H_{2,1}(a_1, a_2, a_3; q; q)} = \frac{P(x)}{P(xq) + \frac{R(x)}{P(xq) + \frac{R(xq)}{P(xq^2) + \ldots}}}.
\]

Several remarks are now in order. First it is clear from (3.17)–(3.20) that (1.4) is the case \( a_1 = a_2 = a_3 = a_4 = a_5 = +\infty \). Also the form for the \( H_{k,i} \) given in (2.2) will yield the right-hand side of (1.3) and not the right-hand side of (1.2). However, for completeness, we note we may deduce from (2.2), (2.3) and (1.5) that

\[
H_{2,1}(a_1, a_2, a_3; x; q) = \left( \frac{xq}{a_1a_3} \right)_\infty \left( \frac{xq}{a_2} \right)_\infty \sum_{n=0}^\infty \frac{(a_1)_n(a_3)_n}{(a_1a_3)_n} \left( \frac{xq}{a_1a_3} \right)_n.
\]
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and this last expression reduces to the right-hand side of (1.2) as \( a_1, a_2 \) and \( a_3 \to \infty \).

To conclude we examine the simplest case involving three finite parameters. We replace \( q \) by \( q^4 \) and then set \( a_1 = q, a_2 = q^2, a_3 = q^3 \) in Corollary 1. By (3.21) we see that

\[
\lim_{x \to 1^-} H_{2,1}(q, q^2, q^3; x; q^4) = \lim_{x \to 1^-} (x; q^4) \frac{(q, q^2)_{2n} x^n}{(q^4; q^4)_{n} (x q^2; q^4)_{n}} = (q; q^2)_{\infty},
\]

and if we define

\[
f(q) = H_{2,1}(q, q^2, q^3; q^4, q^4),
\]

then by (3.21) after simplification

\[
f(q) = \frac{(q^2; q^4)_{\infty}}{q(1-q)} \sum_{n=0}^{\infty} \frac{(q; q^2)_{2n+1} q^{2n+1}}{(q^4; q^4)_{n} (q^2; q^2)_{n+1}}
\]

and

\[
f(-q) = \frac{-(q^2; q^4)_{\infty}}{2q(1+q)} (-q)_{\infty} \sum_{n=1}^{\infty} (-1)^n q^{n^2}
\]

Hence by [3, p. 23, eq. (2.2.12)]

\[
f(-q) = \frac{-(q^2; q^4)_{\infty}}{2q(1+q)} (-q)_{\infty} 2 \sum_{n=1}^{\infty} (-1)^n q^{n^2} = (-q^3; q^2)_{\infty} \sum_{n=1}^{\infty} (-1)^n q^{n^2-1},
\]

and so

\[
H_{2,1}(q, q^2, q^3; q^4, q^4) = (q^3; q^2)_{\infty} \sum_{n=0}^{\infty} q^{n^2+2n}.
\]

Consequently Corollary 1 reduces to

\[
(3.22) \quad \frac{1-q}{\sum_{n=0}^{\infty} q^{n^2+2n}} = 1 - q - q^3 + q^6 + \frac{q^4(1-q^3)(1-q^4)(1-q^5)}{1-q^5-q^6-q^7+q^{10}+q^{14}+ \frac{q^8(1-q^7)(1-q^8)(1-q^9)}{1-q^9-q^{10}-q^{11}+q^{14}+q^{15}+}}.
\]
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