ON SOME IDEALS OF NEST ALGEBRAS

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Abstract. The purpose of this paper is to characterize the relations of some ideals of a nest algebra such as the Jacobson radical $R_J$, Larson strong radical $R^\infty_J$, and ideal $\mathcal{F}$, and to describe all the ideals between $R_J$ and $\mathcal{F}$.

1. Preliminaries

Let $\mathcal{N}$ be a nest (i.e., a totally ordered complete set of projections containing $O$ and $I$ which is closed in the strong operator topology) on a separable Hilbert space $H$. We denote by $\text{Alg}\mathcal{N}$ the algebra of all operators in $B(H)$ that leave invariant every element of $\mathcal{N}$. A nest $\mathcal{N}$ is continuous if its core, which is the von Neumann algebra generated by the elements of $\mathcal{N}$, is a nonatomic von Neumann algebra. $K(H)$ denotes the set of all the compact operators in $B(H)$. We shall call a set $\{P_\alpha|\alpha \in \Lambda\}$ of intervals $P_\alpha = M_\alpha - N_\alpha$, $M_\alpha, N_\alpha \in \mathcal{N}$, $N_\alpha < M_\alpha$, of the nest $\mathcal{N}$, a partition of $\mathcal{N}$ if the intervals are pairwise orthogonal and the sum $\sum_{\alpha \in \Lambda} P_\alpha = I$ in the strong topology. Since $H$ is a separable Hilbert space, then every partition is denumerable. Given a nest $\mathcal{N}$, Larson strong radical $R^\infty_J$ is the collection of all operators $X$ in $\text{Alg}\mathcal{N}$ for which, given $\varepsilon > 0$, there exists a partition $\{P_\alpha|\alpha \in \Lambda\}$ of $\mathcal{N}$ such that $\|P_\alpha XP_\alpha\| < \varepsilon$ for all $\alpha \in \Lambda$. If we restrict all partitions to be finite sets of intervals, we will obtain the Jacobson radical $R_J$ of $\text{Alg}\mathcal{N}$ [4]. It is clear that $R^\infty_J$ always contain $R_J$. The ideals $R_J, R^\infty_J$ have been studied by some authors ([3], [4]). In this paper, we will study the ideal $\mathcal{F}$ of $\text{Alg}\mathcal{N}$ and its relations with $R_J$ and $R^\infty_J$. Terms and notation not defined here are taken from [4].

Definition 1. If $E_1, E_2$ are nonzero orthogonal intervals from $\mathcal{N}$ such that $E_1B(H)E_2 \subseteq \text{Alg}\mathcal{N}$, we say $E_1, E_2$ are strictly ordered and write $E_1 \ll E_2$.

A mutually orthogonal family of intervals is said to be strictly ordered if it is linearly ordered by the relation $\ll$. The length of such a family is its cardinality.
**Definition 2.** If \( A \in \text{Alg}_\mathcal{N} \) and \( \varepsilon > 0 \), we define the \( \varepsilon \)-order of \( A \) to be the number

\[
R_\varepsilon(A) = \sup\{n| \text{ there exists a strictly ordered family } \mathcal{F} \text{ of length } n \text{ with } \|EAE\| \geq \varepsilon \text{ for all } E \in \mathcal{F}\}.
\]

**Definition 3.** Let

- \( \mathcal{G} = \{A \in \text{Alg}_\mathcal{N}|R_\varepsilon(A) < +\infty, \text{ for every } \varepsilon > 0\} \),
- \( \mathcal{G}_u = \{A \in \text{Alg}_\mathcal{N}|\sup_{\varepsilon>0} R_\varepsilon(A) < +\infty\} \),
- \( R_\varepsilon^\alpha = \{A \in \text{Alg}_\mathcal{N}|EAE = 0 \text{ for every atom } E \text{ of } \mathcal{N}\} \).

It is easy to verify that \( \mathcal{G} \), \( \mathcal{G}_u \), \( R_\varepsilon^\alpha \) are ideals of \( \text{Alg}_\mathcal{N} \). \( \square \)

### 2. Main results

In the paper [1], it is shown that \( \mathcal{G} \) is a norm-closed ideal of \( \text{Alg}_\mathcal{N} \) which contains \( R_\mathcal{N} \). In this section, we will study the relations of \( \mathcal{G} \), \( \mathcal{G}_u \), \( R_\mathcal{N} \), \( R_\mathcal{N}^\alpha \), and \( R_\mathcal{N}^\alpha \).

Let \( x \otimes y \) denote the rank-one operator acting on \( H \) such that \( x \otimes y(z) = \langle z, x \rangle y \), for all \( z \in \mathcal{H} \).

**Theorem 1.** \( \mathcal{G}_u \) is closed in the norm topology if and only if \( \mathcal{N} \) is a finite nest.

**Proof.** If \( \mathcal{N} \) is finite, then \( \mathcal{G}_u = \mathcal{G} = \text{Alg}_\mathcal{N} \). Thus \( \mathcal{G}_u \) is norm-closed.

Now suppose \( \mathcal{G}_u \) is a norm-closed ideal; we claim that \( \mathcal{N} \) is finite. Otherwise, there exists \( P \in \mathcal{N} \) such that \( P = P_- (P \neq 0) \) or \( P = P_+ (P \neq 1) \). We only need to consider the case: \( P = P_- (P \neq 0) \). Choose \( \{P_n\} \), \( P_n \in \mathcal{N} \), such that \( P_n < P_{n+1} \), \( P = \lim_{n \to -\infty} P_n \) in the strong operator topology. Let \( E_n = P_{n+1} - P_n \), and \( x_n \in E_n \) such that \( ||x_n|| = 1 \), \( n = 1, 2, \ldots \).

Let \( T_n = \sum_{k=1}^{n} \frac{1}{k!} x_{k+1} \otimes x_k \), and \( T = \sum_{k=1}^{\infty} \frac{1}{k!} x_{k+1} \otimes x_k \); then \( \lim_{n \to -\infty} T_n = T \) in the norm topology and \( T_n \in \mathcal{G}_u \). Now we prove that \( T \notin \mathcal{G}_u \). For arbitrary positive integer \( M \), there exists \( \varepsilon > 0 \) such that \( \varepsilon < \frac{1}{(2M-1)!} \). Let \( F_k = P_{2k+1} - P_{2k-1} \) \( (k = 1, 2, \ldots, M) \), thereby, \( \|F_k T F_k\| = \frac{1}{(2k-1)! \geq \frac{1}{(2M-1)!}} > \varepsilon \) \( (k = 1, 2, \ldots, M) \), and \( R_\varepsilon(T) \geq M \). Since \( M \) is arbitrary, we have \( \sup_{\varepsilon>0} R_\varepsilon(T) = +\infty \). Thus \( T \notin \mathcal{G}_u \), which contradicts the closedness of \( \mathcal{G}_u \).

**Theorem 2.** Let \( \mathcal{N} \) be a nest. Then

1. \( R_\mathcal{N}^\alpha \cap \mathcal{G} = R_\mathcal{N}^\alpha \cap \mathcal{G} = R_\mathcal{N} \).
2. \( \mathcal{G} = R_\mathcal{N} \) if and only if \( \mathcal{N} \) is continuous.
3. \( \mathcal{G} \subset R_\mathcal{N}^\alpha \) if and only if \( \mathcal{N} \) is continuous.
4. \( \mathcal{G} \not\subset R_\mathcal{N}^\alpha \) if and only if \( \mathcal{N} \) is finite.

For arbitrary nest \( \mathcal{N} \), \( \mathcal{G} \neq R_\mathcal{N}^\alpha \).

**Proof.** (1) The chain \( R_\mathcal{N} \subset R_\mathcal{N}^\alpha \cap \mathcal{G} \subset R_\mathcal{N}^\alpha \cap \mathcal{G} \) is evident. It remains to show that each \( X \in R_\mathcal{N}^\alpha \cap \mathcal{G} \) belongs to \( R_\mathcal{N} \). Apply Ringrose's Criterion, using the seminorms \( i_\mathcal{N}^X(X) \) and \( i_\mathcal{N}^X(N) \) (see [4]). Let \( \varepsilon > 0 \) and \( N \in \mathcal{N} \). By symmetry it suffices to show that \( i_\mathcal{N}^X(N) \leq \varepsilon \). If \( N^+ > N \), then \( N^+ - N \) is an atom and

\[
i_\mathcal{N}^X(N) = ||(N^+ - N)X(N^+ - N)|| = 0.
\]

Otherwise there is a sequence \( N_n \) decreasing to \( N \) in \( \mathcal{N} \). Suppose that \( i_\mathcal{N}^X(N) > \varepsilon \). For each fixed \( k \), \( (N_k - N_n)X(N_k - N_n) \) converges strongly to
(N_k - N)x(N_k - N)\), which is greater than \(\varepsilon\) in norm. So, by the strong lower semicontinuity of the norm there is a \(k'\) such that
\[
\|(N_k - N_{k'})x(N_k - N_{k'})\| > \varepsilon.
\]
Passing to a subsequence, we can assume
\[
\|(N_k - N_{k+1})x(N_k - N_{k+1})\| > \varepsilon
\]
for all \(k\). But this means that \(R_\varepsilon(X) = +\infty\), which is a contradiction.

(2) It is easy to prove that \(E \in \mathcal{F}, E \notin R_\varepsilon^c\), for every atom \(E\). By (1), \(\mathcal{F} = R_N\) if and only if \(\mathcal{F} \subseteq R_\varepsilon^c\) if and only if \(\mathcal{N}\) is continuous.

For (3), by (1), we can prove them easily.

**Proposition 1.** Let \(\mathcal{N}\) be a nest. Then

(1) \(R_N \subseteq \mathcal{F}_u\), where the closure is in the norm topology.

(2) \(R_N \subseteq \mathcal{F}_u\) if and only if \(\mathcal{N}\) is finite.

(3) \(\mathcal{F}_u \subseteq R_N\) if and only if \(\mathcal{N}\) is continuous.

For arbitrary nest \(\mathcal{N}\), \(R_N \neq \mathcal{F}_u\).

**Proof.** (1) Since \(R_N = \text{span}\{PAP^\perp | P \in \mathcal{N}, A \in \text{Alg}\mathcal{N}\}\), where the closure is in the norm topology, and \(PAP^\perp \in \mathcal{F}_u\), for every \(P \in \mathcal{N}, A \in \text{Alg}\mathcal{N}\), (1) holds.

(2) If \(\mathcal{N}\) is finite, then \(R_N \subseteq \mathcal{F}_u = \text{Alg}\mathcal{N}\). If \(\mathcal{N}\) is infinite, we have constructed \(T \in R_N\), but \(T \notin \mathcal{F}_u\) in the proof of Theorem 1. This contradicts \(R_N \subseteq \mathcal{F}_u\).

(3) If \(\mathcal{N}\) is continuous, from Theorem 2 (2), \(\mathcal{F} = R_N \supset \mathcal{F}_u\). If \(\mathcal{N}\) is not continuous, then there exists an atom \(E\). Let \(x, y \in E\) such that \(\|x\| = \|y\| = 1\); then \(x \otimes y \notin R_N\), \(x \otimes y \in \mathcal{F}_u\). Thus \(\mathcal{F}_u \notin R_N\).

In the paper [2], Lance gave a standard form for all ideals containing the radical of a nest algebra, but not in a very explicit manner. In the last theorem of this paper, we will, by the index function of atoms, characterize all the norm-closed ideals between \(R_N\) and \(\mathcal{F}\) in an explicit way.

Let \(\{E_\alpha | \alpha \in \Lambda\}\) be all the atoms of \(\mathcal{N}\). Let
\[
\text{ind}(\alpha) = \begin{cases} 
0 & \text{if dim} E_\alpha < \infty, \\
1 & \text{if dim} E_\alpha = \infty
\end{cases}
\]
denote the index function of \(\{E_\alpha | \alpha \in \Lambda\}\).

Let \(\mathcal{F} = \{\varphi | \varphi : \Lambda \to \{0, 1, 2\} \text{ such that } \varphi(\alpha) \leq \text{ind}(\alpha) + 1\}\). For \(\varphi \in \mathcal{F}\), \(\alpha \in \Lambda\), let
\[
M_{\varphi(\alpha)} = \begin{cases} 
0 & \text{if } \varphi(\alpha) = 0, \\
E_\alpha K(H)E_\alpha & \text{if } \varphi(\alpha) = 1, \\
E_\alpha B(H)E_\alpha & \text{if } \varphi(\alpha) = 2.
\end{cases}
\]

**Definition 4.** For \(\varphi \in \mathcal{F}\), let
\[
\mathcal{D}_\varphi = \left\{ T \left| T = \sum_{\alpha \in \Lambda} T_\alpha, T_\alpha \in M_{\varphi(\alpha)} \right. \right\},
\]
where \(\lim \|T_\alpha\| = 0\) means: for every \(\varepsilon > 0\), set \(\{\alpha \in \Lambda | \|T_\alpha\| \geq \varepsilon\}\) is finite, and let
\[
\mathcal{E}_\varphi = R_N \oplus \mathcal{D}_\varphi.
\]
In the next theorem, we will identify $\mathcal{F}$ with $R_{\mathcal{N}} \oplus D_0$, where $D_0$ is the set of operators which are norm convergent sums of the form $\sum E_i T E_i$ (with $E_i$ an enumeration of the atoms). Clearly, $D_0 = D_{\phi_0}$, where

$$\phi_0(\alpha) = \begin{cases} 1 & \text{if ind}(\alpha) = 0, \\ 2 & \text{if ind}(\alpha) = 1. \end{cases}$$

**Theorem 3.** Let $\mathcal{N}$ be a nest. Then

(1) If $I$ is a closed two-sided ideal of $\Alg_{\mathcal{N}}$ in the norm topology and $R_{\mathcal{N}} \subseteq I \subseteq \mathcal{F}$, then there exists $\varphi \in \mathcal{F}$ such that $I = \mathcal{F}_\varphi$. In particular, $\mathcal{F} = R_{\mathcal{N}} \oplus D_0$.

(2) For every $\varphi \in \mathcal{F}$, $\mathcal{F}_\varphi$ is a norm-closed two-sided ideal of $\Alg_{\mathcal{N}}$, and $R_{\mathcal{N}} \subseteq \mathcal{F}_\varphi \subseteq \mathcal{F}$.

(3) $\mathcal{F}_\varphi = \mathcal{F}_{\varphi'}$ if and only if $\varphi = \varphi'$.

**Proof.** (1) For every $\alpha \in \Lambda$, $E_\alpha I E_\alpha$ is an ideal of $E_\alpha(\Alg_{\mathcal{N}})E_\alpha$, which is closed in the norm topology. If $\text{ind}(\alpha) = 0$, then $E_\alpha I E_\alpha = 0$ or $E_\alpha I E_\alpha = E_\alpha B(H)E_\alpha = E_\alpha K(H)E_\alpha$. If $\text{ind}(\alpha) = 1$, then $E_\alpha I E_\alpha = 0$ or $E_\alpha I E_\alpha = E_\alpha K(H)E_\alpha$ or $E_\alpha I E_\alpha = E_\alpha B(H)E_\alpha \neq E_\alpha K(H)E_\alpha$. Let

$$\varphi(\alpha) = \begin{cases} 0 & \text{if } E_\alpha I E_\alpha = 0, \\ 1 & \text{if } E_\alpha I E_\alpha = E_\alpha K(H)E_\alpha, \\ 2 & \text{if } E_\alpha I E_\alpha = E_\alpha B(H)E_\alpha \neq E_\alpha K(H)E_\alpha. \end{cases}$$

Then $\varphi \in \mathcal{F}$ and $M_{\varphi(\alpha)} \subseteq I$. By the definition of $\mathcal{F}_\varphi$, we have $\mathcal{F}_\varphi \subseteq I$. On the other hand, for $T \in I$, let $S = T - \sum_{\alpha \in \Lambda} T_\alpha$, where $T_\alpha = E_\alpha T E_\alpha$, $\alpha \in \Lambda$. Since $T \in \mathcal{F}$, we have $\sum_{\alpha \in \Lambda} T_\alpha \in \mathcal{F}$ and $\lim \|T_\alpha\| = 0$. Therefore $S \in \mathcal{F}$. Furthermore $E_\alpha S E_\alpha = 0$, for $\alpha \in \Lambda$; then $S \in R_{\mathcal{N}}$ and so $S \in R_{\mathcal{N}} \cap \mathcal{F} = R_{\mathcal{N}}$. Thus $T = S + \sum_{\alpha \in \Lambda} T_\alpha \in \mathcal{F}_\varphi$. Since $T$ is arbitrary, we have $I \subseteq \mathcal{F}_\varphi$, which implies that $I = \mathcal{F}_\varphi$.

(2) Letting $\varphi \in \mathcal{F}$, by the definition of $\mathcal{F}_\varphi$, we know that $R_{\mathcal{N}} \subseteq \mathcal{F}_\varphi \subseteq \mathcal{F}$. Let $I$ be the norm-closed two-sided ideal of $\Alg_{\mathcal{N}}$ generated by $\mathcal{F}_\varphi$. Then $R_{\mathcal{N}} \subseteq I \subseteq \mathcal{F}$. By the proof of (1), we have $I = \mathcal{F}_\varphi$. Thus $\mathcal{F}_\varphi$ is a two-sided ideal of $\Alg_{\mathcal{N}}$, which is closed in the norm topology.

(3) If $\varphi \neq \varphi'$, then there exists $\alpha \in \Lambda$ such that $\varphi(\alpha) \neq \varphi'(\alpha)$. Thus $M_{\varphi(\alpha)} \neq M_{\varphi'(\alpha)}$, and therefore, by a trivial argument, $\mathcal{F}_\varphi \neq \mathcal{F}_{\varphi'}$.

The following corollary follows Theorem 3 immediately.

**Corollary 1.** Let $s$ denote the cardinality of set $\{I|I \subseteq \mathcal{F}, I$ is a norm closed two-sided ideal of $\Alg_{\mathcal{N}}\}$. Then

(1) If $\text{ind}(\alpha) = 0$, for every $\alpha \in \Lambda$, then $s = 2^\Lambda$, where $2^\Lambda$ denotes the cardinality of set which consists of all subsets of $\Lambda$.

(2) If $\text{ind}(\alpha) = 1$, for every $\alpha \in \Lambda$, then $s = 3^\Lambda$, where $3^\Lambda$ denotes the cardinality of the set which consists of all maps from $\Lambda$ to $\{0, 1, 2\}$. Particularly, if $\Lambda = n$ (i.e. the cardinality of $\Lambda$) and $\text{ind}(\alpha) = 1$ for every $\alpha \in \Lambda$, then $s = 3^n$.

(3) If $\Lambda = n$, $\Lambda_0 = \{\alpha|\text{ind}(\alpha) = 0\}$, $\Lambda_1 = \{\alpha|\text{ind}(\alpha) = 1\}$, and $\Lambda_0 = m$, $\Lambda_1 = n - m$, then $s = 2^m \cdot 3^{n-m}$.

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