COHOMOLOGY RING OF THE ORBIT SPACE
OF CERTAIN FREE $Z_p$-ACTIONS

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Abstract. In this paper, we consider actions of $G = Z_p$ (with $p$ an odd prime) on spaces $X$ which are of cohomology type $(0, 0)$ (i.e., have the mod-$p$ cohomology of the one-point union of an $n$-sphere, a $2n$-sphere and a $3n$-sphere, $n$ odd). If $X$ is not totally non-homologous to zero in $X_G$ we determine the fixed set, give examples of all possibilities for the fixed set and compute the cohomology ring structure of the orbit space in the case where $G$ acts freely. In [4], we considered fixed sets for related spaces, when $X$ is totally non-homologous to zero in $X_G$.

1. Introduction

Let $X$ be a finite CW complex with cohomology groups satisfying:

$$H^j(X; Z) = \begin{cases} Z, & j = 0, n, 2n, 3n, \\ 0, & \text{otherwise.} \end{cases}$$

If $u_i$ generates $H^{in}(X; Z)$, $i = 1, 2, 3$, we say that $X$ has cohomology type $(a, b)$ when $u_1^a = au_2$ and $u_1u_2 = bu_3$ (terminology due to Toda [7]). Let $G = Z_p$ ($p$ an odd prime) act on $X$. If $b \not\equiv 0 \pmod{p}$, then either $X \cong_p S^n \times S^{2n}$ or $X \cong_p P^3(n)$ depending on whether $a \equiv 0 \pmod{p}$ or $a \not\equiv 0 \pmod{p}$. Here $X \cong_p Y$ means that $X$ and $Y$ have isomorphic mod-$p$ cohomology rings. When $b \not\equiv 0 \pmod{p}$ the nature of the fixed set of $G$ on $X$ has been studied in detail ([5], [6], [8]). In [4] we considered the case $b \equiv 0 \pmod{p}$, when $X$ is totally non-homologous to zero in $X_G$ (mod $p$). The structure of the possible fixed sets was determined, and it was noted that when $n$ is even, $X$ is always totally non-homologous to zero. Here we settle the remaining case where $X$ is not totally non-homologous to zero (mod $p$), so that $n$ is necessarily odd. Since $n$ is odd, we must have $a \equiv 0 \pmod{p}$ and $X$ is of cohomology type $(0, 0)$. We obtain

**Theorem 1.** Let $G = Z_p$, $p$ an odd prime, act on a finite complex $X$ of cohomology type $(0, 0)$ (mod $p$). If $X$ is not totally non-homologous to zero in $X_G$, then the fixed point set $F \cong_p S^q$, $-1 \leq q \leq 3n$, $q$ odd. Moreover, all possibilities for $q$ occur.

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Examples of $G$ acting freely on spaces $X$ of type $(0, 0)$ were constructed in [4]. Here we compute the cohomology of the orbit space of a free $G$ action on $X$ obtaining

**Theorem 2.** Let $X$ be a space of cohomology type $(0, 0)$ with nonzero mod-$p$ cohomology only in dimensions $0$, $n$, $2n$, $3n$ ($n$ odd). Suppose $G = Z_p$, $p$ odd prime, acts freely on $X$. Then as graded commutative algebras,

$$H^*(X/G; Z_p) = Z_p[x, y, z]/(x^2, z^2, zy^{n+1/2}, y^{3n+1/2})$$

where $\deg x = 1$, $\deg y = 2$, $\deg z = n$ and $y = \beta_p(x)$ ($\beta_p$ being the mod-$p$ Bockstein).

## 2. Preliminaries

We will recall here several facts about equivariant cohomology $H^*_G(X) = H^*(X_G)$ (see [2, Chapter 7] for more information).

First of all we will denote cohomology with $Z_p$ coefficients simply by $H^*(X)$ and from now on $Z_p$ coefficients are intended (unless explicitly indicated otherwise). If $G = Z_p$ acts on $X$, $F$ denotes the fixed set. It is well known that if $H^*(X) = 0$ for $* > m$, then the inclusion $F_G \to X_G$ induces a cohomology isomorphism $H^*(X_G) \to H^*(F_G)$ for $* > m$. Recall that $X_G = EG \times_G X$ is the bundle over the classifying space $BG$ with fibre $X$ associated to the principal bundle $EG \to BG$, $F_G = F \times BG$ is a subbundle.

If $X$ is a finite $G$-CW-complex and $G$ acts trivially on $H^*(X; Z)$ (integer coefficients), then for any $k$,

$$\sum_{i \geq 0} \text{rk} H^{k+2i}(F) \leq \sum_{i \geq 0} \text{rk} H^{k+2i}(X).$$

For example, if $H^*(X)$ vanishes in even degrees, so does $H^*(F)$.

We recall that if $\sum \text{rk} H^*(X) < \infty$ (as in the present case), then $X$ is totally non-homologous to zero in $X_G$ (i.e., there is a cohomology extension of the fibre $X \to X_G$ with $Z_p$-coefficients iff $\sum \text{rk} H^*(F) = \sum \text{rk} H^*(X)$ iff $G$ acts trivially on $H^*(X)$ and the Leray-Serre spectral sequence of $X_G \to X$ degenerates.

In computing the cohomology of the orbit space $X/G$, where $G$ acts freely on $X$, we make use of the Leray-Serre spectral sequence of $X_G \to BG$ (in this case the map of orbit spaces $X_G \to X/G$ is a homotopy equivalence, so the cohomology of $X_G$ obtained from this spectral sequence will be the cohomology of $X/G$). If $\pi_1(BG) = Z_p$ acts trivially on $H^*(X)$ (as it does in the present case), then the $E_2$ term is $E_2^{k, l} = H^k(BG) \otimes H^l(X)$. The product structure in the spectral sequence induces a product in the subalgebras $E_2^{*, 0}$ and $E_2^{0, *}$ which coincides with the cup products. Also the edge homomorphisms,

$$H^k(BG) = E_2^{k, 0} \to E_3^{k, 0} \to \cdots \to E_{k+1}^{k, 0} = E_\infty^{k, 0} \subseteq H^k(X_G),$$

$$H^l(X_G) \to E_\infty^{0, l} = E_{l+1}^{0, l} \subset \cdots \subset E_2^{0, l} = H^l(X)$$

are the homomorphisms $\pi^*: H^k(BG) \to H^k(X_G)$ and $i^*: H^l(X_G) \to H^l(X)$, respectively.
Finally recall that
\[ H^*(BG) = \mathbb{Z}_p[s, t]/(s^2) = \Lambda(s) \otimes \mathbb{Z}_p[t] \]
where \( \deg s = 1 \), \( \deg t = 2 \) and \( \beta_p(s) = t \) (\( \beta_p \) is the mod-\( p \) Bockstein associated to \( \mathbb{Z}_p \to \mathbb{Z}_{p^2} \to \mathbb{Z}_p \)).

3. Proof of Theorem 1 and examples

To prove Theorem 1, suppose that \( X \) is not totally non-homologous to zero in \( X_G \pmod{p} \). Then we must have \( n \) odd (by [4]) and \( \text{rk} \, H^*(F) < \text{rk} \, H^*(X) = 4 \). Also \( \chi(F) \equiv \chi(X) \equiv 0 \pmod{p} \). So \( \chi(F) = 0 \) or \( \chi(F) = 3 \) and \( p = 3 \). But \( \chi(F) = 3 \) requires \( \sum \text{rk} \, H^2(F) > 2 \) and, since \( G \) acts trivially on \( H^*(X; \mathbb{Z}) \) (integer coefficients), we must have
\[ \sum \text{rk} \, H^2(F) \leq \sum \text{rk} \, H^2(X) = 2. \]
Therefore \( \chi(F) = 0 \) and \( F \simeq_p S^q \) for \( q \) odd and \( -1 \leq q \leq 3n \). \( \Box \)

We now will give examples to show that all the possibilities for \( q \) are actually realised. The case of \( q = -1 \) (i.e., \( F = \emptyset \)) is [4]. So assume that \( q \) is odd \( 1 \leq q \). To begin with, let \( n \geq 3 \) be odd and \( m = n + 2 \) (so \( m \) is odd). There is a \( \mathbb{Z}_p \) action on \( S^2 \times S^m \) with fixed set \( S^3 \). One can obtain such an action by letting \( \eta \) be the Hopf 2-plane bundle over \( S^2 \), \(-\eta \) its inverse (i.e., \( \eta \oplus -\eta \) is a trivial 4-plane bundle). Let \( \varepsilon \) be a trivial \( m - 3 \) plane bundle. \(-\eta \oplus \varepsilon \) admits a fibre-wise orthogonal action of \( \mathbb{Z}_p \) which leaves only the zero section (i.e., \( S^2 \)) fixed. Consider \( \eta \oplus (-\eta \oplus \varepsilon) \). Let \( \mathbb{Z}_p \) act trivially on \( \eta \). Taking unit sphere bundles yields an action of \( \mathbb{Z}_p \) on \( S^2 \times S^m \) (\( m = n + 2 \)) with fixed set the total space of the sphere bundle of \( \eta \) (i.e., \( S^3 \)) (see [3] for other such examples). Now remove a fixed point to obtain a \( \mathbb{Z}_p \) action on a space \( Y \) which is homotopy equivalent to \( S^2 \vee S^{n+2} \) and has contractible fixed set. Let \( Z_p \) act trivially on \( S^{n-3} \), and take the join of \( S^{n-3} \) and \( Y \). This space \( W \) has \( Z_p \) action with contractible fixed set and is itself homotopy equivalent to \( S^n \vee S^{2n} \). Now let \( Z_p \) act on \( S^{3n} \) with fixed set \( S^q \) for \( q \) odd (e.g., take a linear action) and form the one-point union (at a fixed point) of \( W \) and \( S^{3n} \). This provides all examples.

4. Proof of Theorem 2

By the Universal Coefficient Theorem, we have \( H^i(X) = \mathbb{Z}_p \) for \( i = 0, 1, 2, 3 \). We choose generators \( v_i \in H^i(X) \), \( i = 1, 2, 3 \), respectively, satisfying the relations \( v_1^2 = 0 \) and \( v_1v_2 = 0 \). Consider the Leray-Serre spectral sequence of the map \( \pi: X_G \to BG \) with coefficients in the constant sheaf \( \mathcal{F}_*(X) \) associated to \( G = \mathbb{Z}_p \) (\( G = \mathbb{Z}_p = \pi_1(BG) \) acts trivially on \( H^*(X) \)). The \( E_2 \)-term of the spectral sequence is
\[ E_2^{k, l} = H^k(BG) \otimes H^l(X). \]
Since \( X \) has no fixed points and \( p \) is odd, \( n \) must be odd and \( E_2 \neq E_\infty \). So some differential:
\[ d_r: E_r^{k, l} \to E_r^{k+r,l-r+1} \]
must be nontrivial. This is only possible for \( r = n + 1, 2n + 1 \) and \( 3n + 1 \), and it is easily seen that
\[ d_{n+1}(1 \otimes v_1) = d_{n+1}(1 \otimes v_3) = 0 \quad \text{and} \quad d_{n+1}(1 \otimes v_2) \neq 0. \]
So

\[ E_r^{k,2n} = 0 = E_r^{k+n+1,n} \]

for all \( k \) and \( r > n + 1 \). Obviously then, \( d_{2n+1} = 0 \).

If \( d_{3n+1}(1 \otimes v_3) = 0 \), then the bottom and top lines of the spectral sequence survive to infinity and this contradicts the fact that \( H^*(X_G) = 0 \) for \( * > 3n \) (for \( G \) acts freely, hence \( F_G = \emptyset \)). Therefore \( d_{3n+1}(1 \otimes v_3) \neq 0 \) so that

\[ E_{\infty}^{k,3n} = 0 = E_{\infty}^{k+3n+1,0}. \]

Hence we obtain

\[
H^j(X_G) = \begin{cases} 
Z_p & \text{for } j > 3n, \\
Z_p & \text{for } 0 \leq j \leq n - 1 \text{ and } 2n + 1 \leq j \leq 3n, \\
Z_p \otimes Z_p & \text{for } n \leq j \leq 2n.
\end{cases}
\]

To determine the multiplicative structure, note that for \( k \leq 3n \), \( E_2^{k,0} \subset H^k(X_G) \). Let \( x = s \otimes 1 \in E_\infty^{1,0} \) and \( y = t \otimes 1 \in E_\infty^{2,0} \). Then \( \pi^*(s) = x \), \( \pi^*(t) = y \) and \( \beta_p(x) = y \), by naturality of the Bockstein cohomology operation. The homomorphism

\[
y \cup (-): E_\infty^{k,l} \rightarrow E_\infty^{k+2,l}
\]

is an isomorphism for \( k \leq 3n - 2 \) if \( l = 0 \) and for \( k \leq n - 2 \) if \( l = n \). Therefore multiplication by \( y \in H^2(X_G) \)

\[
y \cup (-): H^k(X_G) \rightarrow H^{k+2}(X_G)
\]

is an isomorphism for \( k \leq 2(n - 1) \). The element \( 1 \otimes v_1 \in E_2^{0,n} \) is a permanent cocycle and determines an element \( w \) in \( E_\infty^{0,n} \cdot E_\infty^*, * \), and \( E_\infty^*, * \) are bigraded commutative algebras and therefore the total complex \( \operatorname{Tot} E_\infty^*, * \) given by

\[
\left( \operatorname{Tot} E_\infty^*, * \right)^m = \bigoplus_{k+l=m} E_\infty^{k,l}
\]

is a graded commutative algebra isomorphic to

\[
Z_p[x, y, w]/(x^2, w^2, wy^{(n+1)/2}, y^{(3n+1)/2}).
\]

Now we choose an element \( z \in H^n(X_G) \) such that \( i^*(z) = v_1 \). Because the composition \( pi \) factors through a point, \( i^* \pi^* \) is zero in positive degrees and hence we can assume that \( zy^{(n+1)/2} = 0 \). Since the multiplication by \( y \) is an isomorphism in degrees less than \( 2(n - 1) \), \( zy^i \neq 0 \) for \( 2i < n - 1 \). Thus we have

\[
H^*(X_G) = Z_p[x, y, z]/(x^2, z^2, zy^{(n+1)/2}, y^{3(n+1)/2})
\]

as graded commutative algebras. \( \pi: X_G \rightarrow X/G \) is a homotopy equivalence and so induces a cohomology isomorphism. This completes the proof. \( \square \)

**References**


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