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NONLINEAR DEGENERATE ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS WITH CRITICAL GROWTH CONDITIONS ON THE GRADIENT

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ABSTRACT. We consider a nonlinear degenerate elliptic partial differential equation $-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = H(x, u, \nabla u)$ with the critical growth condition on $H(x, u, \nabla u) \leq g(x) + |\nabla u|^p$, where g is sufficiently integrable and p is between 1 and ∞. Our first goal of this paper is to prove the existence of the solution in $W_0^{1,p} \cap L^\infty$. The main idea is to obtain the uniform L^∞-estimate of suitable approximate solutions, employing a truncation technique and radially decreasing symmetrization techniques based on rearrangements. We also find an example of unbounded weak solution of $-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = |\nabla u|^p$ for $1 < p \leq n$.

1. INTRODUCTION

We show the existence of solutions to the following degenerate elliptic partial differential equations:

$$-\Delta_p u \equiv -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = H(x, u, \nabla u) \tag{1.1}$$

where u is in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. We suppose that $H(x, s, \xi)$ is a Carathéodory function and satisfies a critical growth condition

$$|H(x, s, \xi)| \leq g(x) + |\xi|^p \tag{1.2}$$

for all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^n$ and a.e. $x \in \Omega$, where $g \in L^q(\Omega)$, $q > \max\{1, \frac{n}{p}\}$.

A main step is to obtain the uniform L^∞-estimates for the solutions of suitable approximate equations, using the truncation technique and the radially decreasing symmetrization techniques based on rearrangement properties ([3], [6]). This method has been successfully applied to strongly elliptic equations by Ferone and Posteraro ([3]). Such an estimate allows us to prove the existence

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of solutions of (1.1) ([1]). The smallness condition on the measure of Ω and some norm of g are essential in the L^∞-estimates.

In section 3 we find some examples of unbounded weak solutions in $W_0^{1,p}$ when $H(x, s, \xi)$ is $|\xi|^p$. Since g of (1.2) is identically zero in this case, a bounded weak solution also exists on any bounded domain by Theorem 2.2. Here we find a spherical symmetric unbounded solution on $B_R(0)$ for $R < 1$ for all p and n such that $1 < p \leq n$.

Boccardo, Murat and Puel have shown the existence of the $W_0^{1,2} \cap L^\infty$ solutions of $-(a_{ij}u_{x_j})_{x_i} + a_0 u = f(x, u, \nabla u)$ with $|f| \leq C_0 + b(|u|)|\nabla u|^2$ where a_{ij} is bounded measurable, $a_0 > 0$ and b is a function on $\mathbb{R}^+$ ([1]). They also showed the same result for $-\Delta_p u + H(x, u, \nabla u) + a_0|u|^{p-1}\operatorname{sign} u = f - \operatorname{div} g$ with $|H| \leq C_0 + C_1|\nabla u|^p$ where a_0, C_0 and C_1 are strictly positive and f and g are suitably integrable ([2]). Ferone and Posteraro showed an existence result for $-(a_{ij}u_{x_j})_{x_i} = H(x, u, \nabla u) - \operatorname{div} f$ with H satisfying (1.2) for $p = 2$ when f is suitably integrable ([3]).

2. The existence of weak solutions

Suppose that Ω is a bounded open set of $\mathbb{R}^n$. If $\phi : \Omega \to \mathbb{R}$ is a measurable function and $\mu(t) = |\{x \in \Omega : \phi > t\}|$, $t \geq 0$, is the distribution function of ϕ, then $\phi^*(s) = \sup\{t \geq 0 : \mu(t) > s\}$, $s \in [0, |\Omega|]$, is called decreasing rearrangement. We recall some properties of decreasing rearrangements.

Lemma 2.1. *Suppose that u is in $L^p(\Omega)$, v is in $L^{p'}(\Omega)$, $\frac{1}{p} + \frac{1}{p'} = 1$ and F is a nonnegative increasing continuous function on $\mathbb{R}^+ \cup \{0\}$. Then the following are true.*

(a) $(F(u))^*(s) = F(u^*(s))$ *a.e.*

(b) *If $\{u_n\}_{n=1}^\infty$ converges to u in L^p, then $\{u_n^*\}_{n=1}^\infty$ also converges to u^* in L^p.*

(c) $\displaystyle\int_\Omega uv\,dx \leq \int_0^{|\Omega|} u^* v^*\,ds$.

The proof of this lemma can be found in [6].

Now we state the main theorem.

Theorem 2.2. *Let us assume that condition* (1.2) *holds and*

$$\int_0^{|\Omega|} \left(\frac{1}{n^p \omega_n^{\frac{p}{n}} s^{p-\frac{p}{n}}(p-1)} \int_0^s g^*(\sigma)\,d\sigma \right)^{\frac{1}{p-1}} ds < 1. \tag{2.1}$$

Then there exists at least one solution of (1.1).

Remark. The left-hand side of (2.1) is bounded if $g \in L^q(\Omega)$, $q > \frac{n}{p}$. When $p = 2$, a similar theorem can be found in [3].

Proof. Let us put, for $\varepsilon > 0$,

$$H_\varepsilon(x, s, \xi) = \frac{H(x, s, \xi)}{1 + \varepsilon|H(x, s, \xi)|}.$$

We still have

$$|H_\varepsilon(x, s, \xi)| \leq g(x) + |\xi|^p.$$

Furthermore $|H_\varepsilon|$ is bounded and hence a solution u_ε exists for the problem

$$-\Delta_p u_\varepsilon = H_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon), \tag{2.2}$$

where $u_\varepsilon \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ (see [7]). We will omit ε ahead for simple presentation.

If u is a solution of (2.2), we have

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\phi\, dx = \int_\Omega H(x, u, \nabla u)\phi\, dx$$

for all $\phi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. Choosing the test function

$$\phi_t(x) = e^{(p-1)|u|}(e^{|u|} - 1 - t)^+ \operatorname{sign} u$$

and putting $w = e^{|u|} - 1$, we have

$$\int_{w>t} |\nabla u|^{p-2}\nabla u \cdot \nabla u e^{(p-1)|u|}\left(pe^{|u|} - 1 - t\right) dx$$
$$= \int_{w>t} H(x, u, \nabla u)e^{(p-1)|u|}(e^{|u|} - 1 - t) \operatorname{sign} u\, dx.$$

For $h > 0$ we also apply the test function ϕ_{t+h} to (2.2). Building the differential quotient, we get

$$\frac{1}{h}\int_{t<w\le t+h} |\nabla u|^p e^{(p-1)|u|}(pe^{|u|} - 1 - t)\, dx + \int_{w>t+h} |\nabla u|^p e^{(p-1)|u|}\, dx$$
$$= \frac{1}{h}\int_{t<w\le t+h} H(x, u, \nabla u)e^{(p-1)|u|}(e^{|u|} - 1 - t) \operatorname{sign} u\, dx$$
$$+ \int_{w>t+h} H(x, u, \nabla u)e^{(p-1)|u|} \operatorname{sign} u\, dx.$$

Sending h to 0, we get that

$$\frac{1}{h}\int_{t<w\le t+h} |\nabla u|^p (p-1)e^{p|u|}\, dx \to (p-1)\left(-\frac{d}{dt}\int_{w>t} |\nabla w|^p\, dx\right)$$

and

$$\frac{1}{h}\int_{t<w\le t+h} |\nabla u|^p e^{(p-1)|u|}(e^{|u|} - 1 - t)\, dx \to 0.$$

Similarly we see that

$$\frac{1}{h}\int_{t<w\le t+h} |H(x, u, \nabla u)|e^{(p-1)|u|}(e^{|u|} - 1 - t) \operatorname{sign} u\, dx \to 0$$

and

$$\int_{w>t+h} H(x, u, \nabla u)e^{(p-1)|u|} \operatorname{sign} u\, dx \le \int_{w>t} (g(x) + |\nabla u|^p)\, e^{(p-1)|u|}\, dx.$$

Considering the above and Lemma 2.1 we get that

$$(p-1)\left(-\frac{d}{dt}\int_{w>t}|\nabla w|^p\,dx\right) \le \int_{w>t} g(w+1)^{p-1}\,dx$$
$$\le \int_0^{\mu(t)} g^*(s)(w^*(s)+1)^{p-1}\,ds.$$

We use the inequality (see [8])

$$n\omega_n^{\frac{1}{n}}\mu(t)^{1-\frac{1}{n}} \le (-\mu'(t))^{1-\frac{1}{p}}\left(-\frac{d}{dt}\int_{w>t}|\nabla w|^p\,dx\right)^{\frac{1}{p}}, \tag{2.3}$$

where $\mu(t)$ is the distribution function of w. Thus we get

$$\frac{(p-1)n^p\omega_n^{\frac{p}{n}}\mu(t)^{p-\frac{p}{n}}}{(-\mu'(t))^{p-1}} \le \int_0^{\mu(t)} g^*(s)(w^*(s)+1)^{p-1}\,ds.$$

Then it is easy to obtain

$$(-w^*(s))' \le \left(\frac{1}{(p-1)n^p\omega_n^{\frac{p}{n}}s^{p-\frac{p}{n}}}\int_0^s g^*(\sigma)(w^*(\sigma)+1)^{p-1}\,d\sigma\right)^{\frac{1}{p-1}}.$$

Integrating both sides with respect to s, we obtain

$$w^*(0) \le \int_0^{|\Omega|}\left(\frac{1}{(p-1)n^p\omega_n^{\frac{p}{n}}s^{p-\frac{p}{n}}}\int_0^s g^*(\sigma)\,d\sigma\right)^{\frac{1}{p-1}} ds\,(w^*(0)+1), \tag{2.4}$$

since $w^*(|\Omega|)=0$ and w^* attains its maximum at 0.

From (2.1) and (2.4) w^* must be bounded. Thus we get a uniform L^∞-estimate of u independent of ε, and this completes the proof. □

Now we prove the uniform $W^{1,p}$-estimates.

Lemma 2.3. *If u_ε is the solution of* (2.2) *and $\|u_\varepsilon\|_{L^\infty}$ are bounded uniformly for all ε, then*

$$\|u_\varepsilon\|_{W_0^{1,p}} \le C(\|g\|_{L^1}, |\Omega|, n, p).$$

Proof. Let us take $u_\varepsilon e^{u_\varepsilon^2}$ as a test function. From (1.2) we get

$$\int_\Omega |\nabla u_\varepsilon|^p e^{u_\varepsilon^2}(1+2u_\varepsilon^2)\,dx \le \int_\Omega g u_\varepsilon e^{u_\varepsilon^2}\,dx + \int_\Omega |\nabla u_\varepsilon|^p u_\varepsilon e^{u_\varepsilon^2}\,dx.$$

By Young's inequality we obtain

$$\int_\Omega |\nabla u_\varepsilon|^p e^{u_\varepsilon^2}(1+2u_\varepsilon^2)\,dx \le \int_\Omega g u_\varepsilon e^{u_\varepsilon^2}\,dx + \int_\Omega |\nabla u_\varepsilon|^p\left(\frac{1}{2}+2u_\varepsilon^2\right)e^{u_\varepsilon^2}\,dx.$$

Consequently we get

$$\frac{1}{2}\int_\Omega |\nabla u_\varepsilon|^p e^{u_\varepsilon^2}\,dx \le \int_\Omega g u_\varepsilon e^{u_\varepsilon^2}\,dx.$$

Note that $1 \le e^{u_\varepsilon^2}$ and $u_\varepsilon e^{u_\varepsilon} \le C(n, p, \|g\|_{L^1}, |\Omega|)$. Using Sobolev's theorem, we obtain

$$\|u_\varepsilon\|_{W_0^{1,p}(\Omega)} \le C(n, p, \|g\|_{L^1}, |\Omega|). \quad \square$$

Since $W_0^{1,p}$ is a reflexive Banach space and u_ε are bounded uniformly in $W_0^{1,p}$, there are u and a sequence $\{u_{\varepsilon_k}\}$ in $W_0^{1,p}$ such that

$$u_{\varepsilon_k} \rightharpoonup u \quad \text{weakly in } W_0^{1,p} \tag{2.5}$$

and

$$u_{\varepsilon_k} \to u \quad \begin{cases} \text{strongly in } L^q & \text{for } q < \frac{np}{n-p} \quad \text{if } p < n, \\ \text{strongly in } L^q & \text{for } q < \infty \quad \text{if } p = n, \\ \text{uniformly} & \text{if } p > n. \end{cases} \tag{2.6}$$

We show that u is a $W^{1,p}(\Omega) \cap L^\infty(\Omega)$ solution of (1.1). The following lemma shows the strong convergence of ∇u_{ε_k} in L^p. We will omit the index k of u_{ε_k} for the sake of convenience.

Lemma 2.4. *Assume the same hypothesis of Lemma* 2.3 *and* $\{u_\varepsilon\}$ *is the sequence obtained above. Then*

$$\nabla u_\varepsilon \to \nabla u \quad \textit{strongly in } L^p. \tag{2.7}$$

Proof. Let $v_\varepsilon = u_\varepsilon - u$. We take $v_\varepsilon e^{v_\varepsilon^2}$ as a test function. From direct calculations and Young's inequality we get

$$\begin{aligned} &\frac{1}{2}\int_\Omega |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \cdot \nabla v_\varepsilon e^{v_\varepsilon^2}\,dx \\ &\quad \le \int_\Omega g v_\varepsilon e^{v_\varepsilon^2}\,dx + \int_\Omega |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \cdot \nabla u\, v_\varepsilon e^{v_\varepsilon^2}\,dx. \end{aligned}$$

Subtracting $\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v_\varepsilon e^{v_\varepsilon^2}\,dx$ from both sides, we obtain

$$\begin{aligned} &\int_\Omega (|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon - |\nabla u|^{p-2}\nabla u)\cdot(\nabla u_\varepsilon - \nabla u)e^{v_\varepsilon^2}\,dx \\ &\quad \le \int_\Omega g v_\varepsilon e^{v_\varepsilon^2}\,dx + \int_\Omega |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \cdot \nabla u\, v_\varepsilon e^{v_\varepsilon^2}\,dx \\ &\qquad - \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v_\varepsilon e^{v_\varepsilon^2}\,dx. \end{aligned} \tag{2.8}$$

For the first term of the right-hand side in (2.8) we obtain

$$\left|\int_\Omega g v_\varepsilon e^{v_\varepsilon^2}\,dx\right| \to 0 \quad \text{as } \varepsilon \to 0$$

for suitable q from (2.6). Let us consider the second term. By (2.6), Egoroff's theorem, Lemma 2.3 and the L^∞-estimate of u_ε we obtain that

$$\left|\int_\Omega |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \cdot \nabla u v_\varepsilon e^{v_\varepsilon^2}\,dx\right| \to \eta$$

as $\varepsilon \to 0$ for any $\eta > 0$. Now we consider the third term. $e^{v_\varepsilon^2} - 1$ goes to zero almost everywhere and is bounded. By (2.5) and the same method used in the

control of the second term we obtain that

$$\left|\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v_\varepsilon e^{v_\varepsilon^2}\, dx\right| \to \eta$$

as $\varepsilon \to 0$ for any $\eta > 0$.

Thus

$$\int_\Omega (|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon - |\nabla u|^{p-2}\nabla u)\cdot(\nabla u_\varepsilon - \nabla u)\, dx \to 0 \quad \text{as } \varepsilon \to 0.$$

If $p \geq 2$, then

$$\begin{aligned}&\int_\Omega |\nabla u_\varepsilon - \nabla u|^p\, dx\\ &\leq C\int_\Omega (|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon - |\nabla u|^{p-2}\nabla u)\cdot(\nabla u_\varepsilon - \nabla u)\, dx.\end{aligned}$$

If $1 < p < 2$, then

$$\begin{aligned}&\int_\Omega |\nabla u_\varepsilon - \nabla u|^p\, dx\\ &\leq C\left(\int_\Omega (|\nabla ue| + |\nabla u|)^p\, dx\right)^{\frac{2}{2-p}}\\ &\qquad \left(\int_\Omega (|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon - |\nabla u|^{p-2}\nabla u)\cdot(\nabla u_\varepsilon - \nabla u)\, dx\right)^{\frac{p}{2}}\\ &\leq C\left(\int_\Omega (|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon - |\nabla u|^{p-2}\nabla u)\cdot(\nabla u_\varepsilon - \nabla u)\, dx\right)^{\frac{p}{2}}.\end{aligned}$$

Therefore

$$\int_\Omega |\nabla u_\varepsilon - \nabla u|^p\, dx \to 0 \quad \text{as } \varepsilon \to 0. \quad \square$$

By Lemma 2.4 and the Lebesgue Convergence Theorem, for any test function $\phi \in C_0^\infty(\Omega)$, we have

$$\begin{aligned}\int_\Omega H(x, u, \nabla u)\phi\, dx &= \lim_{\varepsilon\to 0}\int_\Omega H_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)\phi\, dx\\ &= \lim_{\varepsilon\to 0}\int_\Omega |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \cdot \nabla\phi\, dx\\ &= \int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\phi\, dx.\end{aligned}$$

Therefore a solution satisfying (1.1) exists in $W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$.

3. The examples of unbounded weak solution

In this section we find some unbounded weak solutions for the equations of the form (1.1). We consider the equation

$$-\Delta_p u = |\nabla u|^p \quad \text{for } 1 < p \leq n. \tag{3.1}$$

We find some examples of unbounded weak solutions of (3.1).

Theorem 3.1. *There is an unbounded solution of* (3.1) *on* $B_R(0)$ *for* $0 < R < 1$.

Proof. At first we consider the case of $1 < p < n$. For $R < 1$ we let

$$u_p(x) = \int_R^{|x|} \frac{n-p}{r^{\frac{n-1}{p-1}} - r}\, dr.$$

Note that $u_p(x) = 0$ for $|x| = R$ and

$$\nabla u_p(x) = \frac{n-p}{|x|^{\frac{n-1}{p-1}} - |x|} \frac{x}{|x|}, \tag{3.2}$$

and

$$\begin{aligned}\int_{B_R(0)} |\nabla u_p(x)|^p\, dx &= \int_{S^{n-1}} \int_0^R \left| \frac{n-p}{r^{\frac{n-1}{p-1}} - r} \right|^p r^{n-1}\, dr\, d\omega \\ &\le (n-p)^p \int_{S^{n-1}} 1\, d\omega \int_0^R \frac{r^{n-1}}{r^p(1 - r^{\frac{n-p}{p-1}})}\, dr < \infty.\end{aligned}$$

Thus u_p is contained in $W_0^{1,p}(B_R(0))$. Furthermore we see that u_p is contained in $W_0^{1,q}(B_R(0))$ for any $q < n$. We also observe

$$\begin{aligned}u_p(x) &\ge \int_{|x|}^R \frac{n-p}{r(1 - r^{\frac{n-p}{p-1}})}\, dr \\ &\ge C \int_{|x|}^R \frac{1}{r}\, dr \to \infty \quad \text{as } |x| \to 0,\end{aligned}$$

that is, u_p is unbounded. Now let us calculate $-\Delta_p u_p$. From (3.2)

$$\begin{aligned}-\Delta_p u_p(x) &= \sum_{i=1}^n \left(\frac{(n-p)^{p-1}}{(|x| - |x|^{\frac{n-1}{p-1}})^{p-1}} \frac{x_i}{|x|} \right)_{x_i} \\ &= \frac{(n-p)^{p-1}}{(|x| - |x|^{\frac{n-1}{p-1}})^p |x|} \left(n(|x| - |x|^{\frac{n-1}{p-1}}) - (p|x| - n|x|^{\frac{n-1}{p-1}}) \right) \\ &= \frac{(n-p)^p}{(|x| - |x|^{\frac{n-1}{p-1}})^p} = |\nabla u_p|^p.\end{aligned}$$

Thus $u_p(x)$ satisfies the problem (3.1) for p which is less than n.

Now we consider the next case of $p = n$. For $R < 1$ let

$$\begin{aligned}u_n(x) &= \int_R^{|x|} \frac{n-1}{r \log r}\, dr \\ &= (n-1) \left(\log\log \frac{1}{|x|} - \log\log \frac{1}{R} \right).\end{aligned}$$

Clearly $u_n(x)$ vanishes for $|x| = R$ and is unbounded on $B_R(0)$. From direct calculations we know u_n is contained in $W_0^{1,n}(B_R(0))$. It is also easy to show that u_n satisfies the problem (3.1) for $p = n$. □

Remark. We can find the same example for the case of $p = n = 2$ on pages 61–62 of the book [4].

Remark. If $p < n$ and $R < 1$, for any fixed x in $(0, R)$ $u_p(x)$ converges to $u_n(x)$ as p converges to n.

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