THE GENERAL LOCAL FORM OF AN ANALYTIC MAPPING INTO THE SET OF IDEMPOTENT ELEMENTS OF A BANACH ALGEBRA

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Dedicated to the memory of Professor Jacques Morgenstern

Abstract. This paper gives a general formula which describes any analytical mapping of a suitably small open neighborhood \(U\) in \(\mathbb{C}\) into the set of idempotent elements of any complex Banach algebra \(B\) and an application of this formula to the case when \(B\) is a Calkin algebra.

Let \(B\) be a complex Banach algebra, for any \(X, Y \in B\) let \([X, Y] = XY - YX\), let \(U\) be an open subset of \(\mathbb{C}\) (which we can assume without loss of generality to contain 0), let \(F(z)\) be an analytic mapping of \(U\) into \(B\) and let \(P \in B\) be idempotent \((P \neq 0)\). Then the main result of this paper is the following:

**Theorem 1.** If \([F(0), P] = 0\) there exists an open neighborhood \(V\) of 0 in \(\mathbb{C}\) and two analytic mappings \(P(z), R(z)\) of \(V\) in \(B\) such that:

(i) \(P(0) = P\) and for all \(z \in V\), \(P(z)\) is idempotent,
(ii) for all \(z \in V\), \([R(z), P] = 0\),
(iii) for all \(z \in V\), \(F(z) = P(z) + R(z)\).

Moreover, in a small enough neighborhood of 0, the pair of mappings \(P(z), R(z)\) is uniquely determined by properties (i) to (iii).

**Definition 2.** Let \(z \in \mathbb{C}\), with \(0 < |z| < 1\) and set \(f(z) = \frac{1}{z} - \frac{1}{z^2} - \cdots\). If \(n = 1, 2, \ldots\) define \(c_n\) (the \(n\)th Catalan number, cf. [1], [3]) as follows:

\[
c_n = \frac{f^{(n)}(0)}{n!}
\]

Clearly:

\[
c_n = \frac{(2n - 2)!}{n!(n - 1)!} = \frac{(2n - 2)!}{[2n - (2n - 1)]n!(n - 1)!} = \frac{(2n - 2)!}{n!(n - 1)!(n - 1)!} = 2\frac{(2n - 2)!}{n!(n - 1)!(n - 1)!} \in \mathbb{N}.
\]

**Lemma 3** (see [3] for a slightly weaker version of this lemma). Using the notation introduced above we have:

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(i) The series \( \sum_{j=1}^{\infty} c_j z^j \) is uniformly convergent to \( f(z) \) in \( 0 < |z| \leq \frac{1}{4} \).

(ii) \( \forall n \in \mathbb{N}^*, \quad c_{n+1} = \sum_{j=1}^{n} c_j c_{n+1-j} \).

Proof. (i) Let us show first that

\[
\sum_{j=1}^{\infty} c_j \left( \frac{1}{4} \right)^j \leq \frac{1}{2}.
\]

If \( n = 1, 2, \ldots \) set \( S_n = \sum_{j=1}^{n} c_j (\frac{1}{4})^j \). Then:

\[
S_{n+1} + \left( \frac{1}{4} \right)^{(n+1)} \frac{(2n+1)!}{n!(n+1)!} = S_n + \left( \frac{1}{4} \right)^{(n+1)} \frac{(2n)!}{n!(n+1)!} + \frac{(2n+1)!}{n!(n+1)!} = S_n + \left( \frac{1}{4} \right)^n \frac{(2n-1)!}{n!(n-1)!}.
\]

Hence \( S_n + \left( \frac{1}{4} \right)^n \frac{(2n-1)!}{n!(n-1)!} \) is independent of \( n \) and since for \( n = 1 \) its value is \( \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \), (1) follows at once, and this implies that for \( 0 \leq |z| < \frac{1}{4} \) the function represented by the series is analytic and coincides with \( f(z) \). By continuity this still holds for \( |z| = \frac{1}{4} \) so that (i) is established and (ii) is easily derived from the fact that \( (f(z))^2 = \frac{1}{4} + \frac{1}{4} - z - 2\left(\frac{1}{2}\right)\sqrt\frac{1}{4} - z = f(z) - z \).

Lemma 4. Let \( A \in \mathcal{B} \). Set \( A_1 = [A, P] \), \( A_2 = [A_1, P] \). Then:

(a) \( [A_2, P] = A_1 \).
(b) \( PA_1 + A_1 P = A_1 \).
(c) \( PA_2 + A_2 P = A_2 \).
(d) \( P \) commutes with \( A_1^2 \), with \( A_2^2 \) and with \( A_1A_2 \).
(e) \( A_1A_2 + A_2A_1 = 0 \).
(f) \( A_1^2 + A_2^2 = 0 \).

Proof. (a), (b) and (c) are established by direct computation from the fact that \( A_1 = AP - PA \) and that \( A_2 = AP - PA - 2PAP \).

(d) is an obvious consequence of (b) and (c).

To prove (e) notice that \( A_1A_2 + A_2A_1 = A_1(A_1P - PA_1) + (A_1P - PA_1)A_1 = A_1^2P - PA_1^2 = 0 \), using (d). Finally (f) follows from the fact that \( A_1^2 + A_2^2 = A_1^2 + (A_1P - PA_1)^2 = A_1^2 + A_1PA_1P + PA_1PA_1 - A_1PA_1 - PA_1^2P = A_1^2 - (I - P)A_1^2 - PA_1^2 = 0 \).

Lemma 5. Let \( A \in \mathcal{B} \) be such that \( \|A_1\| \leq 1/(4\|P\|) \). Then

\[
Q = P + A_2 + \sum_{j=1}^{\infty} c_j A_1 A_2^{2j-1} \text{ is an idempotent element of } \mathcal{B}.
\]

Proof. Under the given hypothesis \( \|A_2\| \leq 2\|A_1\|\|P\| \leq \frac{1}{2} \). Hence using
Lemma 3, the series that defines $Q$ is uniformly convergent. Furthermore:

\[
Q^2 = P + PA_2 + \sum_{j=1}^{\infty} c_j PA_1 A_2^{j-1} + A_2 P + A_2^2 + \sum_{j=1}^{\infty} c_j A_2 A_1 A_2^{2j-1}
\]

\[
+ \sum_{j=1}^{\infty} c_j A_2 A_1 A_2^{j-1} P + \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} c_j c_k A_1 A_2^{j-1} A_1 A_2^{k-1}
\]

\[
= P + A_2 + 2 \sum_{j=1}^{\infty} c_j PA_1 A_2^{j-1} + A_2^2
\]

\[
+ \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} c_j c_k A_1 A_2^{j-1} A_1 A_2^{k-1} \quad \text{(using (b), (c) and (e))}
\]

\[
= P + A_2 + 2PA_1 \sum_{j=1}^{\infty} c_j A_2^{j-1} + A_2^2 - \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} c_{i+k} A_1 A_2^{j-1} A_1 A_2^{k-1}
\]

\[
= P + A_2 + (A_1 - A_2) \sum_{j=1}^{\infty} c_j A_2^{j-1} + A_2^2
\]

\[
+ \sum_{i=1}^{\infty} c_{i+1} A_2^{i+1} \quad \text{(using (f) and Lemma 3)}
\]

\[
= P + A_2 + \sum_{j=1}^{\infty} c_j A_1 A_2^{2j-1} = Q.
\]

We are now in a position to prove Theorem 1.

**Existence of the decomposition.** Let $F(z)$ be analytic mapping of $U$ into $B$ such that $[F(0), P] = 0$. Then there exists a neighborhood $V \subseteq U$ such that $\forall z \in V, ||[F(z), P]|| < 1/(4||P||)$ and therefore the mapping of $V$ into $B$ defined by

\[
P(z) = P + [[F(z), P], P] + \sum_{j=1}^{\infty} c_j [[F(z), P], [F(z), P], P]^{2j-1}
\]

is analytic on $V$. Note that $P(0) = P$ and that $\forall z \in V, P(z)$ is idempotent. Then, if $R(z) = F(z) - P(z)$,

\[
[R(z), P] = [F(z), P] - [P(z), P]
\]

\[
= [F(z), P] - [F(z), P]
\]

\[
+ \sum_{j=1}^{\infty} c_j [[F(z), P], [F(z), P], P]^{2j-1} P
\]

\[
- P[F(z), P][[F(z), P], P]^{2j-1}
\]

\[
= 0
\]

using Lemma 4. Hence the existence of the decomposition is proved.

**Uniqueness of the decomposition.** Assume that $F(z) = P_1(z) + R_1(z) = P_2(z) + R_2(z)$ with $P_1(0) = P_2(0) = P$. Set $D(z) = P_1(z) - P_2(z) = R_1(z) - R_2(z)$.
Then $D(z)$ commutes with $P$ and $D(z) = P^2_1(z) - P^2_2(z) = D(z)P_1(z) + P_2(z)D(z)$. Let us show by induction that

\[(3) \quad \forall n \in \mathbb{N}, \quad D^{(n)}(0) = 0.\]

For $n = 0$ we see that $D(0) = P_1(0) - P_2(0) = 0$. Assume that (3) has been proved for $n = 0, 1, \ldots, r$. Then using Leibniz's formula we get

\[
D^{(r+1)}(0) = \sum_{j=0}^{r+1} \binom{r+1}{j} \{D^{(r+1-j)}(0)P_j^1(0) + P_j^2(0)D^{(r+1-j)}(0)\}
\]

\[
= D^{(r+1)}(0)P + PD^{(r+1)}(0),
\]

using the induction hypothesis. So $D^{(r+1)}(0) = 2PD^{(r+1)}(0)$, whence $PD^{(r+1)}(0) = 2PD^{(r+1)}(0)$ so that $PD^{(r+1)}(0) = 0$, therefore $D^{(r+1)}(0) = 0$ and (3) is proved. Hence $D(z) = 0$, which concludes the proof of the theorem.

Remark. (2) gives the general local form of any analytic mapping of a neighborhood of 0 into the set of idempotent elements of $B$. Indeed suppose that $P$ is such a mapping defined on a neighborhood $U$ of 0 into $B$ and set $P = P(0)$. Then $P(0)P = PP(0)$ and by Theorem 1 there exists a neighborhood $V$ of \{0\} such that

\[
Q(z) = P + \sum_{j=1}^{\infty} c_j [P(z), P][[P(z), P], P]^{2j-1}
\]

is an analytical mapping of a neighborhood $V$ of 0 into the set of the idempotent elements of $B$ and we have

\[
P(z) = Q(z) + R(z) = P(z) + 0
\]

and since the decomposition is unique, it follows that $Q(z) = P(z)$ and consequently that $P(z)$ is locally of the form given by (2).

Application. Let $H$ be a complex Hilbert space, and let $L(H)$ denote the algebra of all bounded linear operators on $H$ and $K(H)$ the closed bilateral ideal constituted by the compact operators of $L(H)$. Then $A(H) = L(H)/K(H)$ is the Calkin algebra associated to $L(H)$. Let $\pi$ denote the natural mapping of $L(H)$ onto $A(H)$.

Theorem 2. Let $U$ be a neighborhood of 0 in $\mathbb{C}$, and let $p(z)$ be an analytic mapping of $U$ into the space of the idempotent elements of $A(H)$. Then there exists a neighborhood $V$ of 0 in $\mathbb{C}$ and an analytic mapping $P(z)$ of $V$ into the set of the idempotent elements of $L(H)$ such that

\[
\forall z \in V, \quad p(z) = \pi(P(z)).
\]

Proof. By assumption $\forall z \in U, p(z) = \sum_{j=0}^{\infty} p_j z^j$ where $p_0$ is idempotent.

According to [2], Proposition 7, there exists $F_0 \in L(H)$, idempotent, such that $\pi(F_0) = p_0$. Furthermore for every $j \geq 1$ there exists $F_j \in L(H)$ such that $\pi(F_j) = p_j$ with $\|F_j\| \leq 2\|p_j\|$ (this is an obvious consequence of the definition of the norm in $A(H)$). Hence $F(z) = \sum_{j=0}^{\infty} F_j z^j$ converges and is analytic in some neighborhood $V$ of 0 and we have $\forall z \in V, \pi(F(z)) = p(z)$. But $F(0) = F_0$. So using Theorem 1, taking $P = F_0$, $F(z) = P(z) +
$R(z)$, and therefore $p(z) = \pi(P(z)) + \pi(R(z))$ where $\pi(P(z))$ is idempotent with $\pi(P(0)) = p_0$ and $\pi(R(z))$ commutes with $p_0$. Hence, because of the uniqueness of the decomposition, $\pi(R(z)) = 0$ and we have shown that there exists an idempotent-valued mapping $P(z)$, analytic on $V$, such that

$$\forall z \in V, \quad p(z) = \pi(P(z)).$$

**References**


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