THE TATE CONJECTURE FOR t-MOTIVES

YUICHIRO TAGUCHI

(Communicated by Eric Friedlander)

Abstract. A version of the Tate conjecture is proved for \( \varphi \)-modules of "t-motive type".

In this note, we formulate a version of the Tate conjecture for \( \varphi \)-modules, and give a proof of it in a special but essential case. Similar results have been obtained independently by Tamagawa [3] in a more general setting.

Let \( K \) be an algebraic function field in one variable over a finite field, whose field of constants is \( \mathbb{F}_q \), \( \pi \) a place of \( K \), and \( K_\pi \) the completion of \( K \) at \( \pi \). Let \( k \) be any field containing \( \mathbb{F}_q \). We set \( K_k := k \otimes_{\mathbb{F}_q} K \), and denote by \( K_k, \pi \) the completion of \( K_k \) with respect to the \( \pi \)-adic topology (these may not be fields). Let \( \sigma \) be the endomorphism \((q\text{-th power Frobenius of } k) \otimes (\text{id}_K)\) of \( K_k \), and also its natural extension to \( K_k, \pi \). By a \( \varphi \)-module \((M, \varphi)\) (or simply \( M \)) over \( K_k \) (resp. over \( K_k, \pi \)), we mean a free \( K_k \)-module (resp. \( K_k, \pi \)-module) \( M \) of finite rank equipped with a \( \sigma \)-semi-linear map \( \varphi : M \to M \). Morphisms of \( \varphi \)-modules are defined naturally. Tensor products \( M \otimes N \) (with diagonal \( \varphi \)-action) exist. Internal horns \( \text{Hom}(M, A) \) (with \( \varphi \)-action) may or may not exist. For a \( \varphi \)-module \( M \), let \( M^\varphi \) denote the fixed part of \( M \) by \( \varphi \). This is a \( K \)-subspace (resp. \( K_\pi \)-subspace) of \( M \). If the internal hom \( H = \text{Hom}(M, N) \) exists, then \( H^\varphi \) is the space \( \text{Hom}_\varphi(M, N) \) of \( \varphi \)-module homomorphisms of \( M \) to \( N \). Now the Tate conjecture in our context is

**Conjecture.** Let \( M \) be a \( \varphi \)-module over \( K_k \). If \( k \) is of finite type over \( \mathbb{F}_q \), then the natural map of \( K_\pi \)-vector spaces

\[
K_\pi \otimes_K M^\varphi \to (K_k, \pi \otimes_k M)^\varphi
\]

is an isomorphism.

In general, this is not true (cf. [3]).

Equivalently, by fixing a \( K_k \)-basis of \( M \), the conjecture can be stated also as follows: Let \( A \) be a matrix in \( M_r(K_k) \). Consider the linear Frobenius equation

\[
(*) \quad A X^\sigma = X
\]

Received by the editors April 28, 1994.

1991 Mathematics Subject Classification. Primary 11G09.

Partially supported by JSPS Postdoctoral Fellowships for Research Abroad.

©1995 American Mathematical Society

3285
where the indeterminate $X$ is considered in $K_{k, \pi}^{\otimes r}$, on which $\sigma$ acts component-wise. Let $V$ (resp. $V_\pi$) be the space of solutions of (\(\ast\)) in $K_{k}^{\otimes r}$ (resp. $K_{k, \pi}^{\otimes r}$). Then the natural map of $K_{\pi}$-vector spaces $K_{\pi} \otimes_K V \rightarrow V_\pi$ is an isomorphism. These problems can (and should) be considered also with $K_k$ replaced by certain localization of it.

The injectivity is easy to see, so the essence is in the surjectivity.

A reduction can be made: suppose $K'$ is a subfield of $K$ which contains $\mathbb{F}_q$ and over which $K$ is finite. Let $\pi'$ be the restriction of $\pi$ to $K'$. A $\varphi$-module $M$ over $K_k$ can be regarded as a $\varphi$-module over $K'_k$. If the conjecture is true for $(K', \pi', M)$, then it is also true for $(K, \pi, M)$ (use the identification $K_{\pi'} \otimes_{K'} K = \prod_{\pi | \pi'} K_{\pi}$, etc.). So we may and do assume $K = \mathbb{F}_q(t)$ and identify $\pi$ with a monic irreducible element of $\mathbb{F}_q[t]$. Replacing $K$ again by the subfield $\mathbb{F}_q(\pi)$, we may assume $\pi = t$ (so $K_k$ is the polynomial ring $k[t]$ to which the inverses of all monic polynomials in $\mathbb{F}_q[t]$ have been adjoined, and $K_{k, \pi} = k((t))$).

Now we prove the conjecture assuming that $k$ is a function field in one variable over $\mathbb{F}_q$ (this is not essential) and that $M$ comes from $t$-motives of characteristic different from $\pi$, by which we mean the following (cf. [1]): there exist a non-zero element $\theta$ of $k$ and positive integers $d$ and $d'$ such that the map $\varphi$ is represented with respect to some $K_k$-basis of $M$ by a matrix $A$ of the form

$$A = (t - \theta)^d B^{-1},$$

where $B$ is a matrix in $M_r(k[t])$ with $\det B$ of the form $u(t - \theta)^{d'}$, $u \in k^\times$ (so we allow any $A \in \text{GL}_r(k[t], \frac{1}{t-\theta})$ in (\(\ast\)), which may not be in $M_r(K_k)$).

We will show that, if the equation (\(\ast\)) has a solution $\hat{x}$ in $K_{k, \pi}^{\otimes r}$, then it has a solution $x$ in $K_{k}^{\otimes r}$ which is sufficiently close (in the $t$-adic= $\pi$-adic topology) to $\hat{x}$ (so that, if $(\hat{x}_i)$ is a basis for $V_\pi$, so is $(x_i)$ for $V$). By assumption, we have

$$(\ast\ast) \quad (t - \theta)^d \hat{x}^\sigma = B \hat{x}, \quad \theta \neq 0.$$

Write $(t - \theta)^d = \sum_{i=0}^d \theta^i t^i$ (with $\theta^i \in k$); $B = \sum_{i=0}^N B_it^i$ (with $B_i \in M_r(k)$); and $\hat{x} = \sum_{i \geq 0} x_it^i$ (with $x_i \in k^{\otimes r}$). Then (\(\ast\ast\)) yields

$$(\ast\ast\ast) \quad \theta^0 x_0^\sigma + \cdots + \theta_d x_d^\sigma = B_0 x_0 + \cdots + B_N x_{-N}, \quad i \geq 0.$$

(Here negatively indexed terms are zero.) For any valuation $v$ of $k$, let $v(x_i)$ denote the minimum of the valuations of the entries of $x_i$, and $v(B)$ the minimum of the valuations of the entries of $B_i$ for all $i \geq 0$. If $v(\theta) \leq 0$, then

$$v(x_i) \geq \min\{v(x_{i-1}), \cdots, v(x_{i-d}), \frac{v(x_i) + v(B)}{q}, \cdots, \frac{v(x_{i-N}) + v(B)}{q}\},$$

so we see recursively that $v(x_i) \geq v(B)/(q - 1)$ for all $i \geq 0$. If $v(\theta) > 0$, we replace $X$ in (\(\ast\ast\)) by $\theta^{-e} X$ (resp. $B$ by $\theta^{e(q-1)}B$) for some $e$ to have $v(\theta^{e(q-1)}B) \geq 0$. Then Anderson's arguments\(^1\) in §4 of [2] imply the

---

\(^1\)His arguments there show in particular the following: let $\mathcal{O}$ be the integer ring of the completion $k_v$ of $k$ at the place $v$. Let $B$ be a matrix in $M_r(\mathcal{O}[t])$ such that $\det B = u(t - \theta)^{d'}$ with a non-zero $u \in \mathcal{O}$. Then any solution $X$ to the equation $(t - \theta)^d X^\sigma = BX$ in $k_v[t][\mathcal{O}]^{\otimes r}$ is

---
holomorphy of the new solution $\theta^e \hat{x}$, hence the old solution $\hat{x}$ satisfies $v(x_i) \geq -ev(\theta)$ for all $i \geq 0$. Thus the values $v(x_i), i \geq 0$, are bounded below for all valuations $v$ of $k$, by constants which are non-negative for almost all $v$. By the bounded height theorem, there appear in fact only finitely many $x_i$'s in the sequence $x_0, x_1, \ldots$. Accordingly, there appear only finitely many equations (***) and one can choose a "periodic" (except for finitely many terms) solution $\hat{x} = \sum x_i t^i$ in $K^{\text{br}}_{k_{\pi}}$ to (***) closely enough to the original $\hat{x}$. Periodicity implies that $\hat{x}$ is rational with denominator in $\mathbb{F}_q[t]$, hence $\hat{x} \in K^{\text{br}}_{k}$.

Acknowledgment

I thank Greg Anderson for a careful reading of the manuscript and helpful comments. I also thank Akio Tamagawa for his encouragement, and the Institute for Advanced Study for its hospitality.

References


Tokyo Metropolitan University, Hachioji, Tokyo 192-03, Japan
E-mail address: taguchi@math.metro-u.ac.jp

automatically in $\mathcal{O}_\ell^\text{br}$. (The proof is found in the last page of [2], except that we need to use his Lemma 7 for a more general $\Phi = B$ as above, with the maximal ideal $\mathcal{M}^{\text{sep}}$ in the statement replaced by the integer ring $\mathcal{O}^{\text{sep}}$ of a separable closure of $k_v$. This can be proved easily by looking at each term of the $t$-adic expansion of the given equation, just as we did in (***)�)