THE BEREZIN SYMBOL AND MULTIPLIERS
OF FUNCTIONAL HILBERT SPACES

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Abstract. This paper focuses on a multiplicative property of the Berezin symbol $\tilde{A}$, of a given linear map $A : \mathcal{H} \to \mathcal{H}$, where $\mathcal{H}$ is a functional Hilbert space of analytic functions. We show $\tilde{A}B = \tilde{AB}$ for all $B$ in $\mathcal{B}(\mathcal{H})$ if and only if $A$ is a multiplication operator $M_\varphi$, where $\varphi$ is a multiplier. We also present a version of this result for vector-valued functional Hilbert spaces.

1. Introduction

Let $n$ be a fixed positive integer and let $\Omega$ be a region in $\mathbb{C}^n$. A functional Hilbert space $\mathcal{H}$ is a Hilbert space of analytic functions on $\Omega$ such that the point evaluations are bounded, linear functionals. By the Riesz-representation theorem there exists, for each $z$ in $\Omega$, a unique element $K_z$ of $\mathcal{H}$ such that $f(z) = \langle f, K_z \rangle$ for all $f$ in $\mathcal{H}$. The function $K$ on $\Omega \times \Omega$, defined by $K(z, w) = K_w(z)$, is called the reproducing kernel function of $\mathcal{H}$. Let $k_z = \frac{K_z}{\|K_z\|}$ be the normalized reproducing kernel function. For a given linear map $A : \mathcal{H} \to \mathcal{H}$, the Berezin symbol $\tilde{A}$ (see [1]) of a map $A$ of $\mathcal{H}$ into itself is defined by

$$\tilde{A}(z) = \langle Ak_z, k_z \rangle.$$

It is known that the map $A \mapsto \tilde{A}$ is injective (see [3]). A function $\varphi$ defined on $\Omega$ is a multiplier of $\mathcal{H}$ if $\varphi \cdot f$ is in $\mathcal{H}$, for all $f$ in $\mathcal{H}$. Let $\mathcal{B}(\mathcal{H})$ denote the set of all bounded, linear operators from $\mathcal{H}$ into $\mathcal{H}$. The multiplication operator $M_\varphi : \mathcal{H} \to \mathcal{H}$ defined by $M_\varphi f = \varphi \cdot f$ is in $\mathcal{B}(\mathcal{H})$, when $\varphi$ is a multiplier of $\mathcal{H}$.

2. The multiplicative property of the Berezin symbol on a functional Hilbert space

Theorem 1. Let $A$ be a bounded operator on $\mathcal{H}$. Then

$$\tilde{AB}(z) = \tilde{A}(z)\tilde{B}(z)$$
for all $B$ in $\mathcal{B}(H)$ if and only if $A$ is a multiplication operator, $M_{\varphi}$, where $\varphi$ is a multiplier. Moreover, $\varphi = \tilde{A}$.

Before proceeding with the proof, we need the following:

**Lemma 1.** When $\varphi$ is a multiplier of $H$, $\tilde{M}_{\varphi}(z) = \varphi(z)$.

**Proof.** $\tilde{M}_{\varphi}(z) = (M_{\varphi}k_z, k_z) = (\varphi k_z, k_z) = \varphi(z)$.

**Lemma 2.** The Berezin symbol of $f \otimes g$, for $f, g$ in $H$, is

$$(f \otimes g)(z) = \frac{g(z)}{\|K_z\|^2} f(z), \quad z \in \Omega.$$

**Proof.** For $f$ and $g$ in $H$ and $z$ in $\Omega$,

$$(f \otimes g)(z) = \left\langle (f \otimes g) \frac{K_z}{\|K_z\|^2} \frac{K_z}{\|K_z\|^2} \right\rangle = \frac{1}{\|K_z\|^2} \langle K_z, g \rangle \langle f, K_z \rangle.$$

By the reproducing property of the kernel function, we have

$$(f \otimes g)(z) = \frac{g(z)}{\|K_z\|^2} f(z), \quad f, q \in H.$$

**Proof of Theorem 1.** Suppose $\tilde{A}B(z) = \tilde{A}(z)B(z)$ for all $B$ in $\mathcal{B}(H)$. Let $B = f \otimes g$ for $f$ and $g$ in $H$. Then, by Lemma 2,

$$\tilde{A}B(z) = (Af \otimes g)(z) = \frac{g(z)}{\|K_z\|^2} (Af)(z).$$

By the hypothesis, we have

$$\frac{g(z)}{\|K_z\|^2} (Af)(z) = \frac{g(z)}{\|K_z\|^2} \tilde{A}(z)f(z),$$

which reduces to

$$(Af)(z) = \tilde{A}(z)f(z)$$

for all $f$ in $H$. Hence $A = M_{\tilde{A}}$.

Conversely, if $A$ is a multiplication operator, $M_{\varphi}$, where $\varphi$ is a multiplier,

$$\tilde{M}_{\varphi}B = (M_{\varphi}Bk_z, k_z) = \varphi(z) \frac{Bk_z}{\|K_z\|}(z)$$

for all $B$ in $\mathcal{B}(H)$. By Lemma 1, we have

$$\tilde{M}_{\varphi}B(z) = \tilde{M}_{\varphi}(z)B(z)$$

for all $B$ in $\mathcal{B}(H)$.

**Corollary 1.** Let $B$ be in $\mathcal{B}(H)$. Then

$$\tilde{A}B(z) = \tilde{A}(z)B(z)$$

for all $A$ in $\mathcal{B}(H)$ if and only if $B = M_{\psi}^*$, where $\psi$ is a multiplier.

**Proof.** The assertion follows from Theorem 1 and the fact that $\tilde{T}^*(z) = \tilde{T}(z)$, for all $T$ in $\mathcal{B}(H)$.
The Hardy space $H^2$ consists of the complex-valued analytic functions on the unit disk $D$ such that the Taylor coefficients are square summable. A calculation shows that $K_z = \frac{1}{1 - z \overline{w}}$ has the reproducing property (see [4]). Let $P$ denote the orthogonal projection of $L^2(\partial D)$ onto $H^2$, and let $\phi$ be a bounded measurable function. Then the Toeplitz operator, $T_\phi$, induced by $\phi$ is defined by $T_\phi f = P(\phi f)$, for all $f$ in $H^2$.

**Corollary 2.** Let $A$ be a bounded operator on $H^2$. Then

$$\widehat{AB}(z) = \widehat{A}(z)\widehat{B}(z)$$

for all $B$ in $\mathcal{B}(H^2)$ if and only if $A$ is a Toeplitz operator, $T_\phi$, induced by $\phi$ in $H^\infty$. Moreover $\phi = \widehat{A}$.

**Proof.** The multiplication operators on $H^2$ are the analytic Toeplitz operators.

We should mention that Corollary 2 is also true if one replaces $H^2$ by the Bergman space or any of the weighted Bergman spaces. (For analytic Toeplitz operators on weighted Bergman spaces see [6].)

3. The multiplicative property of the Berezin symbol on the analytic reproducing kernel space, $\mathcal{H} = \mathcal{H}_0 \otimes \mathbb{C}$

Let $\mathcal{H}_0$ be a functional Hilbert space of (scalar-valued) analytic functions on $\Omega$ with the reproducing kernel function $K_z$, for each fixed $z$ in $\Omega$. Let $\mathbb{C}$ be a separable Hilbert space, and let $\mathcal{H}$ be the functional Hilbert space of $\mathbb{C}$-valued functions, $\mathcal{H} = \mathcal{H}_0 \otimes \mathbb{C}$. The reproducing kernel function of $\mathcal{H}$, $J_z: \mathbb{C} \rightarrow \mathcal{H}$, is defined by $J_z(u) = K_z \otimes u$, where $u$ is in $\mathbb{C}$.

The evaluation functional $E_z: \mathcal{H} \rightarrow \mathbb{C}$, defined by $E_z f = f(z)$, for $z$ in $\Omega$, is bounded (see [2], Lemma 3.2). For $f \in \mathcal{H}$, $u$ in $\mathbb{C}$, we have

$$\langle f, E_z^* u \rangle_\mathcal{H} = \langle f(z), u \rangle_\mathbb{C}.$$ 

We also have the reproducing property of the kernel function, that is

$$\langle f, J_z(u) \rangle_\mathcal{H} = \langle f(z), u \rangle_\mathbb{C}.$$ 

Therefore, $E_z^* u = J_z(u)$, for all $u$ in $\mathbb{C}$. By the reproducing property of the kernel function, we have $\|J_z(u)\|^2 = K_z(z)\|u\|^2$, where $u$ is in $\mathbb{C}$, and hence $\|J_z\| = \sqrt{K_z(z)} = \|E_z\|$.

Let $\mathcal{H}_z = \frac{1}{\|J_z\|}$ be the normalized reproducing kernel function, and let $A$ be a bounded linear operator on $\mathcal{H}$. Then the Berezin symbol $\widehat{A}$ of $A$ is defined by

$$\widehat{A}(z) = \mathcal{H}_z^* A \mathcal{H}_z.$$ 

**Lemma 3.** An operator $A$ is a multiplication operator if and only if, for each fixed $z$ in $\Omega$, $A^* E_z^* = E_z^* \Phi(z)$, for some operator $\Phi(z)$ in $\mathcal{B}(\mathbb{C})$. Moreover, in this case, $A$ is the operator of multiplication by the function $z \mapsto \Phi(z)$.

**Proof.** Let $z$ be fixed in $\Omega$. Suppose $A$ is a multiplication operator, $M_\Phi$, induced by $\Phi: \Omega \rightarrow \mathcal{B}(\mathbb{C})$. We observe that

$$E_z M_\Phi f = M_\Phi f(z) = \Phi(z)f(z) = \Phi(z)E_z f$$

for all $f$ in $\mathcal{H}$.

Then we have $E_z M_\Phi = \Phi(z)E_z$, for some operator $\Phi(z)$ in $\mathcal{B}(\mathbb{C})$.  


Conversely, let $A$ be a bounded operator on $\mathscr{H}$ such that $A^*E_z^* = E_z^*\Phi(z)^*$ for some operator $\Phi(z)$ in $\mathscr{B}(\mathbb{C})$. For $u$ in $\mathscr{C}$, we have

$$\langle f, A^*E_z^*u \rangle_{\mathscr{H}} = \langle Af, E_z^*u \rangle_{\mathscr{H}} = \langle (Af)(z), u \rangle_{\mathscr{H}}$$

for all $f$ in $\mathscr{H}$. On the other hand, for $u$ in $\mathscr{C}$, we have $\langle f, E_z^*\Phi(z)^*u \rangle = \langle \Phi(z)f(z), u \rangle$, for all $f$ in $\mathscr{H}$. Then $\langle (Af)(z), u \rangle = \langle \Phi(z)f(z), u \rangle$, for all $f$ in $\mathscr{H}$ and $u$ in $\mathscr{C}$. Therefore, $(Af)(z) = \Phi(z)f(z)$, for all $f$ in $\mathscr{H}$.

**Theorem 2.** Let $A$ be a bounded operator on $\mathscr{H}$. Then

$$\widetilde{AB}(z) = \tilde{A}(z)\tilde{B}(z)$$

for all $B$ in $\mathscr{B}(\mathscr{H})$ if and only if $A = M_\Phi$, where $\Phi: \Omega \mapsto \mathscr{B}(\mathbb{C})$.

**Proof.** We observe that $E_zM_\Phi f = \Phi(z)f(z)$, for all $f$ in $\mathscr{H}$. Then $E_zM_\Phi E_z^* = \Phi(z)E_zE_z^*$ and $E_zM_\Phi BE_z^* = \Phi(z)E_zB E_z^*$, for all $B$ in $\mathscr{B}(\mathscr{H})$. Since $E_zE_z^* = K_z(z)I_\mathscr{H}$, we have $M_\Phi = \Phi(z)$ and

$$M_\Phi B(z) = \Phi(z)\frac{E_zBE_z^*}{\|z\|^2} = \tilde{M}_\Phi(z)\tilde{B}(z)$$

for all $B$ in $\mathscr{B}(\mathscr{H})$.

Conversely, suppose that $A$ is a bounded operator such that $\widetilde{AB}(z) = \tilde{A}(z)\tilde{B}(z)$ for all $B$ in $\mathscr{B}(\mathscr{H})$. Then from the definitions, we get

$$E_zABE_z^* = \frac{1}{\|z\|^2}E_zAE_zE_zBE_z^*$$

for all $B$ in $\mathscr{B}(\mathscr{H})$.

For $u$ and $v$ in $\mathscr{C}$, we have

$$\langle E_zABE_z^* u, v \rangle = \left( \frac{E_zAE_z^*}{\|z\|^2}E_zBE_z^* u, v \right) = \langle \tilde{A}(z)E_zBE_z^* u, v \rangle.$$ 

Then we have

$$\langle BE_z^* u, A^*E_z^* v \rangle = \langle BE_z^* u, E_z^*\tilde{A}(z)^* v \rangle.$$ 

For each fixed nonzero $u$, $BE_z^* u$ runs through all vectors in $\mathscr{H}$ as $B$ runs through all elements of $\mathscr{B}(\mathscr{H})$. Thus we see that $A^*E_z^* = E_z^*\tilde{A}(z)^*$, for all $z$ in $\Omega$. Therefore $A$ is a multiplication operator, $M_\tilde{A}$, by Lemma 3.

Let us note that if we take $\mathscr{C}$ to be $\mathbb{C}$ and define $\tilde{A}_z = k_z \otimes 1$, the sufficiency proof of Theorem 2 will also work for Theorem 1, the scalar-valued case.

Let $\mathbb{N} = \{0, 1, 2, \ldots \}$ denote the set of nonnegative integers. The set $\mathbb{N}^n$ is partially ordered by setting $I = (i_1, i_2, \ldots, i_n) \geq (j_1, j_2, \ldots, j_n) = J$ if and only if $i_k \geq j_k$ for $k = 1, 2, \ldots, n$. If $z = (z_1, z_2, \ldots, z_n) \in \Omega$, then we set $z^I = z_1^{i_1}z_2^{i_2} \cdots z_n^{i_n}$. We denote by $H^2(n) \otimes \mathbb{C}$, where $H^2(n) = H^2 \otimes H^2 \otimes \cdots \otimes H^2$ (n copies), the set of all vector-valued analytic functions $f: D^n \mapsto \mathbb{C}$ with power series expansion $f(z) = \sum_{I \in \mathbb{N}^n} z^Iv_I$, with $v_I$ in $\mathscr{C}$ and $z$ in $D^n$, such that $\sum_{I \in \mathbb{N}^n} \|v_I\|_{\mathscr{F}}^2 < \infty$.

The space $H^2(n) \otimes \mathbb{C}$ is a Hilbert space with the reproducing kernel function, $J_z: \mathscr{C} \mapsto H^2(n) \otimes \mathbb{C}$, for $z$ in $D^n$, defined by $J_z(u) = K_z \otimes u$, where $u$ is in $\mathscr{C}$ and $K_z(u) = \sum_{I \in \mathbb{N}^n} z^I w_I$ is the reproducing kernel function for $H^2(n)$ (see [5]). Let $H^\infty(n)(\mathscr{B}(\mathbb{C}))$ denote the Banach space of all bounded analytic functions $\Phi: D^n \mapsto \mathscr{B}(\mathbb{C})$ with the norm $\|\Phi\|_\infty = \sup\{\|\Phi(z)\|, \text{ for } z \in D^n\}$.
For every $\Phi$ in $H^\infty(n)(\mathcal{B}(\mathbb{C}))$, we can define the analytic Toeplitz operator $T_\Phi$ in $\mathcal{B}(H^2(n) \otimes \mathbb{C})$ as follows:

$$(T_\Phi f)(z) = \Phi(z)f(z), \quad z \in \mathbb{D}^n, f \in H^2(n) \otimes \mathbb{C}.$$  

For the boundedness of the map $T_\Phi$ see [2].

**Corollary 3.** Let $A$ be a bounded operator on $H^2(n) \otimes \mathbb{C}$. Then

$$A\overline{B}(z) = \overline{A}(z)B(z)$$

for all $B$ in $\mathcal{B}(H^2(n) \otimes \mathbb{C})$ if and only if $A = T_\Phi$, where $\Phi$ is in $H^\infty(n)(\mathcal{B}(\mathbb{C}))$.

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**References**

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