ACTIONS OF REGULAR MULTIPLICATIVE UNITARIES
ON C*-ALGEBRAS

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ABSTRACT. We establish a covariant representation theory for actions of regular multiplicative unitaries, in the sense of S. Baaj and G. Skandalis, on C*-algebras.

In [1] S. Baaj and G. Skandalis have established a formulation for locally compact quantum groups by using the notion of multiplicative unitaries. By adopting this formulation we will extend the notion of a C*-covariant system which usually consists of a C*-algebra and an action of a locally compact group on that C*-algebra. This extension gives a formulation for actions of locally compact quantum groups on C*-algebras which include actions and coactions of locally compact groups on C*-algebras. First we will state a definition of the crossed product and give some related results which are well known in the locally compact group case. One of the results is that there is a one-to-one correspondence between the covariant representations of a C*-covariant system and the nondegenerate *-representations of its crossed product. Secondly we will state a definition of a reduced crossed product and treat some amenable cases.

According to [1] we shall review multiplicative unitaries and those related things which will be used later. All Hilbert spaces and C*-algebras are assumed to be separable, and tensor products of C*-algebras are taken to be spatial ones throughout this paper. For a C*-algebra A and its closed two-sided ideal J, M(A) denotes the multiplier algebra of A and M(A; J) denotes \{a \in M(A) | aA + Aa \subseteq J\}. Let H be a Hilbert space. A unitary operator V on H \otimes H is said to be multiplicative, if it satisfies the so-called pentagon equation: V_{12}V_{13}V_{23} = V_{23}V_{12}. Furthermore V is called regular, if \{(id \otimes \omega)(\Sigma V) | \omega \in B(H)_*\} is norm-dense in K(H), where \Sigma denotes the flip operator on H \otimes H and K(H) is the compact operator on H. We have the following facts for a regular multiplicative unitary V.

(1) Define

$$S_V = \{(\omega \otimes id)(V) | \omega \in B(H)_*\}^{-1}_{\| \cdot \|}$$

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and 
\[ \hat{S}_V = \{ (\text{id} \otimes \omega)(V) \mid \omega \in B(H) \} \].

Then \( S_V \) and \( \hat{S}_V \) are \( C^* \)-subalgebras of \( B(H) \).

2. Define \( \delta_V(s) = V(s \otimes 1)V^* \) for \( s \in S_V \) and \( \delta_V(t) = V^*(1 \otimes t)V \) for \( t \in \hat{S}_V \). Then \( \delta_V \) and \( \hat{\delta}_V \) are coproducts on \( S_V \) and \( \hat{S}_V \) respectively, i.e. \( \delta_V: S_V \to M(\hat{S}_V \otimes S_V + S_V \otimes \hat{S}_V; S_V \otimes S_V) \) is a nondegenerate \( * \)-homomorphism such that
\[ (\delta_V \otimes \text{id})\delta_V = (\text{id} \otimes \delta_V)\delta_V, \]
\( \delta_V \) is injective,

and
\[ \delta_V(S_V)(1 \otimes S_V) \text{ and } \delta_V(S_V)(S_V \otimes 1) \text{ are total in } S_V \otimes S_V, \]
and similar properties hold for \( \hat{\delta}_V \) on \( \hat{S}_V \).

We will start with some definitions.

**Definition 1.** Let \( V \) be a regular multiplicative unitary on a Hilbert space \( H \). Then \( (A, V, \sigma) \) is called a \( C^* \)-covariant system, if

1. \( A \) is a \( C^* \)-algebra,
2. \( \sigma: A \to M(A \otimes S_V; A \otimes S_V) \) is a nondegenerate \( * \)-homomorphism such that
\[ (\sigma \otimes \text{id})\sigma = (\text{id} \otimes \delta_V)\sigma, \]
\( \sigma \) is injective,

and
\[ \sigma(A)(1 \otimes S_V) \text{ is total in } A \otimes S_V. \]

**Definition 2.** Let \( (A, V, \sigma) \) be a \( C^* \)-covariant system. \( (\pi, u) \) is called a covariant representation of \( (A, V, \sigma) \), if

1. \( \pi \) is a nondegenerate \( * \)-representation of \( A \) on some Hilbert space \( H_\pi \),
2. \( u \) is a unitary element of \( M(K(H_\pi) \otimes S_V) \) such that \( (\text{id} \otimes \delta_V)(u) = u_{12}u_{13} \),
3. \( (\pi \otimes \text{id})\sigma(x) = u(\pi(x) \otimes 1)u^* \) for all \( x \in A \).

**Definition 3.** Let \( (A, V, \sigma) \) be a \( C^* \)-covariant system. We define a seminorm on \( A \otimes_{\text{alg.}} (S_V^\omega)_\ast \) by
\[ \left\| \sum_{i=1}^{N} x_i \otimes \omega_i \right\| \]
\[ \equiv \sup \left\{ \left\| \sum_{i=1}^{N} \pi(x_i)(\text{id} \otimes \omega_i)(u) \right\| \mid (\pi, u) \text{ is a covar. rep. of } (A, V, \sigma) \right\} \]
\[ \leq \sum_{i=1}^{N} \| x_i \|_A \| \omega_i \|_{(S_V^\omega)_\ast} < \infty \]
for $x_i \in A$, $\omega_i \in (\overline{S_V^{\omega}})_*$ . We define

$$A \times_\sigma \hat{S}_V \equiv \| \cdot \|\text{-completion of } A \otimes_{\text{alg.}} (\overline{S_V^{\omega}})_* / (\| \cdot \| = 0),$$

which is a Banach space, and call it the crossed product of $(A, V, \sigma)$. Finally for a covariant representation $(\pi, u)$ of $(A, V, \sigma)$, we define a bounded linear mapping $\pi \times u$ of $A \times \hat{S}_V$ into $B(H_\pi)$ such that

$$(\pi \times u) \left( \sum_{i=1}^N x_i \otimes \omega_i \right) = \sum_{i=1}^N \pi(x_i)(\text{id} \otimes \omega_i)(u).$$

**Proposition 4.** $\pi \times u$ is nondegenerate for any covariant representation $(\pi, u)$ of $(A, V, \sigma)$.

*Proof.* By the nondegeneracy we mean that $(\pi \times u)[A \times_\sigma \hat{S}_V] \subset H_\pi$ is dense in $H_\pi$. For this it suffices to show that $\{(\text{id} \otimes \omega)(u)\omega \in (\overline{S_V^{\omega}})_* \}$. $H_\pi$ is dense in $H_\pi$. In fact, for any $x, n \in H_\pi \setminus \{0\}$, there exist $a, \beta \in H$ such that $\langle x, a \beta \rangle \neq 0$. But the left-hand side equals $\langle x, (\text{id} \otimes \omega)(u)a \rangle$, as we have shown. $\Box$

We will give a $C^*$-algebra structure on $A \times_\sigma \hat{S}_V$. For this purpose we need some lemmata.

**Lemma 5.** For $x_1, x_2 \in \hat{A}$, at least one of which belongs to $A$, and $\omega_1, \omega_2 \in (\overline{S_V^{\omega}})_*$, there exists a unique $X \in A \times_\sigma \hat{S}_V$ such that for any covariant representation $(\pi, u)$ of $(A, V, \sigma)$

$$(\pi \times u)(X) = (\pi \times u)(x_1 \otimes \omega_1) \cdot (\pi \times u)(x_2 \otimes \omega_2)$$

where we denote $(\pi \times u)(1 \otimes \omega_i) \equiv (\text{id} \otimes \omega_i)(u)$.

Therefore by setting $X$ to be the product of $(x_1 \otimes \omega_1)$ and $(x_2 \otimes \omega_2)$ we can define a multiplication on $A \times_\sigma \hat{S}_V$ with respect to which $A \times_\sigma \hat{S}_V$ is a Banach algebra.

*Proof.* If $x_2 \in C^1_A$, then we define $X \equiv x_1 \otimes (\omega_1 \otimes \omega_2)(\delta_V(\cdot)) \cdot \lambda_2$ and the statement is easily checked. So we assume $x_2 \in \hat{A}$. Let $(s_\mu)$ be an approximate identity of $S_V$. Since $\sigma(x_2) \in M(\hat{A} \otimes S_V ; A \otimes S_V)$,

$$\sigma(x_2)(1 \otimes s_\mu)(eA \otimes S_V) \rightarrow \sigma(x_2) \text{ strictly in } M(\hat{A} \otimes S_V).$$

So there exists $(\sum_{i=1}^{N_i} x_i^* \otimes s_i^*)| A \in \Lambda)$, a norm bounded net of $A \otimes_{\text{alg.}} S_V \subseteq A \otimes S_V$, such that

$$\sum_{i=1}^{N_i} x_i^* \otimes s_i^* \rightarrow \sigma(x_2) \text{ strictly in } M(\hat{A} \otimes S_V).$$

We next show

$$\sum_{i=1}^{N_i} x_i^* x_i \otimes (\omega_1 \otimes \omega_2)((s_i^* \otimes 1)\delta_V(\cdot)) \text{ converges in } A \times_\sigma \hat{S}_V.$$
For this it is sufficient to show that for any covariant representation \((\pi, u)\) of \((A, V, \sigma)\)

\[
(\pi \times u) \left( \sum_{i=1}^{N} x_i x_i^* \otimes (\omega_1 \otimes \omega_2)((s_i^2 \otimes 1)\delta_V(\cdot)) \right)
\]

converges in norm in \(B(H_n)\) uniformly in \((\pi, u)\). In fact, we have

\[
\forall \varepsilon > 0 \exists \lambda_0 \in \Lambda \\forall \lambda \geq \lambda_0 \forall (\pi, u)\text{(covariant rep.)}
\]

\[
\left\| (\pi \times u) \left( \sum_{i=1}^{N} x_i x_i^* \otimes (\omega_1 \otimes \omega_2)((s_i^2 \otimes 1)\delta_V(\cdot)) \right) - (\pi \times u)(x_1 \otimes \omega_1) \cdot (\pi \times u)(x_2 \otimes \omega_2) \right\|_{B(H_n)} < \varepsilon,
\]

by the estimation

\[
= \left\| \left( \sum_{i=1}^{N} \pi(x_i x_i^*)(\id \otimes \omega_1 \otimes \omega_2)((1_A \otimes s_i^2 \otimes 1)(\id \otimes \delta_V(u)) \right.ight.
\]

\[
- (\id \otimes \omega_1 \otimes \omega_2)((\pi \id \otimes \id)((x_1 \otimes 1 \otimes 1)(\sigma(x_2 \otimes 1))(\id \otimes \delta_V(u))\right)
\]

\[
\leq \left\|((\id \otimes \omega_1 \otimes \omega_2) - (\id \otimes \omega_1 s_1 \otimes \omega_2))
\]

\[
\cdot \left((\pi(x_1) \otimes 1 \otimes 1)(\pi \id \otimes \id) \left( \sum_{i=1}^{N} x_i x_i^* \otimes s_i^2 \otimes 1 \right)(\id \otimes \delta_V(u)) \right)
\]

\[
+ \left\|((\id \otimes \omega_1 \otimes \omega_2)(\pi \id \otimes \id)
\]

\[
\cdot \left(\left( (x_1 \otimes s_1) \left( \sum_{i=1}^{N} x_i x_i^* \otimes s_i^2 \right) - \sigma(x_2) \right) \otimes 1 \right)(\id \otimes \delta_V(u)) \right)
\]

\[
+ \left\|((\id \otimes \omega_1 s_1 \otimes \omega_2) - (\id \otimes \omega_1 \otimes \omega_2))
\]

\[
\cdot ((\pi \id \otimes \id)((x_1 \otimes 1 \otimes 1)(\sigma(x_2 \otimes 1))(\id \otimes \delta_V(u))\right)
\]

for any \(s_1 \in S_V\). Hence there exists

\[
X \equiv \lim_{N \to \infty} \sum_{i=1}^{N} x_i x_i^* \otimes (\omega_1 \otimes \omega_2)((s_i^2 \otimes 1)\delta_V(\cdot)) \in A \times_\sigma \hat{S}_V,
\]

which satisfies that for any covariant representation \((\pi, u)\) of \((A, V, \sigma)\),

\[
(\pi \times u)(X) = (\pi \times u)(x_1 \otimes \omega_1) \cdot (\pi \times u)(x_2 \otimes \omega_2).
\]

We have been able to define a multiplication on \(A \times_\sigma \hat{S}_V\) as above. Next we consider a *-operation on \(A \times_\sigma \hat{S}_V\).
Definition 6. For a covariant representation \((\pi, u)\) of \((A, V, \sigma)\),
\[
\hat{S}_{(\pi, u)} = ((\pi \times u)[A \otimes_{\text{alg}} (S_V^w)])^{-1}\|u\|_{B(H_\pi)}.
\]

Lemma 7. For any covariant representation \((\pi, u)\) of \((A, V, \sigma)\), \(\hat{S}_{(\pi, u)}\) is a C*-subalgebra of \(B(H_\pi)\).

Proof. By Lemma 5 \(\hat{S}_{(\pi, u)}\) is a closed subalgebra of \(B(H_\pi)\), so it suffices to show that \(\hat{S}_{(\pi, u)}\) is selfadjoint. Note that
\[
\hat{S}_{(\pi, u)} = \text{c.l.h.} \{\pi(x)(\text{id} \otimes \omega \otimes \omega')(\Sigma_{23} V_{23} u_{12})| x \in A, \omega, \omega' \in B(H)_*\}.
\]
Hence we obtain
\[
(*) \quad (\hat{S}_{(\pi, u)})^* = \text{c.l.h.} \{(\text{id} \otimes \omega')(u(1 \otimes k)u^*)\pi(x)| x \in A, \omega' \in B(H)_*, k \in K(H)\}
\]
(by Definition 2(2), and the regularity of \(V\))
\[
= \text{c.l.h.} \{(\pi(x)(\text{id} \otimes \omega))(u(1 \otimes k)u^*)| x \in A, \omega \in B(H)_*, k \in K(H)\}
\]
(by Definition 1(2)).

Since it also follows from (*) above that
\[
\hat{S}_{(\pi, u)} = \text{c.l.h.} \{\pi(x)(\text{id} \otimes \omega')(u(1 \otimes k)u^*)| x \in A, \omega \in B(H)_*, k \in K(h)\},
\]
we obtain that \((\hat{S}_{(\pi, u)})^* = \hat{S}_{(\pi, u)}\). □

Proposition 8. \(A \times_\sigma \hat{S}_V\) has a C*-algebra structure.

Proof. Clearly \(A \times_\sigma \hat{S}_V\) is isomorphic to the projective limit C*-algebra of \(\{\hat{S}_{(\pi, u)}| (\pi, u)\text{ a covar. rep. of } (A, V, \sigma)\}\) as a Banach algebra. So \(A \times_\sigma \hat{S}_V\) has also a C*-algebra structure. □

Proposition 9. \(u \in m(\hat{S}_{(\pi, u)} \otimes S_V)\) for any covariant representation \((\pi, u)\) of \((A, V, \sigma)\).

Proof. First we show that \(u \in M(\hat{S}_{(\pi, u)} \otimes K(H))\). For \(x \in A, \omega \in (S_V^w)_*\) and \(k \in K(H)\), by [1, Proposition 3.2b]), \(u \cdot ((\pi(x)(\text{id} \otimes \omega)(u))^* \otimes k)\) belongs to
\[
\text{c.l.h.} \{(\text{id} \otimes \omega^* \otimes \text{id})(u_{13}(1 \otimes k' \otimes 1)V_{23}(1 \otimes 1 \otimes k)u_{12}^*(\pi(x^*) \otimes 1 \otimes 1)) | x \in A, \omega \in B(H)_*, k, k' \in K(H)\}.
\]
This set equals \((\hat{S}_{(\pi, u)})^* \otimes K(H)\) by condition (2) of Definition 2, and similarly \((\pi(x)(\text{id} \otimes \omega)(u))^* \otimes k \cdot u\) belongs to \(\hat{S}_{(\pi, u)} \otimes K(H)\). Therefore we have \(u_{12} \in M(\hat{S}_{(\pi, u)} \otimes K(H) \otimes S_V)\). Also \(V_{23} \in M(\hat{S}_{(\pi, u)} \otimes K(H) \otimes S_V)\) by [1, Proposition 3.6a]), so we get \(U_{13} \in M(\hat{S}_{(\pi, u)} \otimes K(H) \otimes S_V)\), i.e. \(u \in M(\hat{S}_{(\pi, u)} \otimes S_V)\). □

Definition 10. For \(x, y \in A\) and \(\eta \in (S_V^w)_*\), we set \(x * (y \otimes \eta) \equiv xy \otimes \eta\), and thus we define a left action of \(A\) on \(A \times_\sigma \hat{S}_V\) by continuity.

Proposition 11. For \(x \in A\) and \(\omega \in (S_V^w)_*\),

1. \(x, 1 \otimes \omega \in M(A \times_\sigma \hat{S}_V)\) and \(\|x\|^2_{M(A \times_\sigma \hat{S}_V)} = \|x\|^2_A, \|1 \otimes \omega\|^2_{M(A \times_\sigma \hat{S}_V)} \leq \|\omega\|^2_{(S_V^w)_*}\),
2. \((\pi \times u)(x) = \pi(x), (\pi \times u)(1 \otimes \omega) = (\text{id} \otimes \omega)(u)\) for any covariant representation \((\pi, u)\) of \((A, V, \sigma)\)
Proof. (1) It is clear that \( x \in B(A \times_\sigma \tilde{S}_V) \), \( \|x\|_{B(A \times_\sigma \tilde{S}_V)} \leq \|x\| \), and \( x^* \) is the adjoint of \( x \) in \( B(A \times_\sigma \tilde{S}_V) \), so we have \( x \in \mathcal{M}(A \times_\sigma \tilde{S}_V) \). If \( \pi \) is a faithful \( \ast \)-representation of \( A \), we can easily check that \( (\pi \otimes \text{id})\sigma, 1 \otimes V \) is a covariant representation of \( (A, V, \sigma) \) and that \( (\pi \otimes \text{id})\sigma \) is faithful. Hence we have \( \|x\|_{\mathcal{M}(A \times_\sigma \tilde{S}_V)} = \|x\|_A \). The statement about \( 1 \otimes \omega \) follows immediately from Lemma 5.

(2) By nondegeneracy of \( \pi \times u \), the statement is checked as follows. For \( y \in A, \eta \in (\tilde{S}_V^{-w})_* \), and \( k \in K(H_k) \),

\[
(\pi \times u)(x) \cdot (\pi \times u)(y \otimes \eta)k = \pi(x)(\pi \times u)(y \otimes \eta)k,
\]

which implies \( (\pi \times u)(x) = \pi(x) \). Similarly by Lemma 5, we obtain

\[
(\pi \times u)(1 \otimes \omega) = (\text{id} \otimes \omega)(u). \quad \Box
\]

By Definition 3, there exists a covariant representation \( (\Pi_0, W_0) \) of \( (A, V, \sigma) \) such that \( \Pi_0 \times W_0 \) is a faithful \( \ast \)-representation of \( A \times_\sigma \tilde{S}_V \). Let us denote \( W \equiv ((\Pi_0 \times W_0) \otimes \text{id})^{-1}(W_0) \). Then \( W \in \mathcal{M}(A \times_\sigma \tilde{S}_V \otimes S_V) \) by Proposition 9.

Proposition 12. Let \( W \) be as above. We have

(1) \( 1 \otimes \omega = (\text{id} \otimes \omega)(W) \) for any \( \omega \in (\tilde{S}_V^{-w})_* \),

(2) \( (\pi \times u \otimes \text{id})(W) = u \) for any covariant representation \( (\pi, u) \) of \( (A, V, \sigma) \).

Proof. (1) By Proposition 11, \( (\Pi_0 \times W_0)(1 \otimes \omega) = (\text{id} \otimes \omega)(W_0) \); therefore we have

\[
1 \otimes \omega = (\Pi_0 \times W_0)^{-1}((\text{id} \otimes \omega)(W_0)) = (\text{id} \otimes \omega)(W).
\]

(2) By (1), for any \( \omega \in (\tilde{S}_V^{-w})_* \),

\[
(\pi \times u)(1 \otimes \omega) = (\pi \times u)(\text{id} \otimes \omega)(W) = (\text{id} \otimes \omega)((\pi \times u \otimes \text{id})(W)).
\]

By Proposition 11(2) \( (\pi \times u)(1 \otimes \omega)(u) \). So we have \( (\pi \times u \otimes \text{id})(W) = u \). \quad \Box

Theorem 13. There exists a one-to-one correspondence between the equivalence classes of all covariant representations of \( (A, V, \sigma) \) and the equivalence classes of all nondegenerate \( \ast \)-representations of \( A \times_\sigma \tilde{S}_V \) such that

\[
\{(\pi, u)\mid \text{covar. rep. of } (A, V, \sigma)\} \rightarrow \{L\mid \text{nondeg. } \ast\text{-rep. of } A \times_\sigma \tilde{S}_V\}
\]

by \( (\pi, u) \mapsto \pi \),

\[
(\pi_L, u_L) \leftrightarrow L,
\]

where \( \pi_L(x) = L(x), u_L = (L \otimes \text{id})(W) \).

Proof. We first show that the above mappings are well defined. If \( (\pi, u) \) is a covariant representation of \( (A, V, \sigma) \), then we have already shown that \( \pi \times u \) is a nondegenerate \( \ast \)-representation of \( A \times_\sigma \tilde{S}_V \) by Proposition 4. Conversely suppose that \( L \) is a nondegenerate \( \ast \)-representation of \( A \times_\sigma \tilde{S}_V \). Let \( \xi \in H_L \) be such that \( \pi_L(x)\xi = 0 \) for any \( x \in A \). We obtain

\[
L(1 \otimes \omega)^*L(x)\xi = L(x^* \ast (1 \otimes \omega))^*\xi = 0
\]

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for any $x \in A$, $\omega \in (\overline{S_V^{\omega}})_\ast$. Therefore by nondegeneracy of $L$, it follows that $\xi = 0$, i.e., $\pi_L$ is nondegenerate. Further, we can check the covariance condition of $(\pi_L, u_L)$ by

$$u_L(\pi_L(x) \otimes 1)u_L^* = (L \otimes \text{id})(W(x \otimes 1)W^*),$$

and, by Proposition 11(2),

$$W(x \otimes 1)W^* = (\Pi_0 \times W_0 \otimes \text{id})^{-1}(\Pi_0 \otimes \text{id})\sigma(x) = \sigma(x).$$

Hence we obtain $u_L(\pi_L(x) \otimes 1)u_L^* = (L \otimes \text{id})\sigma(x) = (\pi_L \otimes \text{id})\sigma(x)$.

Secondly we prove that the mappings are the inverses of each other. Let $(\pi, u)$ be a covariant representation of $(A, V, \sigma)$. For any $x \in A$, $\pi_{\pi \times u}(x) = (\pi \times u)(x) = \pi(x)$ by Proposition 11 and since $u_{\pi \times u} = (\pi \times u \otimes \text{id})(W)$, we have $u_{\pi \times u} = u$ by Proposition 12(2). Conversely let $L$ be a nondegenerate $\ast$-representation of $(A, V, \sigma)$. For any $x \in A$, $\omega \in (\overline{S_V^{\omega}})_\ast$ we have

$$(\pi_L \times u_L)(x \otimes \omega) = \pi_L(x)(\text{id} \otimes \omega)(u_L) = L(x)(\text{id} \otimes \omega)((L \otimes \text{id})(W)) = L(x \ast (\text{id} \otimes \omega)(W)) = L(x \ast (1 \otimes \omega)) = L(x \otimes \omega),$$

and hence $\pi_L \times u_L = L$. □

So far we have stated a definition and some properties of crossed products. Next we will treat reduced ones.

**Definition 14.** Let $(A, V, \sigma)$ be a covariant system. For a nondegenerate $\ast$-representation $\pi$ of $A$, $((\pi \otimes \text{id})\sigma, 1_{B(H_A)} \otimes V)$ is a covariant representation of $(A, V, \sigma)$ as is easily verified. We define $\text{Ind} \pi \equiv ((\pi \otimes \text{id})\sigma) \times (1 \times V)$ and call it the induced representation of $A \times_{\sigma} \hat{S}_V$ from $\pi$. We define a seminorm $\| \cdot \|_r$ on $A \otimes_{\text{alg}} (\overline{S_V^{\omega}})_\ast$ by

$$\left\| \sum_{i=1}^N x_i \otimes \omega_i \right\|_r = \sup \left\{ \left\| (\text{Ind} \pi) \left( \sum_{i=1}^N x_i \otimes \omega_i \right) \right\| \right\} \text{ is a nondeg. } \ast\text{-rep. of } A$$

for $x_i \in A$, $\omega_i \in (\overline{S_V^{\omega}})_\ast$. Note that if $\pi$ is a faithful $\ast$-representation of $A$,

$$\left\| \sum_{i=1}^N x_i \otimes \omega_i \right\|_r = \left\| (\text{Ind} \pi) \left( \sum_{i=1}^N x_i \otimes \omega_i \right) \right\|,$$

and if $A \subseteq B(H_A)$ for some Hilbert space $H_A$,

$$\left\| \sum_{i=1}^N x_i \otimes \omega_i \right\|_r = \left\| \sum_{i=1}^N \sigma(x_i)(1 \otimes \rho(\omega_i)) \right\|_{B(H_A \otimes H)}$$

where $\rho(\omega) \equiv (\omega \otimes \text{id})(V)$. Finally we define

$$A \times_{\sigma} \hat{S}_V \equiv \| \cdot \|_r\text{-completion of } A \otimes_{\text{alg}} (\overline{S_V^{\omega}})_\ast/(\| \cdot \|_r = 0)$$

and call it the reduced crossed product of $(A, V, \sigma)$. Note that if $\pi$ is a faithful $\ast$-representation of $A$,

$$A \times_{\sigma} \hat{S}_V \equiv (\text{Ind} \pi)[A \times_{\sigma} \hat{S}_V],$$

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and if $A \subseteq B(H_A)$ for some Hilbert space $H_A$,

$$A \times_{\sigma} \hat{S}_V \cong \text{c.l.h.} \{\sigma(x)(1 \otimes \rho(\omega))|x \in A, \omega \in (\hat{S}_V^{w})_*\}.$$ 

**Theorem 15.** Let $V$ be a regular multiplicative unitary on a Hilbert space $H$. Then the following are equivalent.

1. For any $C^*$-covariant system $(A, V, \sigma)$, $A \times_{\sigma} \hat{S}_V$ is isomorphic to $A \times_{\sigma} \hat{S}_V$.

2. There exists a nonzero $*-$homomorphism $\varphi$ from $\hat{S}_V$ to $C$.

In this case $V$ is said to be amendable.

**Proof.** Suppose that $\varphi$ is a nonzero $*-$homomorphism from $\hat{S}_V$ to $C$. Let $(A, V, \sigma)$ be a $C^*$-covariant system. We may assume that $A \subseteq B(H_A)$ for some Hilbert space $H_A$. For any covariant representation $(\pi, u)$ of $(A, V, \sigma)$, we consider the mapping

$$(\text{id} \otimes \varphi)(Ad u^*)(\pi \otimes \text{id}): A \times_{\sigma} \hat{S}_V \to B(H_\pi).$$

For any $x \in A$, $\omega \in (\hat{S}_V^{w})_*$

$$(Ad u^*)(\pi \otimes \text{id})(\sigma(x)(1 \otimes \rho(\omega))) = u^*((\pi \otimes \text{id})\sigma(x) \cdot (1 \otimes \rho(\omega)))u$$

$$= (\pi(x) \otimes 1)u^*(1 \otimes \rho(\omega))u$$

$$= (\pi(x) \otimes 1)(\text{id} \otimes \text{id} \otimes \omega)(u_{12}^*V_{23}u_{12})$$

$$= (\pi(x) \otimes 1)(\text{id} \otimes \text{id} \otimes \omega)(u_{13}V_{23}).$$

Noticing $V \in M(\hat{S}_V \otimes S_V)$ [1, Proposition 3.6(c)], we apply $(\text{id} \otimes \varphi)$ to the above equality and obtain

$$(\text{id} \otimes \varphi)(Ad u^*)(\pi \otimes \text{id})(\sigma(x)(1 \otimes \rho(\omega))) = \pi(x)(\text{id} \otimes \omega)(u(1 \otimes (\varphi \otimes \text{id})(V))) ,$$

and thus

$$\left\| \sum_{i=1}^{N} \sigma(x_i)(1 \otimes \rho(\omega_i)) \right\|_{A \times_{\sigma} \hat{S}_V} \geq \left\| \sum_{i=1}^{N} \pi(x_i)(\text{id} \otimes \omega_i)(u(1 \otimes (\varphi \otimes \text{id})(V))) \right\|_{A \times_{\varphi} \hat{S}_V}.$$ 

It is easily verified that for any covariant representation of $(\pi, u)$ of $(A, V, \sigma)$, $(\pi, u(1 \otimes (\varphi \otimes \text{id})(V)))$ and $(\pi, u(1 \otimes (\varphi \otimes \text{id})(V))^*)$ are also covariant representations of $(A, V, \sigma)$. Thus

$$\left\| \sum_{i=1}^{N} \sigma(x_i)(1 \otimes \rho(\omega_i)) \right\|_{A \times_{\sigma} \hat{S}_V} \geq \left\| \sum_{i=1}^{N} x_i \otimes \omega_i \right\|_{A \times_{\varphi} \hat{S}_V} ,$$

and therefore the equality follows.

Conversely suppose that for any $C^*$-covariant system $(A, V, \sigma)$, $A \times_{\sigma} \hat{S}_V$ is isomorphic to $A \times_{\sigma} \hat{S}_V$. Then in particular,

$$\hat{S}_V = C \times_t \hat{S}_V \cong C \times_t \hat{S}_V,$$

where $t$ is the trivial action. Also we have a trivial covariant representation $(\text{id}_C, 1_C \otimes 1_{S_V})$, so $\text{id}_C \times (1_C \otimes 1_{S_V})$ is a nonzero $*-$homomorphism from $C \times_t \hat{S}_V$ to $C$. Hence (2) is established. □

We give some examples of amenable regular multiplicative unitaries.
Corollary 16. (1) Let $G$ be a locally compact group with the right Haar measure $\mu$ and the modular function $\Delta$, and let $V_G$ be the regular multiplicative unitary associated with $G$, defined by $(V_G\xi)(s, t) \equiv \xi(st, t)$ for all $\xi \in L^2(G \times G, \mu \times \mu)$. Then $\overline{V}_G$, the dual of $V_G$, is amenable ([3, Theorem 3.7]), where

$$(\overline{V}_G\xi)(s, t) \equiv \Delta(s)^{1/2}\xi(s, s^{-1}t) \quad \text{for all } \xi \in L^2(G \times G, \mu \times \mu).$$

(An action of $\overline{V}_G$ on a $C^*$-algebra $A$ is usually called a coaction of $G$ on $A$.)

(2) Let $G_c = (A, \delta, \mu)$ be a compact matrix pseudogroup with the Haar state $h$ ([4]) and $V_{G_c}$ be the regular multiplicative unitary associated with $G_c$, defined by

$$V_{G_c}(a_h \otimes b_h) \equiv (\delta(a)(1 \otimes b))_{h \otimes h} \quad \text{for all } a, b \in A,$$

where $a_h$ is $a$ as a vector of the Hilbert space in the GNS construction by $h$. Then $V_{G_c}$ is amenable ([2, Theorem 7.6]).

Proof. (1) We can verify that $S_{\overline{V}_G} = S_V = C_0(G)$, the set of all continuous functions on $G$ vanishing at $\infty$. So if we define $\varphi(f) = f(e)$ for all $f \in C_0(G)$, where $e$ is the unit or any element of $G$, then $\varphi$ is a $\ast$-homomorphism.

(2) By [2, Theorem 7.7] $S_{V_{G_c}}$ is isomorphic to $\bigoplus_{\alpha \in G_c} \hat{M}_{\dim(\alpha)}(C)$, where $\hat{G}_c$ are the equivalence classes of all irreducible unitary representations of $G_c$. Take the trivial representation $1$; then a canonical projection from $\bigoplus_{\alpha \in G_c} \hat{M}_{\dim(\alpha)}(C)$ to $M_{\dim(1)}(C) \cong C$ is a nonzero $\ast$-homomorphism. □

References


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