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ACTIONS OF REGULAR MULTIPLICATIVE UNITARIES ON C^*-ALGEBRAS

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ABSTRACT. We establish a covariant representation theory for actions of regular multiplicative unitaries, in the sense of S. Baaj and G. Skandalis, on C^*-algebras.

In [1] S. Baaj and G. Skandalis have established a formulation for locally compact quantum groups by using the notion of multiplicative unitaries. By adopting this formulation we will extend the notion of a C^*-covariant system which usually consists of a C^*-algebra and an action of a locally compact group on that C^*-algebra. This extension gives a formulation for actions of locally compact quantum groups on C^*-algebras which include actions and coactions of locally compact groups on C^*-algebras. First we will state a definition of the crossed product and give some related results which are well known in the locally compact group case. One of the results is that there is a one-to-one correspondence between the covariant representations of a C^*-covariant system and the nondegenerate $*$-representations of its crossed product. Secondly we will state a definition of a reduced crossed product and treat some amenable cases.

According to [1] we shall review multiplicative unitaries and those related things which will be used later. All Hilbert spaces and C^*-algebras are assumed to be separable, and tensor products of C^*-algebras are taken to be spatial ones throughout this paper. For a C^*-algebra A and its closed two-sided ideal J, $M(A)$ denotes the multiplier algebra of A and $M(A;J)$ denotes $\{a \in M(A) | aA + Aa \subseteq J\}$. Let H be a Hilbert space. A unitary operator V on $H \otimes H$ is said to be multiplicative, if it satisfies the so-called pentagon equation: $V_{12}V_{13}V_{23} = V_{23}V_{12}$. Furthermore V is called regular, if $\{(\mathrm{id} \otimes \omega)(\Sigma V) | \omega \in B(H)_*\}$ is norm-dense in $K(H)$, where Σ denotes the flip operator on $H \otimes H$ and $K(H)$ is the compact operator on H. We have the following facts for a regular multiplicative unitary V.

(1) Define

$$S_V = \{(\omega \otimes \mathrm{id})(V) | \omega \in B(H)_*\}^{-\|\cdot\|}$$

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and

$$\widehat{S}_V = \{(\mathrm{id}\otimes\omega)(V) | \omega \in B(H)_*\}^{-\|\cdot\|}.$$

Then S_V and $\widehat{S}_V$ are C^*-subalgebras of $B(H)$.

(2) Define $\delta_V(s) = V(s\otimes 1)V^*$ for $s \in S_V$ and $\hat{\delta}_V(t) = V^*(1\otimes t)V$ for $t \in \widehat{S}_V$. Then δ_V and $\hat{\delta}_V$ are coproducts on S_V and $\widehat{S}_V$ respectively, i.e. $\delta_V : S_V \to M(\widetilde{S}_V \otimes S_V + S_V \otimes \widetilde{S}_V\,;\, S_V \otimes S_V)$ is a nondegenerate $*$-homomorphism such that

$$(\delta_V \otimes \mathrm{id})\delta_V = (\mathrm{id}\otimes\delta_V)\delta_V\,,$$
$$\delta_V \quad \text{is injective,}$$

and

$$\delta_V(S_V)(1\otimes S_V) \text{ and } \delta_V(S_V)(S_V\otimes 1) \quad \text{are total in } S_V\otimes S_V\,,$$

and similar properties hold for $\hat{\delta}_V$ on $\widehat{S}_V$.

We will start with some definitions.

Definition 1. Let V be a regular multiplicative unitary on a Hilbert space H. Then (A, V, σ) is called a C^*-covariant system, if

(1) A is a C^*-algebra,

(2) $\sigma : A \to M(\tilde{A}\otimes S_V\,;\, A\otimes S_V)$ is a nondegenerate $*$-homomorphism such that

$$(\sigma\otimes\mathrm{id})\sigma = (\mathrm{id}\otimes\delta_V)\sigma\,,$$
$$\sigma \text{ is injective,}$$

and

$$\sigma(A)(1\otimes S_V) \quad \text{is total in } A\otimes S_V.$$

Definition 2. Let (A, V, σ) be a C^*-covariant system. (π, u) is called a covariant representation of (A, V, σ), if

(1) π is a nondegenerate $*$-representation of A on some Hilbert space H_π,

(2) u is a unitary element of $M(K(H_\pi)\otimes S_V)$ such that $(\mathrm{id}\otimes\delta_V)(u) = u_{12}u_{13}$,

(3) $(\pi\otimes\mathrm{id})\sigma(x) = u(\pi(x)\otimes 1)u^*$ for all $x \in A$.

Definition 3. Let (A, V, σ) be a C^*-covariant system. We define a seminorm on $A \otimes_{\mathrm{alg.}} (\overline{S_v}^w)_*$ by

$$\begin{aligned}
&\left\|\sum_{i=1}^N x_i\otimes\omega_i\right\| \\
&\quad \equiv \sup\left\{\left\|\sum_{i=1}^N \pi(x_i)(\mathrm{id}\otimes\omega_i)(u)\right\| \,\middle|\, (\pi, u) \text{ is a covar. rep. of } (A, V, \sigma)\right\} \\
&\quad \le \sum_{i=1}^N \|x_i\|_A \|\omega_i\|_{(\overline{S_V}^w)_*} < \infty
\end{aligned}$$

for $x_i \in A$, $\omega_i \in (\overline{S_V}^w)_*$. We define

$$A \times_\sigma \widehat{S}_V \equiv \|\cdot\|\text{-completion of } A \otimes_{\text{alg.}} (\overline{S_V}^w)_*/(\|\cdot\| = 0),$$

which is a Banach space, and call it the crossed product of (A, V, σ). Finally for a covariant representation (π, u) of (A, V, σ), we define a bounded linear mapping $\pi \times u$ of $A \times \widehat{S}_V$ into $B(H_\pi)$ such that

$$(\pi \times u)\left(\sum_{i=1}^N x_i \otimes \omega_i\right) = \sum_{i=1}^N \pi(x_i)(\mathrm{id} \otimes \omega_i)(u).$$

Proposition 4. *$\pi \times u$ is nondegenerate for any covariant representation (π, u) of (A, V, σ).*

Proof. By the nondegeneracy we mean that $(\pi \times u)[A \times_\sigma \widehat{S}_v]$. H_π is dense in H_π. For this it suffices to show that $\{(\mathrm{id} \otimes \omega)(u) | \omega \in (\overline{S_V}^w)_*\}$. H_π is dense in H_π. In fact, for any $\xi, \eta \in H_\pi \backslash \{0\}$, there exist $\alpha, \beta \in H$ such that $\langle \xi \otimes \eta, u(\alpha \otimes \beta)\rangle \neq 0$. But the left-hand side equals $\langle \xi, (\mathrm{id} \otimes \omega_{\eta,\beta})(u)\alpha\rangle$, as we have shown. □

We will give a C^*-algebra structure on $A \times_\sigma \widehat{S}_V$. For this purpose we need some lemmata.

Lemma 5. *For $x_1, x_2 \in \tilde{A}$, at least one of which belongs to A, and $\omega_1, \omega_2 \in (\overline{S_V}^w)_*$, there exists a unique $X \in A \times_\sigma \widehat{S}_V$ such that for any covariant representation (π, u) of (A, V, σ)*

$$(\pi \times u)(X) = (\pi \times u)(x_1 \otimes \omega_1) \cdot (\pi \times u)(x_2 \otimes \omega_2)$$

where we denote $(\pi \times u)(1 \otimes \omega_i) \equiv (\mathrm{id} \otimes \omega_i)(u)$.

Therefore by setting X to be the product of $(x_1 \otimes \omega_1)$ and $(x_2 \otimes \omega_2)$ we can define a multiplication on $A \times_\sigma \widehat{S}_V$ with respect to which $A \times_\sigma \widehat{S}_V$ is a Banach algebra.

Proof. If $x_2 \in C1_A$, then we define $X \equiv x_1 \otimes (\omega_1 \otimes \omega_2)(\delta_V(\cdot)) \cdot \lambda_2$ and the statement is easily checked. So we assume $x_2 \in A$. Let (s_μ) be an approximate identity of S_V. Since $\sigma(x_2) \in M(\tilde{A} \otimes S_V; A \otimes S_V)$,

$$\sigma(x_2)(1 \otimes s_\mu)(\varepsilon A \otimes S_V) \to \sigma(x_2) \quad \text{strictly in } M(\tilde{A} \otimes S_V).$$

So there exists $(\sum_{i=1}^{N_\lambda} x_1^\lambda \otimes s_i^\lambda | \lambda \in \Lambda)$, a norm bounded net of $A \otimes_{\text{alg.}} S_V \subseteq A \otimes S_V$, such that

$$\sum_{i=1}^{N_\lambda} x_1^\lambda \otimes s_i^\lambda \to \sigma(x_2) \quad \text{strictly in } M(\tilde{A} \otimes S_V).$$

We next show

$$\sum_{i=1}^{N_\lambda} x_1 x_i^\lambda \otimes (\omega_1 \otimes \omega_2)((s_i^\lambda \otimes 1)\delta_V(\cdot)) \quad \text{converges in } A \times_\sigma \widehat{S}_V.$$

For this it is sufficient to show that for any covariant representation (π, u) of (A, V, σ)

$$(\pi \times u)\left(\sum_{i=1}^{N_\lambda} x_1 x_i^\lambda \otimes (\omega_1 \otimes \omega_2)((s_i^\lambda \otimes 1)\delta_V(\cdot))\right) \quad \text{converges in norm in } B(H_\pi)$$

uniformly in (π, u). In fact, we have

$$\begin{aligned}
&\forall \varepsilon > 0\ \exists \lambda_0 \in \Lambda\ \forall \lambda \geq \lambda_0\ \forall (\pi, u)(\text{covariant rep.}) \\
&\quad \Bigg\| (\pi \times u)\left(\sum_{i=1}^{N_\lambda} x_1 x_i^\lambda \otimes (\omega_1 \otimes \omega_2)((s_i^\lambda \otimes 1)\delta_V(\cdot))\right) \\
&\qquad\qquad - (\pi \times u)(x_1 \otimes \omega_1) \cdot (\pi \times u)(x_2 \otimes \omega_2) \Bigg\|_{B(H_\pi)} < \varepsilon,
\end{aligned}$$

by the estimation

$$\begin{aligned}
&= \Bigg\| \Bigg(\sum_{i=1}^{N_\lambda} \pi(x_1 x_i^\lambda)(\mathrm{id} \otimes \omega_1 \otimes \omega_2)((1_A \otimes s_i^\lambda \otimes 1)(\mathrm{id} \otimes \delta_V(u)) \\
&\qquad - (\mathrm{id} \otimes \omega_1 \otimes \omega_2)((\pi \otimes \mathrm{id} \otimes \mathrm{id})((x_1 \otimes 1 \otimes 1)(\sigma(x_2 \otimes 1))(\mathrm{id} \otimes \delta_V)(u)\Bigg) \Bigg\| \\
&\leq \Bigg\| ((\mathrm{id} \otimes \omega_1 \otimes \omega_2) - (\mathrm{id} \otimes \omega_1 s_1 \otimes \omega_2)) \\
&\qquad \cdot \left((\pi(x_1) \otimes 1 \otimes 1)(\pi \otimes \mathrm{id} \otimes \mathrm{id}) \left(\sum_{i=1}^{N_\lambda} x_i^\lambda \otimes s_i^\lambda \otimes 1 \right) (\mathrm{id} \otimes \delta_V)(u) \right) \Bigg\| \\
&\quad + \Bigg\| (\mathrm{id} \otimes \omega_1 \otimes \omega_2)(\pi \otimes \mathrm{id} \otimes \mathrm{id}) \\
&\qquad \cdot \left(\left((x_1 \otimes s_1) \left(\sum_{i=1}^{N_\lambda} x_1^\lambda \otimes s_i^\lambda - \sigma(x_2) \right) \otimes 1 \right) (\mathrm{id} \otimes \delta_V)(u) \right) \Bigg\| \\
&\quad + \| ((\mathrm{id} \otimes \omega_1 s_1 \otimes \omega_2) - (\mathrm{id} \otimes \omega_1 \otimes \omega_2)) \\
&\qquad \cdot ((\pi \otimes \mathrm{id} \otimes \mathrm{id})((x_1 \otimes 1 \otimes 1)(\sigma(x_2 \otimes 1))(\mathrm{id} \otimes \delta_V)(u)) \|
\end{aligned}$$

for any $s_1 \in S_V$. Hence there exists

$$X \equiv \lim \sum_{i=1}^{N_\lambda} x_1 x_i^\lambda \otimes (\omega_1 \otimes \omega_2)((s_i^\lambda \otimes 1)\delta_V(\cdot)) \in A \times_\sigma \widehat{S}_V,$$

which satisfies that for any covariant representation (π, u) of (A, V, σ), $(\pi \times u)(X) = (\pi \times u)(x_1 \otimes \omega_1) \cdot (\pi \times u)(x_2 \otimes \omega_2)$. $\square$

We have been able to define a multiplication on $A \times_\sigma \widehat{S}_V$ as above. Next we consider a $*$-operation on $A \times_\sigma \widehat{S}_V$.

Definition 6. For a covariant representation (π, u) of (A, V, σ),

$$\widehat{S}_{(\pi,u)} \equiv ((\pi \times u)[A \otimes_{\text{alg.}} (\overline{S_V}^{w})_*])^{-\|\cdot\|_{B(H_\pi)}}.$$

Lemma 7. *For any covariant representation (π, u) of (a, V, σ), $\widehat{S}_{(\pi,u)}$ is a C^*-subalgebra of $B(H_\pi)$.*

Proof. By Lemma 5 $\widehat{S}_{(\pi,u)}$ is a closed subalgebra of $B(H_\pi)$, so it suffices to show that $\widehat{S}_{(\pi,u)}$ is selfadjoint. Note that

$$\widehat{S}_{(\pi,u)} = \text{c.l.h. } \{\pi(x)(\text{id} \otimes \omega \otimes \omega')(\Sigma_{23} V_{23}^* u_{12}) | x \in A, \omega, \omega' \in B(H)_*\}.$$

Hence we obtain

$(*)$

$$\begin{aligned}(\widehat{S}_{(\pi,u)})^* &= \text{c.l.h. } \{(\text{id} \otimes \omega')(u(1 \otimes k)u^*)\pi(x) | x \in A, \omega' \in B(H)_*, k \in K(H)\} \\ &\qquad \text{(by Definition 2(2), and the regularity of } V) \\ &= \text{c.l.h. } \{(\pi(x)(\text{id} \otimes \omega')(u(1 \otimes k)u^*) | x \in A, \omega' \in B(H)_*, k \in K(H)\} \\ &\qquad \text{(by Definition 1(2)).}\end{aligned}$$

Since it also follows from $(*)$ above that

$$\widehat{S}_{(\pi,u)} = \text{c.l.h. } \{\pi(x)(\text{id} \otimes \omega')(u(1 \otimes k)u^*) | x \in A, \omega' \in B(H)_*, k \in K(h)\},$$

we obtain that $(\widehat{S}_{(\pi,u)})^* = \widehat{S}_{(\pi,u)}$. □

Proposition 8. *$A \times_\sigma \widehat{S}_V$ has a C^*-algebra structure.*

Proof. Clearly $A \times_\sigma \widehat{S}_V$ is isomorphic to the projective limit C^*-algebra of $\{\widehat{S}_{(\pi,u)} | (\pi, u)$ is a covar. rep. of $(A, V, \sigma)\}$ as a Banach algebra. So $A \times_\sigma \widehat{S}_V$ has also a C^*-algebra structure. □

Proposition 9. *$u \in m(\widehat{S}_{(\pi,u)} \otimes S_V)$ for any covariant representation (π, u) of (A, V, σ).*

Proof. First we show that $u \in M(\widehat{S}_{(\pi,u)} \otimes K(H))$. For $x \in A$, $\omega \in (\overline{S_V}^{w})_*$ and $k \in K(H)$, by [1, Proposition 3.2b)], $u \cdot ((\pi(x)(\text{id} \otimes \omega)(u))^* \otimes k)$ belongs to

$$\text{c.l.h. } \{(\text{id} \otimes \omega^* \otimes \text{id})(u_{13}(1 \otimes k' \otimes 1)V_{23}(1 \otimes 1 \otimes k)u_{12}^*(\pi(x^*) \otimes 1 \otimes 1)) \\ | x \in A, \omega \in B(H)_*, k, k' \in K(H)\}.$$

This set equals $(\widehat{S}_{(\pi,u)})^* \otimes K(H)$ by condition (2) of Definition 2, and similarly $(\pi(x)(\text{id} \otimes \omega)(u) \otimes k) \cdot u$ belongs to $\widehat{S}_{(\pi,u)} \otimes K(H)$. Therefore we have $u_{12} \in M(\widehat{S}_{(\pi,u)} \otimes K(H) \otimes S_V)$. Also $V_{23} \in M(\widehat{S}_{(\pi,u)} \otimes K(H) \otimes S_V)$ by [1, Proposition 3.6a)], so we get $U_{13} \in M(\widehat{S}_{(\pi,u)} \otimes K(H) \otimes S_V)$, i.e. $u \in M(\widehat{S}_{(\pi,u)} \otimes S_V)$. □

Definition 10. For $x, y \in A$ and $\eta \in (\overline{S_V}^{w})_*$, we set $x * (y \otimes \eta) \equiv xy \otimes \eta$, and thus we define a left action of A on $A \times_\sigma \widehat{S}_V$ by continuity.

Proposition 11. *For $x \in A$ and $\omega \in (\overline{S_V}^{w})_*$,*

(1) *$x, 1 \otimes \omega \in M(A \times_\sigma \widehat{S}_V)$ and $\|x\|_{M(A \times_\sigma \widehat{S}_V)} = \|x\|_A$, $\|1 \otimes \omega\|_{M(A \times_\sigma \widehat{S}_V)} \leq \|\omega\|_{(\overline{S_V}^{w})_*}$,*

(2) *$(\pi \times u)(x) = \pi(x)$, $(\pi \times u)(1 \otimes \omega) = (\text{id} \otimes \omega)(u)$ for any covariant representation (π, u) of (A, V, σ).*

Proof. (1) It is clear that $x \in B(A \times_\sigma \widehat{S}_V)$, $\|x\|_{B(A\times_\sigma \widehat{S}_V)} \leq \|x\|$, and x^* is the adjoint of x in $B(A \times_\sigma \widehat{S}_V)$, so we have $x \in M(A \times_\sigma \widehat{S}_V)$. If π is a faithful $*$-representation of A, we can easily check that $((\pi \otimes \mathrm{id})\sigma, 1 \otimes V)$ is a covariant representation of (A, V, σ) and that $(\pi \otimes \mathrm{id})\sigma$ is faithful. Hence we have $\|x\|_{M(A\times_\sigma \widehat{S}_V)} = \|x\|_A$. The statement about $1 \otimes \omega$ follows immediately from Lemma 5.

(2) By nondegeneracy of $\pi \times u$, the statement is checked as follows. For $y \in A$, $\eta \in (\overline{S_V}^w)_*$, and $k \in K(H_\pi)$,

$$(\pi \times u)(x) \cdot (\pi \times u)(y \otimes \eta)k = \pi(x)(\pi \times u)(y \otimes \eta)k,$$

which implies $(\pi \times u)(x) = \pi(x)$. Similarly by Lemma 5, we obtain

$$(\pi \times u)(1 \otimes \omega) = (\mathrm{id} \otimes \omega)(u). \quad \square$$

By Definition 3, there exists a covariant representation (Π_0, W_0) of (A, V, σ) such that $\Pi_0 \times W_0$ is a faithful $*$-representation of $A \times_\sigma \widehat{S}_V$. Let us denote $W \equiv ((\Pi_0 \times W_0) \otimes \mathrm{id})^{-1}(W_0)$. Then $W \in M(A \times_\sigma \widehat{S}_V \otimes S_V)$ by Proposition 9.

Proposition 12. *Let W be as above. We have*

(1) $1 \otimes \omega = (\mathrm{id} \otimes \omega)(W)$ *for any* $\omega \in (\overline{S_V}^w)_*$,

(2) $(\pi \times u \otimes \mathrm{id})(W) = u$ *for any covariant representation* (π, u) *of* (A, V, σ).

Proof. (1) By Proposition 11, $(\Pi_0 \times W_0)(1 \otimes \omega) = (\mathrm{id} \otimes \omega)(W_0)$; therefore we have

$$1 \otimes \omega = (\Pi_0 \times W_0)^{-1}((\mathrm{id} \otimes \omega)(W_0)) = (\mathrm{id} \otimes \omega)(W).$$

(2) By (1), for any $\omega \in (\overline{S_V}^w)_*$,

$$(\pi \times u)(1 \otimes \omega) = (\pi \times u)(\mathrm{id} \otimes \omega)(W) = (\mathrm{id} \otimes \omega)((\pi \times u \otimes \mathrm{id})(W)).$$

By Proposition 11(2) $(\pi \times u)(1 \otimes \omega)(u)$. So we have $(\pi \times u \otimes \mathrm{id})(W) = u$. $\square$

Theorem 13. *There exists a one-to-one correspondence between the equivalence classes of all covariant representations of (A, V, σ) and the equivalence classes of all nondegenerate $*$-representations of $A \times_\sigma \widehat{S}_V$ such that*

$$\{(\pi, u) |\ \textit{covar. rep. of}\ (A, V, \sigma)\} \to \{L |\ \textit{nondeg.}\ *\textit{-rep. of}\ A \times_\sigma \widehat{S}_V\}$$
$$\textit{by}\ (\pi, u) \mapsto \pi \times u$$
$$(\pi_L, u_L) \leftarrow\!\shortmid L,$$

where $\pi_L(x) \equiv L(x)$, $u_L \equiv (L \otimes \mathrm{id})(W)$.

Proof. We first show that the above mappings are well defined. If (π, u) is a covariant representation of (A, V, σ), then we have already shown that $\pi \times u$ is a nondegenerate $*$-representation of $A \times_\sigma \widehat{S}_V$ by Proposition 4 and Proposition 8. Conversely suppose that L is a nondegenerate $*$-representation of $A \times_\sigma \widehat{S}_V$. Let $\xi \in H_L$ be such that $\pi_L(x)\xi = 0$ for any $x \in A$. We obtain

$$L(1 \otimes \omega)^* L(x)\xi = L(x^* * (1 \otimes \omega))^* \xi = 0$$

for any $x \in A$, $\omega \in (\overline{S_V}^w)_*$. Therefore by nondegeneracy of L, it follows that $\xi = 0$, i.e., π_L is nondegenerate. Further, we can check the covariance condition of (π_L, u_L) by

$$u_L(\pi_L(x) \otimes 1)u_L^* = (L \otimes \mathrm{id})(W(x \otimes 1)W^*),$$

and, by Proposition 11(2),

$$W(x \otimes 1)W^* = (\Pi_0 \times W_0 \otimes \mathrm{id})^{-1}(\Pi_0 \otimes \mathrm{id})\sigma(x) = \sigma(x).$$

Hence we obtain $u_L(\pi_L(x) \otimes 1)u_L^* = (L \otimes \mathrm{id})\sigma(x) = (\pi_L \otimes \mathrm{id})\sigma(x)$.

Secondly we prove that the mappings are the inverses of each other. Let (π, u) be a covariant representation of (A, V, σ). For any $x \in A$, $\pi_{\pi \times u}(x) = (\pi \times u)(x) = \pi(x)$ by Proposition 11 and since $u_{\pi \times u} = (\pi \times u \otimes \mathrm{id})(W)$, we have $u_{\pi \times u} = u$ by Proposition 12(2). Conversely let L be a nondegenerate $*$-representation of (A, V, σ). For any $x \in A$, $\omega \in (\overline{S_V}^w)_*$ we have

$$\begin{aligned}(\pi_L \times u_L)(x \otimes \omega) &= \pi_L(x)(\mathrm{id} \otimes \omega)(u_L) = L(x)(\mathrm{id} \otimes \omega)((L \otimes \mathrm{id})(W)) \\ &= L(x * (\mathrm{id} \otimes \omega)(W)) = L(x * (1 \otimes \omega)) \\ &= L(x \otimes \omega),\end{aligned}$$

and hence $\pi_L \times u_L = L$. □

So far we have stated a definition and some properties of crossed products. Next we will treat reduced ones.

Definition 14. Let (A, V, σ) be a covariant system. For a nondegenerate $*$-representation π of A, $((\pi \otimes \mathrm{id})\sigma, 1_{B(H_\pi)} \otimes V)$ is a covariant representation of (A, V, σ) as is easily verified. We define $\mathrm{Ind}\,\pi \equiv ((\pi \otimes \mathrm{id})\sigma) \times (1 \times V)$ and call it the induced representation of $A \times_\sigma \widehat{S}_V$ from π. We define a seminorm $\|\cdot\|_r$ on $A \otimes_{\mathrm{alg.}} (\overline{S_V}^w)_*$ by

$$\left\|\sum_{i=1}^N x_i \otimes \omega_i\right\|_r \equiv \sup\left\{\left\|(\mathrm{Ind}\,\pi)\left(\sum_{i=1}^N x_i \otimes \omega_i\right)\right\| \;\middle|\; \pi \text{ is a nondeg. } *\text{-rep. of } A\right\}$$

for $x_i \in A$, $\omega_i \in (\overline{S_V}^w)_*$. Note that if π is a faithful $*$-representation of A,

$$\left\|\sum_{i=1}^N x_i \otimes \omega_i\right\|_r = \left\|(\mathrm{Ind}\,\pi)\left(\sum_{i=1}^N x_i \otimes \omega_i\right)\right\|,$$

and if $A \subseteq B(H_A)$ for some Hilbert space H_A,

$$\left\|\sum_{i=1}^N x_i \otimes \omega_i\right\|_r = \left\|\sum_{i=1}^N \sigma(x_i)(1 \otimes \rho(\omega_i))\right\|_{B(H_A \otimes H)},$$

where $\rho(\omega) \equiv (\omega \otimes \mathrm{id})(V)$. Finally we define

$$A \times_{\sigma r} \widehat{S}_V \equiv \|\cdot\|_r\text{-completion of } A \otimes_{\mathrm{alg.}} (\overline{S_V}^w)_*/(\|\cdot\|_r = 0)$$

and call it the reduced crossed product of (A, V, σ). Note that if π is a faithful $*$-representation of A,

$$A \times_{\sigma r} \widehat{S}_V \cong (\mathrm{Ind}\,\pi)[A \times_\sigma \widehat{S}_v],$$

and if $A \subseteq B(H_A)$ for some Hilbert space H_A,

$$A \times_{\sigma r} \widehat{S}_V \cong \text{ c.l.h. } \{\sigma(x)(1 \otimes \rho(\omega)) | x \in A,\ \omega \in (\overline{S_V}^w)_*\}.$$

Theorem 15. *Let V be a regular multiplicative unitary on a Hilbert space H. Then the following are equivalent.*

(1) *For any C^*-covariant system (A, V, σ), $A \times_{\sigma r} \widehat{S}_V$ is isomorphic to $A \times_\sigma \widehat{S}_V$.*

(2) *There exists a nonzero $*$-homomorphism φ from $\widehat{S}_V$ to C.*

In this case V is said to be amendable.

Proof. Suppose that φ is a nonzero $*$-homomorphism from $\widehat{S}_V$ to C. Let (A, V, σ) be a C^*-covariant system. We may assume that $A \subseteq B(H_A)$ for some Hilbert space H_A. For any covariant representation (π, u) of (A, V, σ), we consider the mapping

$$(\mathrm{id} \otimes \varphi)(Adu^*)(\pi \otimes \mathrm{id}) \colon A \times_{\sigma r} \widehat{S}_V \to B(H_\pi).$$

For any $x \in A$, $\omega \in (\overline{S_V}^w)_*$

$$\begin{aligned}(Adu^*)(\pi \otimes \mathrm{id})(\sigma(x)(1 \otimes \rho(\omega))) &= u^*((\pi \otimes \mathrm{id})\sigma(x) \cdot (1 \otimes \rho(\omega)))u \\ &= (\pi(x) \otimes 1)u^*(1 \otimes \rho(\omega))u \\ &= (\pi(x) \otimes 1)(\mathrm{id} \otimes \mathrm{id} \otimes \omega)(u_{12}^* V_{23} u_{12}) \\ &= (\pi(x) \otimes 1)(\mathrm{id} \otimes \mathrm{id} \otimes \omega)(u_{13} V_{23}).\end{aligned}$$

Noticing $V \in M(\widehat{S}_V \otimes S_V)$ [1, Proposition 3.6(c)], we apply $(\mathrm{id} \otimes \varphi)$ to the above equality and obtain

$$(\mathrm{id} \otimes \varphi)(Adu^*)(\pi \otimes \mathrm{id})(\sigma(x)(1 \otimes \rho(\omega))) = \pi(x)(\mathrm{id} \otimes \omega)(u(1 \otimes (\varphi \otimes \mathrm{id})(V))),$$

and thus

$$\left\| \sum_{i=1}^N \sigma(x_i)(1 \otimes \rho(\omega_i)) \right\|_{A \times_{\sigma r} \widehat{S}_V} \geq \left\| \sum_{i=1}^N \pi(x_i)(\mathrm{id} \otimes \omega_i)(u(1 \otimes (\varphi \otimes \mathrm{id})(V))) \right\|.$$

It is easily verified that for any covariant representation of (π, u) of (A, V, σ), $(\pi, u(1 \otimes (\varphi \otimes \mathrm{id})(V)))$ and $(\pi, u(1 \otimes (\varphi \otimes \mathrm{id})(V))^*)$ are also covariant representations of (A, V, σ). Thus

$$\left\| \sum_{i=1}^N \sigma(x_i)(1 \otimes \rho(\omega_i)) \right\|_{A \times_{\sigma r} \widehat{S}_V} \geq \left\| \sum_{i=1}^N x_i \otimes \omega_i \right\|_{A \times_\sigma \widehat{S}_V},$$

and therefore the equality follows.

Conversely suppose that for any C^*-covariant system (A, V, σ), $A \times_{\sigma r} \widehat{S}_V$ is isomorphic to $A \times_\sigma \widehat{S}_V$. Then in particular,

$$\widehat{S}_V = C \times_{\iota r} \widehat{S}_V \cong C \times_\iota \widehat{S}_V,$$

where ι is the trivial action. Also we have a trivial covariant representation $(\mathrm{id}_C,$
$1_C \otimes 1_{S_V})$, so $\mathrm{id}_C \times (1_C \otimes 1_{S_V})$ is a nonzero $*$-homomorphism from $C \times_\iota \widehat{S}_V$ to C. Hence (2) is established. □

We give some examples of amenable regular multiplicative unitaries.

Corollary 16. (1) *Let G be a locally compact group with the right Haar measure μ and the modular function Δ, and let V_G be the regular multiplicative unitary associated with G, defined by $(V_G\xi)(s, t) \equiv \xi(st, t)$ for all $\xi \in L^2(G\times G, \mu\times\mu)$. Then $\widehat{V_G}$, the dual of V_G, is amenable* ([3, Theorem 3.7]), *where*

$$(\widehat{V_G}\xi)(s, t) \equiv \Delta(s)^{1/2}\xi(s, s^{-1}t) \quad \textit{for all } \xi \in L^2(G \times G, \mu \times \mu).$$

(*An action of $\widehat{V_G}$ on a C^*-algebra A is usually called a coaction of G on A.*)

(2) *Let $G_c = (A, \delta, u)$ be a compact matrix pseudogroup with the Haar state h* ([4]) *and V_{G_c} be the regular multiplicative unitary associated with G_c, defined by*

$$V_{G_c}(a_h \otimes b_h) \equiv (\delta(a)(1 \otimes b))_{h\otimes h} \quad \textit{for all } a, b \in A,$$

where a_h is a as a vector of the Hilbert space in the GNS construction by h. Then V_{G_c} is amenable ([2, Theorem 7.6]).

Proof. (1) We can verify that $\widehat{S}_{\widehat{V_G}} = S_{V_G} = C_0(G)$, the set of all continuous functions on G vanishing at ∞. So if we define $\varphi(f) = f(e)$ for all $f \in C_0(G)$, where e is the unit or any element of G, then φ is a $*$-homomorphism.

(2) By [2, Theorem 7.7] $\widehat{S}_{V_{G_c}}$ is isomorphic to $\bigoplus_{\alpha\in\widehat{G_c}} M_{\dim(\alpha)}(C)$, where $\widehat{G_c}$ are the equivalence classes of all irreducible unitary representations of G_c. Take the trivial representation ι; then a canonical projection from $\bigoplus_{\alpha\in\widehat{G_c}} M_{\dim(\alpha)}(C)$ to $M_{\dim(\iota)}(C) \cong C$ is a nonzero $*$-homomorphism. □

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