SEMI-FREDHOLM OPERATORS WITH FINITE ASCENT OR DESCENT AND PERTURBATIONS

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(Communicated by Palle E. T. Jorgensen)

Abstract. In this note we prove that the collection of upper (lower) semi-Fredholm operators with finite ascent (descent) is closed under commuting operator perturbations that belong to the perturbation class associated with the set of upper (lower) semi-Fredholm operators. Then, as a corollary we get the main result of S. Grabiner (Proc. Amer. Math. Soc. 71 (1978), 79-80).

Let $X$ be an infinite-dimensional complex Banach space and denote the set of bounded (compact) linear operators on $X$ by $B(X)$ ($K(X)$). For $T$ in $B(X)$ throughout this paper $N(T)$ and $R(T)$ will denote, respectively, the null space and the range space of $T$. Set $N^∞(T) = \bigcup_n N(T^n)$, $R^∞(T) = \bigcap_n R(T^n)$, $\alpha(T) = \dim N(T)$ and $\beta(T) = \dim X/R(T)$. Recall that an operator $T \in B(X)$ is semi-Fredholm if $R(T)$ is closed and at least one of $\alpha(T)$ and $\beta(T)$ is finite. For such an operator we define an index $i(T)$ by $i(T) = \alpha(T) - \beta(T)$. Let $\Phi_+(X)$ ($\Phi_-(X)$) denote the set of upper (lower) semi-Fredholm operators, i.e., the set of semi-Fredholm operators with $\alpha(T) < \infty$ ($\beta(T) < \infty$). The perturbation classes associated with $\Phi_+(X)$ and $\Phi_-(X)$ are denoted, respectively, by $P(\Phi_+(X))$ and $P(\Phi_-(X))$, i.e.,

$$P(\Phi_+(X)) = \{T \in B(X): T + S \in \Phi_+(X) \text{ for all } S \in \Phi_+(X)\}$$

and

$$P(\Phi_-(X)) = \{T \in B(X): T + S \in \Phi_-(X) \text{ for all } S \in \Phi_-(X)\}.$$

Recall that $a(T)$ ($d(T)$), the ascent (descent) of $T \in B(X)$, is the smallest non-negative integer $n$ such that $N(T^n) = N(T^{n+1})$ ($R(T^n) = R(T^{n+1})$). If no such $n$ exists, then $a(T) = \infty$ ($d(T) = \infty$). For a subset $M$ of $X$ let $\overline{M}$ denote the closure of $M$. The main result of this note is the following theorem.

**Theorem 1.** Suppose that $T$, $K \in B(X)$ and $TK = KT$. Then

(1.1) \hspace{1em} $T \in \Phi_+(X)$, $a(T) < \infty$ and $K \in P(\Phi_+(X)) \Rightarrow a(T + K) < \infty$,

(1.2) \hspace{1em} $T \in \Phi_-(X)$, $a(T) < \infty$ and $K \in P(\Phi_-(X)) \Rightarrow d(T + K) < \infty$.

Received by the editors March 24, 1994 and, in revised form, June 16, 1994.

1991 Mathematics Subject Classification. Primary 47A53, 47A55.

Key words and phrases. Ascent, descent, semi-Fredholm.

Proof. To prove (1.1) suppose that $T \in \Phi_+(X)$, $a(T) < \infty$, $K \in P(\Phi_+(X))$ and $TK = KT$. Set

$$T_\lambda = T + \lambda K, \quad \lambda \in [0, 1].$$

For each $\lambda \in [0, 1]$, $T_\lambda \in \Phi_+(X)$. By [2, Theorem 3], there exists $\varepsilon = \varepsilon(\lambda) > 0$ such that

$$N^\infty(T_\lambda) \cap R^\infty(T_\lambda) = N^\infty(T_\mu) \cap R^\infty(T_\mu)$$

in the open disc $S(\lambda)$ with center $\lambda$ and radius $\varepsilon$. Since $[0, 1]$ is compact, we can obtain a finite set $\{\lambda_0, \lambda_1, \ldots, \lambda_n\}$ of points on $[0, 1]$ such that $\lambda_0 = 0$, $\lambda_n = 1$ and $[0, 1] \subset \bigcup_{i=0}^n S(\lambda_i)$ with $S(\lambda_i) \cap S(\lambda_{i+1}) \cap [0, 1] \neq \emptyset$ for $i = 0, 1, \ldots, n-1$. Now since $a(T) < \infty$, it follows that $N^\infty(T) \cap R^\infty(T) = N^\infty(T) \cap R^\infty(T) = \{0\}$ [6, Proposition 1.6. (i)], and by (1.3) we have that $N^\infty(T_\mu) \cap R^\infty(T_\mu) = \{0\}$ for all $\mu$ in $S(\lambda_i)$. Hence, because $S(\lambda_i)$ overlaps $S(\lambda_0)$, we conclude that $N^\infty(T_\mu) \cap R^\infty(T_\mu) = \{0\}$ for all $\mu$ in $S(\lambda_i)$. By proceeding along the family of disc, we finally deduce that $N^\infty(T_\mu) \cap R^\infty(T_\mu) = \{0\}$ for all $\mu$ in $S(\lambda_n)$. Thus $N^\infty(T_\lambda) \cap R^\infty(T_\lambda) = \{0\}$, and again by [6, Proposition 1.6. (i)] it follows that $a(T + K) < \infty$. This completes the proof of (1.1).

To prove (1.2) suppose that $T \in \Phi_-(X)$, $d(T) < \infty$, $K \in P(\Phi_-(X))$ and $TK = KT$. Then $T^* \in \Phi_+(X^*)$, $a(T^*) < \infty$, $T^*K^* = K^*T^*$ and $T^* + \lambda K^* \in \Phi_+(X^*)$, $\lambda \in [0, 1]$ [1, pp. 7–8]. Part (1.2) now follows directly from the proof of part (1.1). This completes the proof.

Let us remark that the commutativity condition in Theorem 1 is essential, even for compact $K$ [1, pp. 13–14]. In order to prove Theorem 1 we need the hypothesis that $K$ commutes with $T$ in the place where we invoke [2, Theorem 3].

Now as a corollary, we get the main result of S. Grabiner [3, Theorem 2] (see also [4, Theorem 7.9.2]). Our formulation of that result is somehow different from that of S. Grabiner's, but appropriate to Theorem 1.

Corollary 2. Suppose that $T \in B(X)$, $K \in K(X)$ and $TK = KT$. Then

$$T \in \Phi_+(X) \text{ and } a(T) < \infty \Rightarrow a(T + K) < \infty,$$

$$T \in \Phi_-(X) \text{ and } d(T) < \infty \Rightarrow d(T + K) < \infty.$$

Proof. By Theorem 1 and the fact that $K(X) \subset P(\Phi_+(X)) \cap P(\Phi_-(X))$ [1, Theorem 5.6.9].

To help readers understand how far our Theorem 1 extends the result in [3], we refer them to the discussion of perturbation ideals in Sections 5.5 and 5.6, pages 95–102 in [1]. Let us mention in particular that $P(\Phi_+(X))$ includes all strictly singular operators.

Let $\sigma_a(T)$ and $\sigma_d(T)$ denote, respectively, the approximate point spectrum and the approximate defect spectrum of an element $T$ of $B(X)$. Set

$$\sigma_{ab}(T) = \bigcap_{T K = K T} \sigma_a(T + K) \text{ and } \sigma_{db}(T) = \bigcap_{T K = K T} \sigma_d(T + K).$$

We call $\sigma_{ab}(T)$ and $\sigma_{db}(T)$, respectively, Browder’s essential approximate point spectrum of $T$ and Browder’s essential approximate defect spectrum of $T$ [5], [7]. Recall that by [5, Theorem 2.1] a complex number $\lambda \not\in \sigma_{ab}(T)$ ($\sigma_{db}(T)$)
if and only if \( T - \lambda \in \Phi_+(X) \), \( i(T) \leq 0 \) and \( a(T - \lambda) < \infty \) (\( T - \lambda \in \Phi_-(X) \), \( i(T) \geq 0 \) and \( d(T - \lambda) < \infty \)).

Finally, as a second application of Theorem 1 we have

**Corollary 3.** Suppose that \( T \in B(X) \). Then

\[
\sigma_{ab}(T) = \bigcap_{TK=KT} \sigma_d(T + K) 
\]

and

\[
\sigma_{db}(T) = \bigcap_{TK=KT} \sigma_d(T + K). 
\]

**Proof.** By Theorem 1 and [5, Theorem 2.1].

**ACKNOWLEDGMENT**

I am grateful to Professor Sandy Grabiner for helpful correspondence and conversations. The author also thanks the referee for helpful comments and suggestions concerning the paper.

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