THE TEICHMULLER FLOW IS HAMILTONIAN

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Abstract. It is shown that the Teichmüller flow on the cotangent bundle over Teichmüller space coincides with the Hamiltonian flow defined by the function which gives the length of a cotangent vector.

Introduction

Suppose $M$ is a smooth manifold with local coordinates $(q_1, \ldots, q_n)$. Then the set of 1 forms $dq_1, \ldots, dq_n$ form a basis for the cotangent space at each point and so any cotangent vector $v^*$ can be written as $p_1 dq_1 + \ldots + p_n dq_n$ for coefficients $p_1, \ldots, p_n$. Then $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ are symplectic coordinates for the cotangent bundle $CTM$. Any smooth function $H : CTM \to \mathbb{R}$ defines Hamilton's equations:

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}$$

and

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}.$$ 

The corresponding flow is called the Hamiltonian flow. Suppose $M$ has a Riemannian metric and $H(v^*) = |v^*|^2/2$ where $|v^*|$ is its length. It is a classical result [2, p. 53] that the Hamiltonian flow and the geodesic flow on $CTM$ coincide.

In this paper we consider the Teichmüller space $T_g$ of closed Riemann surfaces of genus $g \geq 2$. It is a fundamental result that $T_g$ is a complex manifold and that the cotangent space at a point $X \in T_g$ is the vector space $Q(X)$ of holomorphic quadratic differentials on $X$. The Teichmüller space also comes equipped with the Teichmüller metric which is not Riemannian, but rather a Finsler metric, which means it is defined by a norm on the tangent space and a dual norm

$$||\phi|| = \int_X |\phi(z)| \, dz^2$$

on the cotangent space $Q(X)$. Thus the standard equations of Riemannian geometry are not available. Nonetheless the geodesics in this metric are well understood. The geodesics are determined by the family of Teichmüller extremal
maps defined by a fixed quadratic differential and a 1 parameter family of real numbers. At the level of the cotangent bundle $\mathcal{E}$ this leads to a flow called the Teichmuller flow. The question then arises whether this flow is Hamiltonian for the corresponding length function

$$H(\phi) = \frac{||\phi||^2}{2}$$

as in the classical case.

An immediate difficulty arises from consideration of quadratic differentials with higher order zeroes. A result of Royden's [7] says that the vector field

$$(\partial H/\partial p_i, -\partial H/\partial q_i)$$

is not Lipschitz at quadratic differentials with zeroes of order at least 3. Thus the Hamiltonian system may not admit a unique solution. Because of this difficulty we define $\mathcal{E}_1 \subset \mathcal{E}$ to be the subset of quadratic differentials with only simple zeroes. This is a dense subset of $\mathcal{E}$ and is known as the principal stratum. The Teichmuller flow preserves $\mathcal{E}_1$. Our Theorem states

**Theorem.** The Hamiltonian flow is $C^\infty$ on $\mathcal{E}_1$ and coincides with the Teichmuller flow. On $\mathcal{E} - \mathcal{E}_1$, the Teichmuller flow satisfies Hamilton's equations.

In the next section we will show that the flows are $C^\infty$ on $\mathcal{E}_1$. We will then introduce coordinates for $T_g$ that allow us to show that Hamilton's equations are satisfied along the Teichmuller flow lines in $\mathcal{E}_1$. Continuity will allow us to conclude that the Teichmuller flow on $\mathcal{E} - \mathcal{E}_1$ also satisfies Hamilton's equations. In particular this means that at a point on a lower dimensional stratum, the Hamiltonian vector field is tangent to that stratum. However we do not know if the Hamiltonian system is $C^1$ along that stratum and therefore do not know if there are other solutions to the Hamiltonian system other than the Teichmuller flow.

**COORDINATES FOR TEICHMULLER SPACE**

A Riemann surface $X$ can be described by a family $\{U_\mu, z_\mu\}$, where the $U_\mu$ form an open cover of $X$ and $z_\mu : U_\mu \to \mathbb{C}$ are homeomorphisms such that $z_\mu \circ z_\nu^{-1}$ is analytic whenever defined. The maps $z_\mu$ are called local uniformizers. A holomorphic quadratic differential $\phi(z) dz^2$ on $X$ assigns to each local uniformizer $z_\mu$ a holomorphic function $\phi_\mu(z_\mu)$ such that in the overlap

$$\phi_\mu(z_\mu) dz^2_\mu = \phi_\nu(z_\nu) dz^2_\nu.$$

Associated to a quadratic differential are the horizontal and vertical trajectories. These are the arcs along which $\phi(z) dz^2 > 0$ and $\phi(z) dz^2 < 0$ respectively. The set of horizontal and vertical trajectories forms the horizontal and vertical foliations. We denote the latter by $\psi(\phi)$. A quadratic differential $\phi$ also defines a metric $|\phi^{1/2}(z) dz|$ which is locally Euclidean except at the zeroes of $\phi$ which are singularities of the metric. The set of all quadratic differentials on $X$ forms a complex vector space $Q(X)$ of dimension $3g - 3$. As $X$ varies over the Teichmuller space $T_g$, these vector spaces fit together to form a bundle $\mathcal{E}$ over $T_g$. A Beltrami differential on $X$ assigns to each uniformizer $z$ a measurable function $\mu(z)$ such that

$$\mu(z) \frac{dz}{dz}$$
is invariant under changes of coordinates. Then \(|\mu(z)|\) defines a function on \(X\). There is a pairing between \(Q(X)\) and the space \(M(X)\) of \(L^\infty\) Beltrami differentials on \(X\) given by

\[
\langle \phi, \mu \rangle = \text{Re} \int_X \phi \mu.
\]

The infinitesimally trivial Beltrami differentials \(M_0(X)\) are those \(\mu\) for which \(\langle \phi, \mu \rangle = 0\) for all \(\phi \in Q(X)\). It is a classical result in Teichmüller theory that \(T_g\) is a complex manifold, the tangent space at \(X\) is \(M(X)/M_0(X)\) and \(Q(X)\) is the cotangent space at \(X\). We let \(\pi : \mathcal{E} \to T_g\) be the natural projection.

Each \(\phi \in \mathcal{E}\) determines certain topological data \(\kappa = (k_1, \ldots, k_n; \epsilon = \pm 1)\) where \(k_1, \ldots, k_n\) are the orders of the zeroes: \(\epsilon = +1\) if \(\phi\) is the square of an abelian differential; \(\epsilon = -1\) if it is not. A stratum \(\mathcal{E}_\kappa\) consists of all quadratic differentials determining the data \(\kappa\). The principle stratum \(\mathcal{E}_1\) corresponds to \(\kappa = (1, \ldots, 1; -1)\) and its complement has codimension 1.

The quantity

\[
\int_X |\phi(z)d\bar{z}|^2
\]

which defines the Teichmüller cometric is also the area of the quadratic differential. The geodesics in the Teichmüller metric are defined by the Teichmüller maps. For each \(\phi \in Q(X)\) and \(t \in \mathbb{R}\) the Teichmüller map \(f_{\phi, t}\) maps \(X\) to a new Riemann surface \(X_t\). There is a quadratic differential \(\phi_t\) on \(X_t\) with the property that \(f_t\) sends horizontal trajectories of \(\phi\) to horizontal trajectories of \(\phi_t\), expanding lengths by a factor of \(e^t\) and sends vertical trajectories to vertical trajectories contracting lengths by the same factor \(e^t\). We can take \(\phi_t\) so that \(H(\phi_t) = H(\phi)\). It also sends zeroes of \(\phi\) to zeroes of \(\phi_t\) of the same order. At the level of the cotangent bundle \(\mathcal{E}\) this gives a flow \(\phi \to \phi_t\) called the Teichmüller flow and the flow preserves each stratum \(\mathcal{E}_\kappa\). These flows have been studied in [5], [8], and [6].

For any \(\phi_0\) in the principle stratum \(\mathcal{E}_1\) we may triangulate the underlying surface so that the edges of the triangulations are geodesic segments with respect to the metric \(|\phi_0(z)^{1/2}dz|^2\) and the vertices are zeroes of \(\phi_0\). (A canonical triangulation is given in [6].) Each \(\phi\) near \(\phi_0\) in \(\mathcal{E}_1\) has a corresponding triangulation by geodesic edges. For \(\phi\) in a neighborhood of \(\phi_0\) we may continuously choose a branch of \(\phi^{1/2}\) along each edge. To each directed edge \(e\) is associated a holonomy vector \(\text{hol}(e)\) whose components

\[
\text{hol}_1(e) = \int_e \text{Re}(\phi^{1/2}dz)
\]

and

\[
\text{hol}_2(e) = \int_e \text{Im}(\phi^{1/2}dz)
\]

are called the horizontal and vertical components of \(e\). The holonomy vectors of a set of \(6g - 6\) edges serve as analytic coordinates for \(\mathcal{E}_1\) near \(\phi_0\). The area of a triangle in \(\mathbb{R}^2\) is an analytic function of the coordinates of its vertices. Therefore \(H\) is an analytic function on \(\mathcal{E}_1\) and Hamilton’s equations must have a unique solution in a neighborhood of a point in \(\mathcal{E}_1\).
In the holonomy coordinates the Teichmuller flow \((\phi, t) \to \phi_t\) is given by

\[
(\text{hol}_1(e_1), \text{hol}_2(e_1), \ldots, \text{hol}_1(e_{6g-6}), \text{hol}_2(e_{6g-6}), t)
\]

\[
\to (e^t\text{hol}_1(e_1), e^{-t}\text{hol}_2(e_1), \ldots, e^t\text{hol}(e_{6g-6}), e^{-t}\text{hol}_2(e_{6g-6})),
\]

and thus is analytic.

Now fix \(\phi_0 \in \mathcal{E}_1\) which determines the flow line \(\phi_t \in \mathcal{E}_1\). Let \(X_t\) be the corresponding Teichmuller geodesic through \(X_0\). The proof that Hamilton's equations are satisfied along \(\phi_t\) depends on finding a useful set of coordinates in a neighborhood \(U\) of \(X_t\). Recall that \(v(\phi)\) denotes the vertical measured foliation of the quadratic differential \(\phi\).

**Proposition 1.** There are \(C^\infty\) coordinates \((q_1, \ldots, q_{6g-6})\) in a neighborhood \(U\) of \(X_t\) such that

1. For fixed \(q_1\), each point with coordinates \((q_1, \ldots, q_{6g-6})\) has a quadratic differential \(\phi\) such that \(H(\phi) = 1\) and \(v(\phi) = e^{v(\phi_0)}\);
2. For fixed \((q_2, \ldots, q_{6g-6})\), the points with coordinates \((q_1, q_2, \ldots, q_{6g-6})\) parametrize a Teichmuller geodesic with \(q_1\) as arclength parameter.

**Proof.** Let \(F = v(\phi_0)\) be the vertical foliation of \(\phi_0\). We let

\[
E_F = \{\phi \in \mathcal{E}: v(\phi) = F\}.
\]

Then \(E_F \cap \mathcal{E}_1\) is locally described near \(\phi_0\) by a set of equations

\[
\text{hol}_1(e_i) = \text{constant}
\]

and thus is a smooth submanifold of \(\mathcal{E}_1\). In particular \(H\) restricted to \(E_F \cap \mathcal{E}_1\) is smooth. Moreover by the Main Theorem of [4] the projection

\[
\pi: E_F \to T_g
\]

is a local diffeomorphism at \(\phi_0\). (The Main Theorem of [4] says that the map is a homeomorphism. The proof uses the inverse function theorem. The fact that the derivative at \(\phi_0\) is an isomorphism is proved in Lemma 4.4 and Proposition 4.16.) Then \(E_F \cap \mathcal{E}_1 \cap H^{-1}(1)\) is a smooth submanifold of \(\mathcal{E}_1\) near \(\phi_0\) which maps diffeomorphically onto its image \(N_{\phi_0}\) which is a codimension 1 submanifold of \(T_g\). Find local coordinates \((q_2, \ldots, q_{6g-6})\) for \(N_{\phi_0}\) with 0 corresponding to \(X_0\). We now define a map \(f\) from a neighborhood of 0 in \(R^{6g-6}\) into \(T_g\). Given \((q_1, q_2, \ldots, q_{6g-6})\) let \(X \in N_{\phi_0}\) have coordinates \((q_2, \ldots, q_{6g-6})\) and let \(\phi \in E_F \cap \mathcal{E}_1 \cap H^{-1}(1)\) be such that \(\pi(\phi) = X\). Then let \(\phi_{q_1}\) be the quadratic differential found by flowing time \(q_1\) from \(\phi\). Set

\[
f(q_1, \ldots, q_{6g-6}) = \pi(\phi_{q_1}).
\]

If we can show that \(f\) is a local diffeomorphism, then \((q_1, \ldots, q_{6g-6})\) will serve as local coordinates for \(T_g\) near \(X\). Since \(v(\phi_{q_1}) = e^{q_1}v(\phi_0)\), these coordinates will satisfy (1) and (2). To see that \(f\) is smooth note that \(f\) can be written as a composite

\[
(q_1, \ldots, q_{6g-6}) \to (q_1, \phi) \to \phi_{q_1} \to \pi(\phi_{q_1})
\]

of smooth maps. We now show that \(Df\) is an isomorphism at 0 and then apply the inverse function theorem.

First we note that for \(i \geq 2\), \(\mu_i = Df(0)(\partial/\partial q_i)\) are independent vectors in the tangent space to \(N_{\phi_0}\) at \(X_0\). Thus we need to prove that \(\mu_1 = Df(0)(\partial/\partial q_1)\)
is a nonzero vector that is not tangent to $N_{\phi_0}$. But $\mu_1$ is a unit vector tangent to the Teichmuller geodesic determined by $\phi_0$. Thus $\mu_1 = \frac{\phi_0}{|\phi_0|}$ and so

$$\langle \phi_0, \mu_1 \rangle = 1.$$  

We now rely on a result from [3]. We introduce a function $G : T_g \rightarrow \mathbb{R}$. For each $X \in T_g$ by the Main Theorem of [4] there exists a unique $\psi \in Q(X)$ such that $v(\psi) = F$. Define

$$G(X) = \log ||\psi||.$$  

Then

$$G^{-1}(G(X_0)) = G^{-1}(\log||\phi_0||) = G^{-1}(0) = N_{\phi_0}.$$  

Then ([3], Theorem 5, p. 217) $G$ is smooth and the derivative of $G$ at $X$ in the direction of $\mu$ is given by the formula

$$DG(X)[\mu] = 2\langle \psi, \mu \rangle = \Re \int_X 2\mu \psi.$$  

Since $G = 0$ on $N_{\phi_0}$,

$$\langle \phi_0, \mu \rangle = 0$$  

for all $\mu$ tangent to $N_{\phi_0}$. Since $\langle \phi_0, \mu_1 \rangle = 1$, $\mu_1$ is not tangent to $N_{\phi_0}$. □

**Proof of Theorem**

We begin by proving that Hamilton's equations are satisfied along each Teichmuller geodesic in the principle stratum $\mathcal{S}_1$. Introduce the coordinates $(q_1, \ldots, q_{6g-6})$ in a neighborhood of $X_0$ given by Proposition 1. They define symplectic coordinates

$$(q_1, \ldots, q_{6g-6}, p_1, \ldots, p_{6g-6})$$  

for $\mathcal{S}$ in a neighborhood of $\phi_0$. First let $\phi \in E_F \cap \mathcal{S}_1 \cap H^{-1}(1)$. Then $\pi(\phi)$ has coordinates $(0, q_2, \ldots, q_{6g-6})$. An argument similar to that given in Proposition 1 shows that the coordinates of $\phi_{q_1}$ are

$$(2.1) \quad (q_1, \ldots, q_{6g-6}, 1, \ldots, 0).$$  

For by construction, the $q$ coordinates are $q_1, \ldots, q_{6g-6}$. For each $t$ let $F_t = v(\phi_t) = e^t v(\psi)$. For each $X \in T_g$ let

$$G_t(X) = \log ||\psi||,$$  

where $\psi \in Q(X)$ is the unique quadratic differential such that $v(\psi) = F_t$. Then $G_t = 0$ on the fiber $\{(q_1, \ldots, q_{6g-6}) : q_1 = t\}$, so

$$\langle \phi_t, \mu \rangle = 0$$  

for all $\mu$ tangent to the fiber or, in other words for $i \geq 2$,

$$\langle \phi_t, \partial / \partial q_i \rangle = 0.$$  

This implies $\phi_t = rdq_1$ for $r \in \mathbb{R}$. Since $\partial / \partial q_1$ is tangent to the Teichmuller geodesic in the direction of positive time, in fact $r > 0$. However since $\mu_1 = \partial / \partial q_1$ is a unit vector in the Teichmuller metric, $H(dq_1) = 1$. Since $H(\phi_t) =
1, \( \phi_t = dq_1 \) and so \( \phi_t \) has coordinates \((t, q_2, \ldots, q_{6g-6}, 1, 0 \ldots, 0)\), proving (2.1). We also note that

\[
\mu_1 = \frac{\dot{\phi}_t}{|\phi_t|}.
\]

Now let \( \phi_{0,t} \) be the path through \( \phi_0 \), so by (2.1) it has coordinates

\((t, 0, \ldots, 0, 1, 0, \ldots, 0)\).

Then along \( \phi_{0,t} \),

\[
(2.2) \quad dq_i/dt = \begin{cases} 1 & i = 1 \\ 0 & i \neq 1 \end{cases}, \quad dp_i/dt = 0, \quad dq_i/dt = dp_i/dt = 0, \quad i \neq 1.
\]

Since \( H(\phi_{0,t}) = 1 \) where \( \phi_{0,t} \) has coordinates \((q_1, q_2, \ldots, q_{6g-6}, 1, \ldots, 0)\),

\[
(2.3) \quad \partial H/\partial q_i(t, 0, \ldots, 0, 1, 0, \ldots, 0) = 0.
\]

Since \( H((1+s)\phi_{0,t}) = (1+s)^2 H(\phi_{0,t}) = (1+s)^2 \), we have

\[
(2.4) \quad \partial H/\partial p_i(t, 0, \ldots, 0, 1, 0, \ldots, 0) = \frac{1}{2} \frac{d(1+s)^2}{ds}(0) = 1.
\]

Finally we apply a formula of Royden's [7]. For \( \chi, \psi \in Q(X) \),

\[
\frac{d||\chi + t\psi||}{dt}(0) = \text{Re} \int_X \psi \frac{\bar{\chi}}{||\chi||}.
\]

This is applied with

\[
\chi = \phi_{0,t} = (t, 0, \ldots, 0, 1, 0, \ldots, 0)
\]

and

\[
\psi = dq_i = (t, 0, \ldots, 0, 0, \ldots, 1, \ldots, 0).
\]

Since \( ||\phi_{0,t}|| = 1 \) and \( \frac{\dot{\phi}_{0,t}}{|\phi_{0,t}|} = \mu_1 = \partial/\partial q_1 \), for \( i \geq 2 \),

\[
(2.5) \quad \partial H/\partial p_i(t, 0, \ldots, 0, 1, 0, \ldots, 0) = \frac{d}{ds}||\phi_{0,t} + sdq_i||_i(s = 0)
\]

\[
= \text{Re} \int dq_i \mu_1 = (dq_i, \partial/\partial q_1) = 0.
\]

We conclude from (2.2) and (2.3)-(2.5) that Hamilton's equations are satisfied along \( \phi_{0,t} \).

To finish the proof of the Theorem we need to discuss the lower dimensional strata \( \mathcal{S}_k \). We begin by recalling some results proved in [4]. Suppose \( q_0 \in \mathcal{S}_k \) is a quadratic differential on the Riemann surface \( X \). Let \( \Lambda_{q_0} \) be the sheaf of germs of vector fields \( \chi \) such that

\[
q_0(\chi, \chi) = \text{constant}.
\]

For \( k \geq 2 \) let \( P_k \) be the set of polynomials of the form

\[
z^k + a_{k-2}z^{k-2} + \ldots + a_0,
\]

and \( S_k \) the set of polynomials of the form

\[
a_{k-2}z^{k-2} + \ldots + a_0,
\]

the tangent space to \( P_k \) at \( z^k \). Suppose \( \phi_0 \) has zeroes of order \( k_1, \ldots, k_n \).

In a neighborhood of the zero of order \( k_i \) there are coordinates \( z \) such that
\( \phi_0 = z^k d z^2 \). Let \( U \) be a small neighborhood of \( \phi_0 \) in \( \mathcal{C} \). There is an analytic map

\[ f : U \to \prod_{i=1}^n P_{k_i} \]

classifying the deformations of the zeroes of \( \phi_0 \). Then \( \mathcal{C}_K \) is defined near \( \phi_0 \) by

\[ f^{-1}(z^{k_1}, \ldots, z^{k_n}) . \]

If \( \phi_0 \) is not the square of an abelian differential, then by [4], Proposition 4.7, the derivative of \( f \) is onto \( \bigoplus S_{k_i} \) and there is an exact sequence

\[ 0 \to H^1(X, \Lambda_{\phi_0}) \to T_{\phi_0} \mathcal{C} \to \bigoplus S_{k_i} \to 0. \]

If \( \phi_0 \) is the square of an abelian differential, choose a small circle \( \gamma_i \) about the zero and define a map \( \alpha_i : U \to \mathbb{C} \) by \( \phi \to \int_{\gamma_i} \phi^{1/2} d z \). Here the branch of \( \phi^{1/2} \) is chosen to be near \( z^{k_i/2} \) for \( \phi \) near \( \phi_0 \). Then [4], Lemma 4.8, says that the map \( f \) is a submersion onto the submanifold defined by the equation \( \sum \alpha_i(\phi) = 0 \). Now there is an exact sequence

\[ 0 \to H^1(X, \Lambda_{\phi_0}) \to T_{\phi_0} \mathcal{C} \to \bigoplus S_{k_i} \to C \to 0. \]

In either case the implicit function theorem says that \( \mathcal{C}_K \) is an analytic submanifold of \( \mathcal{C} \); \( T_{\phi_0} \mathcal{C} = T_{\phi_0} \mathcal{C}_K \oplus S_{k_i} \) in the first case, and \( T_{\phi_0} \mathcal{C} = T_{\phi_0} \mathcal{C}_K \oplus S \) where \( S \) is codimension 1 subspace of \( \bigoplus S_{k_i} \) in the second.

**Proposition 2.** The Teichmüller flow restricted to \( \mathcal{C}_K \) is real analytic.

**Proof.** We may triangulate the underlying surface of \( \phi_0 \) so that the edges are geodesic segments joining the zeroes of \( \phi_0 \) and the triangles have no zeroes in their interior. Let \( p \) be the dimension of \( \mathcal{C}_K \). There is a choice of \( p \) edges \( e_i \) of the triangulation such that the holonomy vectors \( \text{hol}_{1}(e_1), \text{hol}_{2}(e_1), \ldots, \text{hol}_{1}(e_p), \text{hol}_{2}(e_p) \) serve as local coordinates for \( \mathcal{C}_K \) near \( \phi_0 \). The Teichmüller flow preserves \( \mathcal{C}_K \) and in terms of the holonomy vectors it is described by \( (1.1) \), so is analytic. \( \Box \)

We continue with the proof of the Theorem. Choose \( \phi \) near \( \phi_0 \) which has simple zeroes and such that the critical vertical trajectories of \( \phi \) in each neighborhood of the zeroes of \( \phi_0 \) form a connected set of edges \( e_i \). Again let \( f : \mathcal{C} \to \prod P_{k_i} \) be the map classifying the deformations of the zeroes of \( \phi_0 \) . We may express

\[ f(\phi(z)) = \prod(z - r_i)d z^2 = (z^k + a_{k-1}z^{k-1} + \ldots)d z^2. \]

Let \( p_s \) be the family of polynomials

\[ p_s = \prod(z - s^{1/l}r_i) = z^k + s a_{k-1}z^{k-1} + \ldots , \]

which converge to \( z^k \) as \( s \to 0 \). It is easy to check by a change of variables that this family also has the property that the critical vertical trajectories also form a connected set of edges \( e_i \). Moreover the holonomy vector \( \text{hol}_i(s) \) of \( e_i \) at time \( s \) satisfies

\[ \frac{\text{hol}_i(s_1)}{\text{hol}_i(s_2)} = (\frac{s_1}{s_2})^{(k/2+1)}, \]

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which says in particular that the change in holonomy vector is by a constant
factor independent of $e_i$. From this we see that
\begin{equation}
\lim_{s \to 0} \frac{d}{ds} \text{hol}_i(s) = \lim_{s \to 0} \frac{(k/2 + 1)}{ls} = \infty.
\end{equation}

Let $\phi_s \to \phi_0$ a family of quadratic differentials so that
\begin{equation}
f(\phi_s(z)) = p_s(z).
\end{equation}

We may find a set $e_{i,s}$, $i = 1, \ldots, 6g - 6$, of edges of $\phi_s$ whose holonomy
vectors serve as local coordinates for $\mathcal{C}$ near $\phi_s$ such that for $i \leq p$ the edges
$e_{i,s}$ converge to the edges $e_i$ of $\phi_0$ that determine local coordinates for $\mathcal{C}_K$ and
for $i > p$ are vertical edges in the neighborhood of the zeroes of $\phi_0$. Now for each $s$
and $t$ consider the flow $\phi_s \to \phi_{s,t}$. Since Teichmuller maps
contract the holonomy of the vertical edges $e_{i,s}$, $i \geq p+1$, in the neighborhood
of the zeroes by a constant factor independent of the edge $e_{i,s}$, there must be
$s' = s'(s, t)$ such that
\begin{equation}
f(\phi_{s,t}) = p_{s'}.
\end{equation}

Let $v_{s,t}$ be the tangent vector to the flow $\phi_s \to \phi_{s,t}$ at $\phi_{s,t}$. Then $Df(v_{s,t})$
is tangent to the family $p_s$ at $s = s'$. By (1.1) at time $s'$ we have
\begin{equation}
\frac{D\text{hol}_i(s')}{\text{hol}_i(s')} (Df(v_{s,t})) = -e^{-t}
\end{equation}
and this is independent of $s$; in particular this quantity does not go to infinity
as $s \to 0$. The tangent vector $Df(v_{s,t})$ is a multiple $\lambda(s')$ of the tangent vector
to the family $p_s$ at $s'$, and comparing (2.6) and (2.7) we see that $\lambda(s') \to 0$ as
$s \to 0$. Thus $Df(v_{s,t}) \to 0$ as $s \to 0$ for each $t$ and we conclude that as $s \to 0$
any convergent subsequence of tangent vectors to the flow at $\phi_{s,t}$ converges to
a vector tangent to the stratum $\mathcal{C}_K$; namely an element of $H^1(X, \Lambda_{6g})$. We
may interpret such an element as infinitesimal change in the holonomy of the
edges $e_i$. Since $\text{hol}_i(s) \to \text{hol}_i(e)$ for $i \leq p$, by formula (1.1), the limit must
be tangent to the flow through $\phi_0$. In other words the tangent vector
\begin{equation}
(dq_1/dt, \ldots, dp_{6g-6}/dt)
\end{equation}
to the flow at $\phi_{s,t}$ converges to the tangent vector to the flow through $\phi_0$ at
time $t$ as $s \to 0$. The vector field
\begin{equation}
(\partial H/\partial p_i, -\partial H/\partial q_i)
\end{equation}
is continuous on $\mathcal{C}$ ([7]). Since Hamilton’s equations are satisfied along the
flow through $\psi$, by continuity they are satisfied along the flow through $\phi_0$. □

From the work of [5], [6], and [8], $\mathcal{C}_1$ has an absolutely continous measure
$\rho$ invariant under the Teichmuller flow and invariant under the action of the
mapping class group $\text{Mod}(g)$. In the local coordinates defined by holonomy
vectors \{$\text{hol}_1(e_j), \text{hol}_2(e_j)$\}, $j = 1, 6g - 6$, the measure is described by
\begin{equation}
d\rho = d \text{hol}_1(e_1) \wedge d \text{hol}_2(e_1) \wedge \ldots \wedge d \text{hol}_2(e_{6g-6}).
\end{equation}

**Corollary.** We have $d\rho = dq_1 \wedge \ldots \wedge dq_n \wedge dp_1 \ldots \wedge dp_n$.

**Proof.** The measure $dq_1 \wedge \ldots \wedge dq_n \wedge dp_1 \ldots \wedge dp_n$ is absolutely continous
with respect to $\rho$. Each measure is invariant under the Teichmuller flow on
$\mathcal{C}_1/\text{Mod}(g)$. Since $\rho$ is an ergodic measure for the flow [5], [8], the measures
must be equal. □
REFERENCES


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