ALGEBRAIC AND TRIANGULAR $n$-HYPONORMAL OPERATORS

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Abstract. In this paper we shall prove that if an operator $T \in \mathcal{L}(\mathfrak{D}^n H)$ is a finite triangular operator matrix with hyponormal operators on main diagonal, then $T$ is subscalar. As corollaries we get the following:

1. Every algebraic operator is subscalar.
2. Every operator on a finite-dimensional complex space is subscalar.
3. Every triangular $n$-hyponormal operator is subscalar.

1. Introduction

Let $H$ and $K$ be separable, complex Hilbert spaces and $\mathcal{L}(H, K)$ denote the space of all linear, bounded operators from $H$ to $K$. If $H = K$, we write $\mathcal{L}(H)$ in place of $\mathcal{L}(H, K)$. An operator $T$ in $\mathcal{L}(H)$ is called hyponormal if $TT^* \leq T^*T$ or, equivalently, if $\|T^*h\| \leq \|Th\|$ for each $h$ in $H$.

A linear bounded operator $S$ on $H$ is called scalar of order $m$ if it possesses a spectral distribution of order $m$, i.e., if there is a continuous unital morphism of topological algebras

$$\Phi: C_0^m(C) \to \mathcal{L}(H)$$

such that $\Phi(z) = S$, where as usual $z$ stands for the identity function on $C$ and $C_0^m(C)$ stands for the space of compactly supported functions on $C$, continuously differentiable of order $m$, $0 \leq m \leq \infty$. An operator is subscalar if it is similar to the restriction of a scalar operator to a closed invariant subspace.

This paper is divided into four sections. Section 2 deals with some preliminary facts. In section 3, we shall state the Putinar theorem. In section 4, we shall prove our main theorem and several corollaries.

2. Preliminaries

An operator $T \in \mathcal{L}(H)$ is said to satisfy the single-valued extension property if for any open subset $U$ in $C$, the function

$$z - T: O(U, H) \to O(U, H)$$

defined by the obvious pointwise multiplication is one-to-one where $O(U, H)$ denotes the Fréchet space of $H$-valued analytic functions on $U$ with respect to

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3473
uniform topology. If, in addition, the above function \( z - T \) has closed range on \( O(U, H) \), then \( T \) satisfies the Bishop's condition (\( \beta \)).

In other terms, condition (\( \beta \)) means that, for any open set \( U \) and any sequence of analytic functions \( f_n \in O(U, H) \), \( \lim_{n \to \infty} f_n = 0 \) in \( O(U, H) \) whenever \( \lim_{n \to \infty} (z - T)f_n = 0 \). In particular, \( (z - T)g = 0 \) if and only if \( g = 0 \), where \( g \in O(U, H) \).

2.1 Lemma ([MP], Theorem 5.5). Every hyponormal operator has property (\( \beta \)).

Let \( z \) be the coordinate in the complex plane \( C \), and let \( d\mu(z) \), or simply \( d\mu \), denote the planar Lebesgue measure. Fix a complex (separable) Hilbert space \( H \) and a bounded (connected) open subset \( U \) of \( C \). We shall denote by \( L^2(U, H) \) the Hilbert space of measurable functions \( f: U \to H \), such that

\[
\|f\|_{2,U} = \left( \int_U \|f(z)\|^2 d\mu(z) \right)^{1/2} < \infty.
\]

The space of functions \( f \in L^2(U, H) \) which are analytic functions in \( U \) (i.e., \( \overline{\partial}f = 0 \)) is denoted by

\[
A^2(U, H) = L^2(U, H) \cap O(U, H).
\]

\( A^2(U, H) \) is called the Bergman space for \( U \). Note that \( A^2(U, H) \) is complete (i.e., \( A^2(U, H) \) is a Hilbert space).

Let \( P \) denote the orthogonal projection of \( L^2(U, H) \) onto \( A^2(U, H) \). Let \( L^\infty(U, H) \) denote the Banach space of essentially bounded \( H \)-valued functions on \( U \). Let \( \overline{U} \) be the closure in \( C \) of the open set \( U \), and let \( C^p(\overline{U}, H) \) denote the space of continuously differentiable functions on \( \overline{U} \) of order \( p \), \( 0 \leq p \leq \infty \).

**Cauchy-Pompeiu formula.** For a bounded disk \( D \),

\[
f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(s)}{s - z} ds - \frac{1}{\pi} \int_D \overline{\partial} f(s) d\mu(s)
\]

where \( z \in D \) and \( f \in C^2(\overline{D}, H) \).

**Remark.** The function \( g(z) = \int_{\partial D} \frac{f(s)}{s - z} d\mu(s) \) appearing in the Cauchy-Pompeiu formula is analytic in \( D \) and continuous on \( \overline{D} \), in particular \( g \in A^2(D, H) \) for \( f \in C^2(\overline{D}, H) \).

Let us now define a special Sobolev type space. Let \( U \) again be a bounded open subset of \( C \) and \( m \) be a fixed non-negative integer. The vector-valued Sobolev space \( W^m(U, H) \) with respect to \( \overline{\partial} \) and of order \( m \) will be the space of those functions \( f \in L^2(U, H) \) whose derivatives \( \overline{\partial} f, \ldots, \overline{\partial}^m f \) in the sense of distributions still belong to \( L^2(U, H) \). Endowed with the norm

\[
\|f\|_{W^m} = \sum_{i=0}^m \|\overline{\partial}^i f\|_{2,U}^2
\]

\( W^m(U, H) \) becomes a Hilbert space contained continuously in \( L^2(U, H) \).

Let \( U \) be a (connected) bounded open subset of \( C \), and let \( m \) be a non-negative integer. The linear operator \( M \) of multiplication by \( z \) on \( W^m(U, H) \) is continuous and has a spectral distribution of order \( m \), defined by the relation

\[
\Phi_M: C_0^m(C) \to \mathcal{L}(W^m(U, H)), \quad \Phi_M(f) = Mf.
\]
Therefore, $M$ is a scalar operator of order $m$. Let

$$V : W^m(U, H) \to \bigoplus_0^m L^2(U, H)$$

be the operator $V(f) = (f, z^2 f, \ldots, z^m f)$. Then $V$ is an isometry such that $VM_z = (\bigoplus_0^m M_z) V$. Therefore, $M_z$ is a subnormal operator.

3. PUTINAR’S THEOREM

Let $T$ be in $S(H)$. Then for a given open bounded subset $D$ of $\mathbb{C}$, $z - T$ acts (linearly and) continuously on the space $W^2(D, H)$.

3.1 Lemma ([Pu], Lemma 1.1). If $U$ and $V$ are bounded connected open sets in $\mathbb{C}$, and if $V$ is relatively compact in $U$, then there is a constant $c > 0$, such that

$$\|f\|_{L^\infty(U)} \leq c \|f\|_{L^2(V)}$$

for every $f$ in $A^2(U, H)$.

3.2 Proposition ([Pu], Proposition 2.1). For a bounded disk $D$ in complex plane there is a constant $C_D$, such that for an arbitrary operator $T$ in $S(H)$ and $f$ in $W^2(D, H)$ we have

$$\|(I - P)f\|_{L^2(D)} \leq C_D (\|(z - t)^* \bar{\partial} f\|_{L^2(D)} + \|(z - T)^* \partial^2 f\|_{L^2(D)})$$

where $P$ denotes the orthogonal projection of $L^2(D, H)$ onto the Bergman space $A^2(D, H)$.

3.3 Corollary ([Pu], Corollary 2.2). If $T$ is hyponormal, then

$$\|(I - P)f\|_{L^2(D)} \leq C_D (\|(z - T)^* \partial f\|_{L^2(D)} + \|(z - T)^* \partial^2 f\|_{L^2(D)}).$$

3.4 Theorem ([Pu], Theorem 1). Any hyponormal operator is subscalar of order 2.

Proof. Let $T$ be a hyponormal operator on the Hilbert space $H$. Consider an arbitrary bounded open subset $D$ of $\mathbb{C}$ and the quotient space

$$H(D) = \frac{W^2(D, H)}{\text{cl}(z - T) W^2(D, H)}$$

endowed with the Hilbert space norm. The class of a vector $f$ or an operator $A$ on this quotient will be denoted by $\tilde{f}$, respectively $\tilde{A}$.

Note that $M$, the operator of multiplication by $z$ on $W^2(D, H)$, leaves invariant $\text{ran}(z - T)$, hence $\tilde{M}$ is well defined.

On the other hand, the map

$$\Phi : C_0^2(\mathbb{C}) \to S(W^2(D, H)), \quad \Phi(f) = M_f$$

is a spectral distribution for $M$, of order 2. Thus the operator $M$ is $C^2$-scalar. Since $\text{ran}(z - T)$ is invariant under every operator $M_f$, $f \in C_0^2(\mathbb{C})$, we infer that $\tilde{M}$ is a $C^2$-scalar operator with spectral distribution $\Phi$.

Define

$$V : H \to \frac{W^2(D, H)}{\text{cl}(z - T) W^2(D, H)}$$
by $V(h) = 1 \otimes h$ where $1 \otimes h$ denotes the constant $h$. Then

$$VT = \widetilde{M}V.$$ 

Indeed, $VTh = (1 \otimes Th)^\sim = (z \otimes h)^\sim = \widetilde{M}(1 \otimes h)^\sim = \widetilde{M}Vh$. In particular ran$V$ is an invariant subspace for $\widetilde{M}$. In order to conclude the proof of this theorem, it is enough to show the following lemma.

3.5 Lemma ([Pu], Lemma 2.3). Let $D$ be a bounded disk which contains $\sigma(T)$. Then the operator $V$ is one-to-one and has closed range.

Proof. We have to prove the following assertion: If $h_n$ in $\mathcal{H}$ and $f_n$ in $W^2(D, \mathcal{H})$ are sequences such that

$$\lim_{n \to \infty} \|(z-T)f_n + h_n\|_{W^2} = 0,$$

then $\lim_{n \to \infty} h_n = 0$. The assumption (1) implies

$$\lim_{n \to \infty} \|(z-T)\overline{\partial}f_n\|_{2,D} + \|(z-T)\overline{\partial}^2f_n\|_{2,D} = 0.$$  

By Corollary 3.3,

$$\lim_{n \to \infty} \|(I-P)f_n\|_{2,D} = 0.$$  

Then by (1),

$$\lim_{n \to \infty} \|(z-T)Pf_n + h_n\|_{2,D} = 0.$$  

Let $\Gamma$ be a curve in $D$ surrounding $\sigma(T)$. Then for $z \in \Gamma$

$$\lim_{n \to \infty} \|Pf_n(z) + (z-T)^{-1}h_n\| = 0$$

uniformly by the preceding consequence of Proposition 3.2. Hence,

$$\left\| \frac{1}{2\pi i} \int_{\Gamma} Pf_n(z) dz + h_n \right\| \to 0.$$  

But $\int_{\Gamma} Pf_n dz = 0$. Hence, $\lim_{n \to \infty} h_n = 0$. □

4. Main theorems

In this section, we shall prove that every algebraic and triangular $n$-hyponormal operator is subscalar.

4.1 Definition. An operator $T \in \mathcal{L}(\mathcal{H})$ is algebraic if there is a non-zero polynomial $p$ such that $p(T) = 0$.

4.2 Definition. An operator $T \in \mathcal{L}(\mathcal{H})$ is nilpotent if $T^n = 0$ for some integer $n$.

4.3 Proposition. Every nilpotent operator is an algebraic operator.

An interesting characterization of algebraic operators was given by P. R. Halmos.
4.4 **Theorem** ([Ha]). If $T$ is an algebraic operator and $p$ is a polynomial of minimal degree $n$ such that $p(T) = 0$, then $T$ is unitarily equivalent to a finite operator matrix of the form

$$
\begin{pmatrix}
\alpha_1 & T_{12} & \cdots & \cdots & T_{1n} \\
0 & \alpha_2 & T_{23} & \cdots & \cdots \\
\vdots & \vdots & \ddots & \cdots & \cdots \\
\vdots & \vdots & \cdots & \alpha_{n-1} & T_{n-1,n} \\
0 & \cdots & \cdots & 0 & \alpha_n
\end{pmatrix}
$$

where $\alpha_i$ are the roots of the polynomial $p$.

The following theorem will be proved in this paper.

4.5 **Theorem.** If an operator $T \in \mathcal{L}(\bigoplus_1^n H)$ is a finite operator matrix of the form

$$
T = \begin{pmatrix}
T_{11} & T_{12} & \cdots & \cdots & T_{1n} \\
0 & T_{22} & T_{23} & \cdots & \cdots \\
\vdots & \vdots & \ddots & \cdots & \cdots \\
\vdots & \vdots & \cdots & T_{n-1,n-1} & T_{n-1,n} \\
0 & \cdots & \cdots & 0 & T_{nn}
\end{pmatrix}
$$

where $T_{i,i}$ are hyponormal for $i = 1, 2, \ldots, n$, then $T$ is a subscalar operator of order $2n$.

**Proof.** Consider an arbitrary bounded open subset $D$ of $\mathbb{C}$ which contains $\sigma(T)$ and the quotient space

$$
H(D) = \frac{\oplus_0^n W^{2n}(D, H)}{\text{cl}(T - z) \oplus_0^n W^{2n}(D, H)}
$$

endowed with the Hilbert space norm. Let $M_z$ be the multiplication operator with $z$ on $W^{2n}(D, H)$. Then $\oplus_1^n M_z$ is a $C^{2n}$-scalar subnormal operator and its spectral distribution is

$$
\Phi: \bigoplus_1^n C_0^{2n}(C) \to \mathcal{L}(\bigoplus_1^n W^{2n}(D, H)), \quad \Phi \bigoplus_1^n f_i = \bigoplus_1^n M_{f_i}.
$$

Since $M_z$ commutes with $M_{T - z}$, $\bigoplus_1^n \widetilde{M}_z$ is still a scalar operator of order $2n$, with $\Phi$ as spectral distribution.

Let $\bigoplus_1^n V$ be the operator

$$
\bigoplus_1^n V)(\bigoplus_1^n h_i) = (1 \otimes h_1, \ldots, 1 \otimes h_n)^t + (T - z) \otimes_1^n W^{2n}(D, H),
$$

from $\bigoplus_1^n H$ into $H(D)$, denoting by $(1 \otimes h_1, \ldots, 1 \otimes h_n)^t$ the constant function $\bigoplus_1^n h_i$. Then

$$
\bigoplus_1^n V)T = (\bigoplus_1^n \widetilde{M}_z)(\bigoplus_1^n V)
$$

In particular, $\text{ran}(\bigoplus_1^n V)$ is an invariant subspace for $\bigoplus_1^n \widetilde{M}_z$. In order to conclude the proof of this theorem, it is enough to show Lemma 4.6.

4.6 **Lemma.** Let $D$ be a bounded disk which contains $\sigma(T)$. Then the operator $\bigoplus_1^n V: \bigoplus_1^n H \to H(D)$ is one-to-one and has closed range.
Proof of Lemma 4.6. Let $\bigoplus_i^n h_i^k \in \bigoplus_i^n H$ and $\bigoplus_i^n f_i^k \in \bigoplus_i^n W^{2n}(D, H)$ be sequences (in $k$) such that

$$\lim_{k \to \infty} \| (T - z) \bigoplus_i^n f_i^k + \bigoplus_i^n (1 \otimes h_i^k) \|_{\bigoplus_i^n W^{2n}} = 0.$$ 

It suffices to show that $\lim_{k \to \infty} \bigoplus_i^n h_i^k = 0$. The limit given directly above can be written as

$$\lim_{k \to \infty} \| (T_{11} - z) f_1^k + T_{12} f_2^k + \cdots + T_{1n} f_n^k + 1 \otimes h_1^k \|_{W^{2n}} = 0,$$

$$\vdots$$

$$\lim_{k \to \infty} \| (T_{j,j} - z) f_j^k + T_{j,j+1} f_{j+1}^k + \cdots + T_{j,n} f_n^k + 1 \otimes h_j^k \|_{W^{2n}} = 0,$$

$$\vdots$$

$$\lim_{k \to \infty} \| (T_{n,n} - z) f_n^k + 1 \otimes h_n^k \|_{W^{2n}} = 0.$$

In order to prove Lemma 4.6 we need the following:

Fact. For $t = 1, 2, 3, \ldots, n$,

$$(1,1) \quad \lim_{k \to \infty} \| (T_{11} - z) f_1^k + T_{12} f_2^k + \cdots + T_{1t} f_t^k + 1 \otimes h_1^k \|_{W^{2n}} = 0,$$

$$\vdots$$

$$(1,j) \quad \lim_{k \to \infty} \| (T_{j,j} - z) f_j^k + T_{j,j+1} f_{j+1}^k + \cdots + T_{j,t} f_t^k + 1 \otimes h_j^k \|_{W^{2n}} = 0,$$

$$\vdots$$

$$(1,t) \quad \lim_{k \to \infty} \| (T_{t,t} - z) f_t^k + 1 \otimes h_t^k \|_{W^{2n}} = 0.$$

We prove this fact by induction. We assume that Lemma 4.7 holds for some given $t = 2, 3, \ldots, n$. We only need to verify that

$$\lim_{k \to \infty} \| (T_{11} - z) f_1^k + T_{12} f_2^k + \cdots + T_{1,t-1} f_{t-1}^k + 1 \otimes h_1^k \|_{W^{2(t-1)}} = 0,$$

$$\vdots$$

$$\lim_{k \to \infty} \| (T_{t-1,t-1} - z) f_{t-1}^k + 1 \otimes h_{t-1}^k \|_{W^{2(t-1)}} = 0.$$

However the reader will note that this result follows directly from $(1, 1), \ldots, (1, t)$ provided $\lim_{k \to \infty} \| \partial_i f_i^k \|_{2,D} = 0$ for $i = 0, 1, \ldots, 2(t-1)$. So this will be shown to be true.

Claim 1. $\lim_{k \to \infty} h_i^k = 0$.

The proof of Lemma 3.5 suitably modified to include the higher order partials with $T = T_{t,t}$ shows the claim to be true.

Claim 2. $\lim_{k \to \infty} \| (I - P) \partial_i f_i^k \|_{2,D} = 0$ for $i = 0, \ldots, 2(t-1)$.

By Claim 1 and the equation $(1, t)$, $\lim_{k \to \infty} \| (T_{t,t} - z) f_t^k \|_{W^{2t}} = 0$. Then
we can apply Proposition 3.2 and Corollary 3.3 with $T = T_{i,t}$. In fact,
\[
\| (I - P)(f^{rk}_{t}, \overline{\partial} f^{i}_{t}, \ldots, \overline{\partial}^{2t-2} f^{i}_{t}) \|_{2,D} \\
\leq C_D \| (T_{i,t} - z)^{(\overline{\partial} f^{i}_{t}, \ldots, \overline{\partial}^{2t} f^{i}_{t})} \|_{2,D} \\
+ \| (T_{i,t} - z)^{(\overline{\partial}^{2} f^{i}_{t}, \ldots, \overline{\partial}^{2(t-1)} f^{i}_{t})} \|_{2,D} \\
\leq C_D \| (T_{i,t} - z)(\overline{\partial} f^{i}_{t}, \ldots, \overline{\partial}^{2t} f^{i}_{t}) \|_{2,D} \\
+ \| (T_{i,t} - z)(\overline{\partial}^{2} f^{i}_{t}, \ldots, \overline{\partial}^{2(t-1)} f^{i}_{t}) \|_{2,D},
\]
where $P$ denotes the orthogonal projection of $\bigoplus_{2t-1} L^2(D, H)$ onto the Bergman space $\bigoplus_{2t-1} A^2(D, H)$. Thus Claim 2 follows from (1, $t$).

By Claim 2, for $i = 0, 1, \ldots, 2(t-1)$
\[
\lim_{k \to \infty} \|(T_{i,t} - z)\overline{\partial} f^{i}_{t} - (T_{i,t} - z)P\overline{\partial} f^{i}_{t} \|_{2,D} = 0.
\]
From (1, $t$),
\[
\lim_{k \to \infty} \|(T_{i,t} - z)P\overline{\partial} f^{i}_{t} \|_{2,D} = 0
\]
for $i = 0, 1, \ldots, 2(t-1)$. Since every hyponormal operator has property $(\beta)$ by Lemma 2.1, for $i = 0, 1, \ldots, 2(t-1)$, $P\overline{\partial} f^{i}_{t} \to 0$ uniformly on compact subsets of $D$.

Consider $\sigma(T) \subset B(0, r) \subset \overline{B(0, r)} \subset D$. For $i = 0, 1, \ldots, 2(t-1)$,
\[
\|P\overline{\partial} f^{i}_{t} \|_{2,D}^2 = \int_{D} \|P\overline{\partial} f^{i}_{t}(z)\|^2 d\mu(z) \\
= \int_{D \setminus B(0, r)} \|P\overline{\partial} f^{i}_{t}(z)\|^2 d\mu(z) = \int_{D \setminus B(0, r)} \|P\overline{\partial} f^{i}_{t}(z)\|^2 d\mu(z).
\]
By property $(\beta)$, the first integral converges to 0. Since $T - z$ is invertible on $D \setminus B(0, r)$, the second integral also converges to 0. Therefore,
\[
\lim_{k \to \infty} \|P\overline{\partial} f^{i}_{t} \|_{2,D} = 0.
\]
From Claim 2, we get $\lim_{k \to \infty} \|\overline{\partial} f^{i}_{t} \|_{2,D} = 0$ for $i = 0, 1, \ldots, 2(t-1)$. So this completes the proof of the fact stated above.

Let us come back now to the proof of Lemma 4.6. By the fact, we get the equation
\[
\lim_{k \to \infty} \|(T_{11} - z)f^{k}_{1} + 1 \otimes h^{k}_{1} \|_{W_2} = 0.
\]
By the application of Lemma 3.5 with $T = T_{11}$, we get $\lim_{k \to \infty} h^{k}_{1} = 0$. Since $\lim_{k \to \infty} h^{k}_{j} = 0$ for $j = 1, \ldots, n$, $\lim_{k \to \infty} h^{k} = 0$ where $h^{k} = (h^{k}_{1}, \ldots, h^{k}_{n})$. Thus $\bigoplus_{\alpha} V$ is one-to-one and has closed range. $\Box$

This also concludes the proof of Theorem 4.5, because $\text{ran}(\bigoplus_{\alpha} V)$ is a closed invariant subspace for the scalar operator $(\bigoplus_{\alpha} M_{z})$. $\Box$

4.8 Corollary. If $T$ is an algebraic operator, then $T$ is a subscalar operator.
Proof. It is clear from Theorem 4.4 and Theorem 4.5. $\Box$
4.9 **Corollary.** Every operator on a finite-dimensional complex space is subscalar.

4.10 **Definition.** An operator $T \in \mathcal{L}(H)$ is said to have property $(\alpha)$ if for every (not necessarily strict) contraction $A$, every operator $X$ with dense range such that $XA = TX$, and every $h$ in $H$, there exists a non-zero polynomial $p$ such that $p(T)h \in \text{ran } X$.

4.11 **Lemma** ([ABFP], Proposition 3.9). If a strict contraction $T$ (i.e., $\|T\| < 1$) has the property $(\alpha)$, then $T$ is an algebraic operator.

4.12 **Corollary.** If a strict contraction $T$ has property $(\alpha)$, then $T$ is subscalar.

4.13 **Corollary.** If $A \prec T$ and $T$ is algebraic, then $A$ is subscalar.

**Proof.** By hypothesis, we can show $A$ is algebraic. □

**Remark.** Let

$$B = \begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix}$$

where $T$ is hyponormal. Then $B$ is a non-hyponormal, but $B$ is a subscalar operator of order 4.

4.14 **Definition.** An operator $T$ in $\mathcal{L}(\bigoplus_1^n H)$ is said to be a triangular $n$-hyponormal operator if

$$T = \begin{pmatrix} T_{11} & T_{12} & \cdots & \cdots & T_{1n} \\ 0 & T_{22} & \cdots & \cdots \\ \vdots & \vdots & \ddots & \cdots \\ \cdots & \cdots & \cdots & T_{n-1,n-1} & T_{n-1,n} \\ \cdots & \cdots & \cdots & 0 & T_{nn} \end{pmatrix}$$

where $(T_{ij})$ are commuting hyponormal operators on $H$.

4.15 **Corollary.** Let $T$ be a triangular $n$-hyponormal operator. Then $T$ is a subscalar operator.

4.16 **Question.** Let

$$T = \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{pmatrix}$$

where $\{T_i\}$ are commuting hyponormal operators. Is $T$ subscalar?

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