SOLUTION OF THE BAIRE ORDER PROBLEM OF MAULDIN

MAREK BALCERZAK AND DOROTA ROGOWSKA

(Communicated by C. D. Sogge)

Abstract. Let $X$ be an uncountable Polish space, and let $I$ be a proper $\sigma$-ideal of subsets of $X$ such that $\{x\} \in I$ for each $x \in X$. Denote by $B_\alpha(I)$, $\alpha \leq \omega_1$, the Baire system generated by the family of functions $f : X \to \mathbb{R}$ continuous $I$ almost everywhere. We prove that if $r(I) = \min\{\alpha \leq \omega_1 : B_{\alpha+1}(I) = B_\alpha(I)\}$, then either $r(I) = 1$ or $r(I) = \omega_1$. This answers the problem raised by R. D. Mauldin in 1973.

1. Introduction

Let $X$ be an uncountable separable and complete metric space (briefly called Polish), and let $I$ be a $\sigma$-ideal of subsets of $X$ such that $X \not\in I$. Denote by $C_I$ the family of all functions $f : X \to \mathbb{R}$ whose sets of points of discontinuity are in $I$. Then put $B_0(I) = C_I$ and for each ordinal $\alpha > 0$ define $B_\alpha(I)$ as the family of all pointwise limits of sequences of functions from $\bigcup_{\gamma<\alpha} B_\gamma(I)$.

It is easy to check that the Baire system $B_\alpha(I)$, $\alpha \leq \omega_1$, has the following properties:

- $B_{\omega_1}(I)$ is closed under pointwise limits, i.e. $B_{\omega_1+1}(I) = B_{\omega_1}(I)$,
- for $I = \{\emptyset\}$ we have the classical Baire system (denoted by $B_\alpha$, $\alpha \leq \omega_1$).

Now we define $r(I) = \min\{\alpha \leq \omega_1 : B_{\alpha+1}(I) = B_\alpha(I)\}$ which is called the Baire order of $C_I$.

The following results are known:

1. If $I = \{\emptyset\}$, then $r(I) = \omega_1$ (see [L]).
2. If $I$ is the $\sigma$-ideal of the first category sets in $X$, then $r(I) = 1$ and $B_1(I)$ consists of all functions with the Baire property (see [K1]).
3. If $I$ is the $\sigma$-ideal of sets of Lebesgue measure zero in $[0, 1]$, then $r(I) = \omega_1$ (see [M2]; for some generalizations compare [M3], [B1]).

In [M2] Mauldin posed the following problem: If $0 < \alpha < \omega_1$, is there a $\sigma$-ideal $I_\alpha$ of the first category subsets of $[0, 1]$ which contains all $F_\alpha$ sets of Lebesgue measure 0 such that the family of all functions which are continuous except for a set in this $\sigma$-ideal $I_\alpha$ has Baire order $\alpha$?

Note that in the above question, $\sigma$-ideals are required to contain all singletons $\{x\}$; a $\sigma$-ideal which has that property is called uniform. In the main theorem
we will show that, for each uniform $\sigma$-ideal $I$ of subsets of $X$, we have either $r(I) = 1$ or $r(I) = \omega_1$. It solves Mauldin’s problem in the negative. Observe that always $r(I) > 0$ since the characteristic function of a countable set dense in $X$ belongs to $B_1(I) \setminus B_0(I)$ (cf. [B2]).

Denote by $\mathcal{B}$ the family of all Borel subsets of $X$, and by $\Sigma_0^0$, $\Pi_0^0$ (for $0 < \alpha < \omega_1$) the subclasses of $\mathcal{B}$ defined as in [Mo, 1 F]. In particular, $\Sigma_2^0$ is the family of all $F_\sigma$ sets in $X$.

A $\sigma$-ideal $I$ is called $\Sigma_2^0$ supported if each $A \in I$ is contained in some $B \in I \cap \Sigma_2^0$. For a $\sigma$-ideal $I$, we define

$$I^* = \{A \subset X : (\exists B \in I \cap \Sigma_2^0)(A \subset B)\}.$$

Obviously, $I^*$ is a $\Sigma_2^0$ supported $\sigma$-ideal, and if $I$ is a $\Sigma_2^0$ supported $\sigma$-ideal, then $I = I^*$. Since the set of discontinuity points of an arbitrary function is of type $F_\sigma$, we have $C_1 = C_1^*$ for each $\sigma$-ideal $I$, and thus the Baire order problem may be restricted to $\Sigma_2^0$ supported $\sigma$-ideals.

2. Auxiliary facts

In this section we will assume that $I$ is a uniform $\sigma$-ideal of subsets of $X$. If $\mathcal{F}$ is a family of subsets of $X$, then define $MGR(\mathcal{F})$ as the family of all subsets $B$ of $X$ such that for each $A \in \mathcal{F}$ the set $B \cap A$ is of the first category.

The following deep result plays a key role in the proof of our Main Theorem.

**Proposition 1** [KS, Theorem 2]. Let $I$ be a $\Sigma_2^0$ supported $\sigma$-ideal. Then precisely one of the following possibilities holds:

(i) $I = MGR(\mathcal{F})$ for a countable family $\mathcal{F}$ of closed subsets of $X$ (moreover, it may be assumed that $\mathcal{F} = \{F_\gamma : \gamma < \alpha\}$ where $\alpha < \omega_1$ and $F_\gamma \subset F_\beta$ for $\beta < \gamma < \alpha$, and $F_{\gamma+1}$ is nowhere dense in $F_\gamma$ for $\gamma < \alpha$);

(ii) there exists a homeomorphic embedding $\varphi : 2^\omega \times \omega^\omega \rightarrow X$ such that $\varphi[\{t\} \times \omega^\omega] \notin I$ for each $t \in 2^\omega$.

**Proposition 2.** If a $\sigma$-ideal $I$ satisfies condition (ii) of Proposition 1, then $I$ has the following property:

(M) There exists a Borel function $f : X \rightarrow X$ such that $f^{-1}[\{x\}] \notin I$ for each $x \in X$.

**Proof.** Denote $B = \varphi[2^\omega \times \omega^\omega]$ and consider a continuous function $\psi = p_1 \circ \varphi^{-1} : B \rightarrow 2^\omega$, where $p_1$ is a projection onto the first factor. For each $t \in 2^\omega$ we have

$$\psi^{-1}[\{t\}] = \varphi[p_1^{-1}[\{t\}]] = \varphi[\{t\} \times \omega^\omega] \notin I.$$  

Since $B$ is a Borel set (even of type $G_\delta$, see [K, §35, III]), we can extend $\psi$ to a Borel function $g : X \rightarrow 2^\omega$ and we get $g^{-1}[\{t\}] \notin I$ for each $t \in 2^\omega$. Let $h : 2^\omega \rightarrow X$ be a Borel isomorphism (see [K, §37, II]). Then $f = h \circ g : X \rightarrow X$ is a Borel function and for each $x \in X$ we have

$$f^{-1}[\{x\}] = g^{-1}[h^{-1}[\{x\}]] \notin I.$$  

Define $R(I) = \min\{\alpha \leq \omega_1 : (\forall B \in \mathcal{B})(\exists A \in \Sigma_0^0)(B \Delta A \in I)\}$ where $\Sigma_0^\omega = \mathcal{B}$ and $B \Delta A = (B \setminus A) \cup (A \setminus B)$. Observe that $\Sigma_2^0$ can be replaced by $\Pi_0^\omega$ in the above definition.
Proposition 3 [B3, Corollary 2.2]. If a σ-ideal \( I \) has the property (M), then \( R(I) = \omega_1 \).

Proposition 4 [M2, Theorem 3]. For every σ-ideal \( I \) and each \( \alpha, 0 < \alpha \leq \omega_1 \), we have \( f \in B_\alpha(I) \) if and only if there is \( g \in B_\alpha \) such that \( \{x \in X : f(x) \neq g(x)\} \in I^* \).

Proposition 5 [B2]. If \( I \) is a \( \Sigma^0_2 \) supported σ-ideal and \( R(I) = \omega_1 \), then \( r(I) = \omega_1 \).

Proof. Suppose to the contrary that \( r(I) = \alpha < \omega_1 \), and consider an arbitrary set \( E \in \mathcal{B} \). Then Proposition 4 yields that for each Borel function \( f : X \to \mathbb{R} \) there is a function \( g \in B_\alpha \) satisfying \( \{x \in X : f(x) \neq g(x)\} \in I \). In particular, consider such a function \( g \) for \( f = \chi_E \). Put \( A = g^{-1}([1]) \). Then \( A \in \Pi^0_{\alpha+1} \) (cf. [K, §31, IX]) and \( E \Delta A \subset \{x \in X : f(x) \neq g(x)\} \in I \). Hence \( R(I) \leq \alpha + 1 < \omega_1 \), which gives a contradiction. \( \square \)

3. Main Theorem

Theorem. If \( X \) is an uncountable Polish space and \( I \) is an uniform σ-ideal, then either \( r(I) = 1 \) or \( r(I) = \omega_1 \).

Proof. According to our general remarks in Introduction, we may assume that \( I \) is \( \Sigma^0_2 \) supported and, by Proposition 1, we consider Cases (i) and (ii).

Case (i). Assume that \( \mathcal{F} = \{F_\gamma : \gamma < \alpha\} \) for \( \alpha < \omega_1 \) and that the remaining conditions stated in (i) are fulfilled. Let \( S \) denote the set of all functions \( f : X \to \mathbb{R} \) such that \( f|F_\gamma \) has the Baire property for each \( \gamma < \alpha \). Then \( C_I \subset S \) and \( S \) is closed with respect to pointwise limits. Hence \( B_\gamma(I) \subset S \) for each \( \gamma < \omega_1 \). Thus it suffices to show that \( S \subset B_1(I) \). Let \( f \in S \). By virtue of Kuratowski's result [K1] and Proposition 4, there exist functions \( g_\gamma : F_\gamma \to \mathbb{R}, \gamma < \alpha, \) of the Baire class 1, such that the set \( \{x \in F_\gamma : (f|F_\gamma)(x) \neq g_\gamma(x)\} \) is of the first category in \( F_\gamma \). Define \( g : X \to \mathbb{R} \) by \( g(x) = g_\gamma(x) \) for \( x \in F_\gamma \setminus F_{\gamma+1}, \gamma < \alpha, \) and

\[
g(x) = 0 \quad \text{for } x \in (X \setminus F_0) \cup \bigcup_{\lambda < \alpha, \lambda \text{ limit } \gamma < \lambda} \left( \bigcap_{\lambda} F_\gamma \setminus F_\lambda \right).
\]

Since

\[
\{x \in X : f(x) \neq g(x)\} \cap F_\gamma \subset \{x \in F_\gamma \setminus F_{\gamma+1} : (f|F_\gamma)(x) \neq g_\gamma(x)\} \cup F_{\gamma+1},
\]

therefore \( \{x \in X : f(x) \neq g(x)\} \cap F_\gamma \) is of the first category in \( F_\gamma \). Thus

\[
\{x \in X : f(x) \neq g(x)\} \in MGR(\mathcal{F}) = I.
\]

The function \( g \) is of the Baire class 1 since for each open subset \( U \subset \mathbb{R} \) we have \( g^{-1}[U] \in \Sigma^0_2 \). Indeed,

\[
g^{-1}[U] = \bigcup_{\gamma < \alpha} (g_\gamma^{-1}[U] \setminus F_{\gamma+1}) \quad \text{if } 0 \notin U,
\]

and

\[
(X \setminus F_0) \cup \bigcup_{\lambda < \alpha, \lambda \text{ limit } \gamma < \lambda} \left( \bigcap_{\lambda} F_\gamma \setminus F_\lambda \right).
\]
must be added to the right side of the last equality, if \( 0 \in U \). Consequently, \( f \in B_1(I) \) by Proposition 4.

Case (ii). Use Propositions 2, 3, and 5. \( \square \)

Remark. At this moment we do not know which values between 1 and \( \omega_1 \) can be achieved by \( r(I) \) when \( I \) is not uniform. That question for non-uniform principal \( \sigma \)-ideals \( I_A = \{ E \subseteq X : E \subseteq X \setminus A \} \) where \( A \subseteq X \) is uncountable was considered in [M3]. Note that \( r(I_A) = \omega_1 \) if \( A \) contains a perfect set (see [M3, Theorem 6]). So, it would be interesting to examine the case when \( A \) is uncountable and does not contain perfect sets.

References