ON THE ISOMORPHISM PROBLEM FOR BURNSIDE RINGS

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Dedicated to Professor H.-J. Volrath on his sixtieth birthday

Abstract. Nonisomorphic 3-groups of distinct nilpotency class are constructed with isomorphic Burnside rings.

1. The result

W. Burnside in 1911 (see [4, p. 236 ff]) introduced a ring associated to permutation representations of a finite group $G$ which is now called the Burnside ring $\Omega(G)$ of $G$. Indeed, let $M_1, \ldots, M_t$ be a set of representatives of the transitive permutation representations of $G$. Then every finite $G$-set $M$ decomposes as a disjoint union of transitive $G$-sets, so that we can write $M = \lambda_1 M_1 + \cdots + \lambda_t M_t$ for nonnegative integers $\lambda_i$. Moreover, if $M$ and $N$ are two $G$-sets, then there is a natural action of $G$ on the cartesian product $M \times N$ and, by the above, this can be written as an integral linear combination of the $M_i$. Allowing negative coefficients (i.e. using the usual Grothendieck construction), this yields a ring-structure on the set of all (generalized) permutation representations of $G$.

The isomorphism problem arises naturally. Indeed, let $G$ and $H$ be finite groups and assume that their Burnside rings $\Omega(G)$ and $\Omega(H)$ are ring-isomorphic. What can be said about $G$ if we know $H$ (see the survey article [8] for the analogous problem for group rings). As the additive group of $\Omega(G)$ is free abelian, freely generated by the representatives of transitive $G$-sets, we see that the number $t$ of conjugacy classes of subgroups of $G$ and $H$ are equal. Beyond this trivial remark, a celebrated result of A. Dress [5] says that the solubility of $H$ implies solubility of $G$. Furthermore, a number of other properties can be read off. However, see [9].

The ring $\Omega(G)$ is determined by the table of marks $M(G)$ of $G$ (see [7, Chapter 3]). Thus, for the isomorphism problem for Burnside rings, it is of interest to see what properties can be read off from $M(G)$. Note that the table of marks of $G$ determines the poset $\mathcal{E}(G)$ of conjugacy classes of subgroups of $G$ (see [7, p. 120]). The latter contains much less information about $G$. Indeed, all groups of order $pq$ have order-isomorphic posets of conjugacy classes. However, it was shown in [1] that $\mathcal{E}(G) \cong \mathcal{E}(H)$ and $H$ a noncyclic $p$-group

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implies that \(|G| = |H|\) and if \(H\) is abelian or metacyclic, then \(G\) and \(H\) are isomorphic (see [1] and [3]). Also, the result proved in [2] can be viewed in this context. Indeed, finite groups were classified with precisely one conjugacy class of nonnormal subgroups, and a trivial consequence of this result is that groups of equal order with this property are isomorphic.

It has been conjectured that \(\Omega(G) \cong \Omega(H)\) and \(H\) a \(p\)-group implies that \(G\) and \(H\) are of equal nilpotency class. The objective of this note is to construct a counterexample for this. Indeed, we shall prove:

**Theorem.** Let \(G = \langle x, y, z | x^9 = y^9 = z^9 = [x, z] = [y, z] = 1, [x, y] = z \rangle\) and \(H = \langle x, y, z | x^9 = y^9 = z^9 = [y, z] = 1, [x, z] = z^3, [x, y] = z \rangle\). Then \(\Omega(G) \cong \Omega(H)\). Moreover, \(|G| = 729 = |H|\) and \(G\) is nilpotent of class two and \(H\) is of class three.

In particular, the nilpotency class of a \(p\)-group cannot in general be read off from its Burnside ring (at least for \(p = 3\)). It seems likely that the analogous construction works for all primes \(p \geq 5\). However, it is not clear how to modify the groups for \(p = 2\). Also, it is not known to us whether \(c(G) \leq 2\) implies any bound for the class \(c(H)\) of \(H\).

2. The isomorphism

First of all, the groups \(G\) and \(H\) have been shown in [3] to have isomorphic posets of conjugacy classes, so that they were natural candidates to try. The tables of marks for \(G\) and \(H\) were calculated using the following sequence of GAP commands:

```gap
x := AbstractGenerator("x");;
y := AbstractGenerator("y");;
z := AbstractGenerator("z");;
G := Group(x, y, z);
G.relators := [x^9, y^9, z^9, x^-1*z^-1*x*z, y^-1*z^-1*y*z, x^-1*y^-1*x*y*z^-1];;
p := OperationCosetsFpGroup(G, Subgroup(G, [y]));
pp := TableOfMarks(p);
LogTo("M(G)");
DisplayTom(pp);
LogTo();;
```

and

```gap
a := AbstractGenerator("a");;
b := AbstractGenerator("b");;
c := AbstractGenerator("c");;
H := Group(a, b, c);
H.relators := [a^9, b^9, c^9, b^-1*c^-1*b*c, a^-1*b^-1*a*b*c^-1 - 1, a^-1*c^-1*a*c^-1 - 1*a*c^-2];
q := OperationCosetsFpGroup(H, Subgroup(H, [b]));
qq := TableOfMarks(q);
LogTo("M(H)");
DisplayTom(qq);
LogTo();;
```

This produced in files \(M(G)\) and \(M(H)\) two 87\times 87 lower triangular matrices that describe the multiplication of the corresponding Burnside rings in terms
of the representatives of conjugacy classes of subgroups that will be numbered by \(1 \ldots 87\). Instead of the \texttt{LogTo} and \texttt{DisplayTom} commands we could also use for example

\begin{verbatim}
ppp := MatTom(p);;
PrintTo("M(G)", ppp);;
\end{verbatim}

Clearly, a permutation \(\pi\) of the conjugacy classes of subgroups of \(H\) does not affect the ring structure of \(\Omega(H)\), and our problem was reduced to finding a suitable permutation matrix \(P\) related to \(\pi\) such that \(P^T M(H) P = M(G)\).

To boil down the possibilities for \(\pi\) for a systematic search, we used the following properties for \(G\) and \(M(G)\):

(a) Using CAYLEY at Bologna University, we determined that the automorphism group of \(G\) is of order \(2^4 \cdot 3^9\) and that \(G\) has an automorphism \(\varphi\) of order 8. As \(G\) is a two generator 3-group, a result of Burnside [4] on coprime automorphisms says that \(\varphi\) acts as an isomorphism of order 8 on the Frattini quotient \(G/\Phi(G) \cong \mathbb{Z}_3 \otimes \mathbb{Z}_3\) of \(G\). Hence \(\text{Aut}(G)\) acts transitively on the set of maximal subgroups of \(G\) and so without loss of generality, the representative 83 corresponding to one of the maximal subgroups of \(G\) may be assumed to be fixed.

(b) If the \(n \times n\) matrices \(M(G)\) and \(M(H)\) satisfy

\begin{equation}
M(G) = P^T M(H) P
\end{equation}

where \(P\) is a permutation matrix, then for all mixed products of the form \(q(A, A^T) := A^{k_1}(A^{m_1})^T \cdots A^{k_s}(A^{m_s})^T\) the analogous relation holds

\begin{equation}
q(M(G), M(G)^T) = P^T q(M(H), M(H)^T) P.
\end{equation}

To determine \(P\) with (1) we can consider for example \(q_1(A, A^T) := A^3\) or \(q_2(A, A^T) := A^2 A^T\) and determine properties that must hold for all permutation matrices (2). A solution \(P\) of (1) must satisfy such properties, too. Note that the matrices \(q_1(M(G), M(G)^T)\) have more distinct elements with lower frequency. Thus an inspection of \(q_1(M(G), M(G)^T)\) provides information about permutations satisfying (1).

(c) For given \(n \times n\) matrices \(A\) and \(B\) we look for a permutation \(\pi\) that satisfies \(A = P^T B P\). Each matrix \(A\) and every element \(s = A_{i,j}\) of \(A\) give rise to a graph \(G_A(s)\) defined by the knots \(P_1, \ldots, P_n\) and the set of edges \([(P_i, P_j)\] if \(A_{i,j} = s\). Permutations applied to \(A\) permute the knots of \(G_A(s)\) in the same way. Hence, \(G_A(s)\) and \(G_B(s)\) differ only by the permutation \(\pi\). Comparing these two graphs we can extract information about \(\pi\). As a trivial example let us consider the case that \(s\) appears only once in \(A\) and \(B\). Then \(G_A(s)\) is given by one edge \([(P_i, P_j)\] and \(G_B(s)\) by \([(P_k, P_m)\]. Hence, we can deduce that \(\pi\) satisfies \(\pi(k) = i\) and \(\pi(m) = j\).

Combining (a), (b), and (c) one can determine sufficiently many properties of permutation matrices \(P\) that satisfy (1). To compute the matrices \(q(A, A^T)\) and the related graphs \(G_{q(A, A^T)}(s)\) we used MATLAB. We determined one possible permutation \(\pi\) that transforms the matrix \(M(H)\) into the matrix \(M(G)\). This permutation \(\pi\) is given by

\begin{verbatim}
(25, 26)(30, 32)(44, 45)(50, 51)(56, 57)(66, 77, 72)
(62, 71, 65, 76, 70, 82, 80, 75, 69, 81, 64, 74, 68, 79, 63, 73, 67, 78).
\end{verbatim}
References


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