SOME TRACE INEQUALITIES FOR DISCRETE GROUPS OF MÖBIUS TRANSFORMATIONS

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Abstract. We show that if \( \langle A, B \rangle \) is discrete where \( A, B \in \text{SL}(2,\mathbb{C}) \) and \( \text{tr}(ABA^{-1}B^{-1}) \neq 2 \), \( \text{tr}(ABAB^{-1}) \neq 2 \), and \(|\text{tr}^2(A) - 4| \leq 2(\cos(2\pi/7) + \cos(\pi/7) - 1) = 1.0489\ldots \), then

\[ |\text{tr}(ABA^{-1}B^{-1}) - 2| \geq 2 - 2\cos(\pi/7) = 0.198\ldots \]

It follows from above that if \( \langle X, Y \rangle \) is discrete with \( \text{tr}(X) = \text{tr}(Y) \neq 0 \) and \( \text{tr}(XYX^{-1}Y^{-1}) \neq 2 \), then

\[ |\text{tr}(XYX^{-1}Y^{-1}) - 2| \geq 2 - 2\cos(\pi/7) = 0.198\ldots \]

Both inequalities are sharp.

1. Introduction

A subgroup of \( \text{SL}(2,\mathbb{C}) \) is said to be discrete if it does not contain any convergent sequences of distinct elements. There is an important necessary condition due to Jørgensen [8] for a two-generator group to be discrete.

If \( A \) and \( B \) generate a discrete subgroup of \( \text{SL}(2,\mathbb{C}) \), then

\[ |\text{tr}^2(A) - 4| + |\text{tr}(ABA^{-1}B^{-1}) - 2| \geq 1, \tag{1.1} \]

unless \( BAB^{-1} \in \{ A, A^{-1} \} \), in which case the subgroup is elementary.

The commutator trace is not uniformly bounded away from 2. In other words, there does not exist a positive real number \( K \) such that \( |\text{tr}(ABA^{-1}B^{-1}) - 2| \geq K \) holds whenever \( A \) and \( B \) generate a nonelementary discrete group [10]. However, Jørgensen has shown that

\[ |\text{tr}(XYX^{-1}Y^{-1}) - 2| > 0.125 \tag{1.2} \]

if \( X \) and \( Y \) with equal traces generate a nonelementary discrete subgroup [11]. Inequality (1.2) was sharpened by Gehring and Martin [4] to give

\[ |\text{tr}(XYX^{-1}Y^{-1}) - 2| > 0.193, \tag{1.3} \]

and they conjectured that

\[ |\text{tr}(XYX^{-1}Y^{-1}) - 2| \geq 2 - 2\cos(\pi/7) = 0.198\ldots \tag{1.4} \]
if \( \text{tr} \left( X Y X^{-1} Y^{-1} \right) \neq 2 \), \( \text{tr}^2(X) = \text{tr}^2(Y) \neq 0 \), and \( X, Y \) generate a discrete subgroup.

We show that \( \text{tr} \left( ABA^{-1}B^{-1} \right) - 2 \) is bounded away from zero if \( |\text{tr}^2(A) - 4| \) is not too large. In particular, if \( A \) and \( B \) generate a discrete group and if

\[
|\text{tr}^2(A) - 4| \leq 2(\cos(2\pi/7) + \cos(\pi/7) - 1) = 1.0489 \ldots
\]

then

\[
|\text{tr}(ABA^{-1}B^{-1}) - 2| \geq 2 - 2\cos(\pi/7) = 0.198 \ldots
\]

We show in section 4 that the conjecture (1.4) of Gehring and Martin is true. We then apply these results to get many other inequalities.

2. Notation

Let \( M \) denote the group of all Möbius transformations of the extended complex plane \( \mathbb{C} = \mathbb{C} \cup \{\infty\} \). We associate with each

\[
f = \frac{az + b}{cz + d} \in M, \quad ad - bc = 1,
\]

the matrix

\[
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{C})
\]

and set \( \text{tr}(f) = \text{tr}(A) \), where \( \text{tr}(A) \) denotes the trace of \( A \). Note that \( \text{tr}(f) \) is defined up to sign.

For each \( f \) and \( g \) in \( M \) we let \([f, g]\) denote the multiplicative commutator \( fgf^{-1}g^{-1} \). We call the three complex numbers

\[
\beta(f) = \text{tr}^2(f) - 4, \quad \beta(g) = \text{tr}^2(g) - 4, \quad \gamma(f, g) = \text{tr}([f, g]) - 2
\]

the parameters of the two-generator group \( \langle f, g \rangle \) and write

\[
\text{par}((f, g)) = (\gamma(f, g), \beta(f), \beta(g)).
\]

These parameters are independent of the choice of representative matrices for \( f \) and \( g \), and they determine \( \langle f, g \rangle \) up to conjugacy whenever \( \gamma(f, g) \neq 0 \) [2]. But see [1] for three-generator groups. Note that \( \gamma(f, g) \neq 0 \) if and only if \( f \) and \( g \) do not have a common fixed point in \( \mathbb{C} \).

3. A Sharp Bound

3.1. Theorem. If \( \langle f, g \rangle \) is discrete with \( \gamma(f, g) \neq 0 \) and \( \beta(g) \neq -4 \) and if

\[
|\beta(f)| \leq c = 2(\cos(2\pi/7) + \cos(\pi/7) - 1) = 1.0489 \ldots
\]

then

\[
|\gamma(f, g)| \geq d = 2 - 2\cos(\pi/7) = 0.198 \ldots
\]

Inequality (3.3) holds when \( \langle f, g \rangle \) is the (2,3,7) triangle group with parameters \( \beta(f) = \beta(g) = c, \quad \gamma(f, g) = -d \).

Proof. Set

\[
m = \inf\{|\gamma(f, g)| : |\beta(f)| \leq c, \gamma(f, g) \neq 0, \beta(g) \neq -4, \langle f, g \rangle \text{ discrete}\}.
\]

Suppose that \( m < d = 2 - 2\cos(\pi/7) \). We will obtain a contradiction.
For each \( \epsilon \) with \( 0 < \epsilon < \frac{1}{2}(d - m) \), there exist \( f, g \) such that
\[
|\gamma(f, g)| < m + \epsilon, \quad |\beta(f)| \leq c, \quad \beta(g) \neq -4.
\]
If \( \gamma(f, g) = \beta(f) \), then \( f \) is of order 3, 4 or 6 by [4, Lemma 2.10], and hence
\[
|\gamma(f, g)| = |\beta(f)| \geq 1. \quad \text{This contradicts the assumption that } |\gamma(f, g)| < d.
\]
Thus we may assume that \( \gamma(f, g) \neq \beta(f) \). By [7, Lemma 2.29], there exists an elliptic \( h \) of order 2 such that \( \langle f, h \rangle \) is discrete with \( \gamma(f, h) = \gamma(f, g) \).

We will use the following two formulae for \( F \) and \( G \) in \( \mathbb{M} \):
\[
\gamma([F, G], F[F, G]F^{-1}) = -\gamma^2(F, G)(\gamma(F, G) - \beta(F))(\beta(F) + 4),
\]
\[
\beta([F, G]) = \gamma(F, G)(\gamma(F, G) + 4).
\]

We define
\[
f_0 = f, \quad g_0 = h, \quad f_{n+1} = \gamma(f_n, g_n), \quad g_{n+1} = f_n[f_n, g_n]f_n^{-1}.
\]
If \( |\beta(f_2)| \leq c \) and \( \gamma(f_2, g_2) \neq 0 \), then by the definition of \( m \), \( |\gamma(f_2, g_2)| \geq m \).

Let \( \gamma = \gamma(f, h) \), \( \beta = \beta(f) \). We have
\[
|\gamma(f_1, g_1)| = |\gamma^2(\gamma - \beta)(\beta + 4)| \leq d^2(d + c)(c + 4) = d(d + c),
\]
\[
|\beta(f_2)| = |\gamma(f_1, g_1)(\gamma(f_1, g_1) + 4)| \leq c.
\]

So, \( |\gamma(f_2, g_2)| \geq m \), that is,
\[
(3.4) \quad |\gamma^2(\gamma - \beta)(\beta + 4)^2(\gamma(\gamma - \beta)(\beta + 4) + \gamma + 4)(\gamma + 2)^2| \geq m.
\]

Set \( p(\gamma, \beta) = \gamma^4(\gamma - \beta)^2(\beta + 4)^2(\gamma(\gamma - \beta)(\beta + 4) + \gamma + 4)(\gamma + 2)^2 \).

Consider one of the polynomials in [7, Lemma 2.1],
\[
\gamma(f, h f^{-1} h^{-1} f h f^{-1} h^{-1} f^{-1} h) = \gamma(\gamma^2 - (\beta - 1)\gamma - (\beta - 1))^2.
\]

We consider three cases.

Case 1. Suppose that \( \gamma^2 - (\beta - 1)\gamma - (\beta - 1) = 0 \). Then
\[
\beta = 1 + \frac{\gamma^2}{1 + \gamma}, \quad p(\gamma, \beta) = \frac{\gamma^4(5 + 5\gamma + \gamma^2)^2(\gamma + 2)^4}{(1 + \gamma)^6}.
\]

Let
\[
q(z) = \frac{z^4(5 + 5z + z^2)^2(z + 2)^4}{(1 + z)^6}.
\]
It is easy to check that
\[
\max_{|z| = d} |q(z)| = 1,
\]
and this maximum is obtained when \( z = -d \).

Since \( m + \frac{1}{2}(d - m) < d \), there is a constant \( a \) such that
\[
\max_{|z| \leq m + \frac{1}{2}(d - m)} |q(z)| \leq a < 1.
\]

By (3.4),
\[
(m + \epsilon)a \geq |\gamma p(\gamma, \beta)| \geq m.
\]
Thus
\[
\epsilon \geq (1 - a)m/a.
\]

By [4, Lemma 3.25], \( m > 0.193 \). Taking \( \epsilon < \min\{ (d - m)/2, (1 - a)m/a \} \) will give a contradiction.
Case 2. Suppose that \( \beta(h^{-1}f^{-1}fhfh^{-1}f^{-1}h) = -4 \). If \( \beta = 0 \), then \(|\gamma| \geq 1\) by the Shimizu-Leutbecher inequality [13, II.C.5]. This contradicts the assumption that \(|\gamma| < d\). If \( \beta \neq 0 \), then we may assume that \( f \) and \( h \) are represented by

\[
A = \begin{pmatrix} u & 0 \\ 0 & 1/u \end{pmatrix}, \quad B = \begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix}.
\]

Thus \( \beta = (u - 1/u)^2 \), \( \gamma = -e_{12}e_{21}(u - 1/u)^2 \). Elementary calculations show that

\[
BA^{-1}B^{-1}ABAB^{-1}A^{-1}B = \begin{pmatrix}
  e_{11}((\gamma + 1)^2 - \gamma u^{-2}) & e_{12}(\gamma^2 - (\beta - 1)\gamma - (\beta - 1)) \\
  e_{21}(\gamma^2 - (\beta - 1)\gamma - (\beta - 1)) & e_{22}((\gamma + 1)^2 - \gamma u^2)
\end{pmatrix}.
\]

Since \( hf^{-1}h^{-1}fhfh^{-1}f^{-1}h \) is of order two,

\[
(\beta - 1)^2 = e_{11}((\gamma + 1)^2 - \gamma u^{-2}) + e_{22}((\gamma + 1)^2 - \gamma u^2) = 0.
\]

Notice that \( e_{11} + e_{22} = 0 \) (\( h \) is of order two). If \( e_{11} \neq 0 \), then (3.5) implies that \( \beta(\beta + 4) = 0 \), a contradiction. If \( e_{11} = 0 \), then \( e_{12}e_{21} = -1 \). Hence \( \gamma = \beta \), another contradiction.

Case 3. Suppose that Case 1 and Case 2 do not hold. By the definition of \( m \),

\[
|\gamma(f, hf^{-1}h^{-1}fhfh^{-1}f^{-1}h)| = |\gamma(\gamma^2 - (\beta - 1)\gamma - (\beta - 1))^2| \geq m.
\]

It follows that

\[
|\beta - 1| > \frac{1}{1 + d} (\sqrt{m/(m + \epsilon)} - d^2).
\]

Let

\[
\beta = pe^{i\theta}, \quad -\pi < \theta \leq \pi, \quad s = \frac{1}{1 + d} (\sqrt{m/(m + \epsilon)} - d^2).
\]

From (3.6), we have

\[
\cos \theta \leq \frac{1}{2p}(1 + \rho^2 - s^2).
\]

We apply Jorgensen’s inequality (1.1) to get \( \rho = |\beta| \geq 1 - d \). Thus \( 1 - d \leq \rho \leq c \). It follows from (3.7) that

\[
\cos \theta \leq \frac{1}{2c}(1 + c^2 - s^2).
\]

By taking sufficiently small \( \epsilon \), we obtain \(|\theta| > 0.8\). We now expand \( p(\gamma, \beta) \) to get

\[
p(\gamma, \beta) = \sum_{n=4}^{10} p_n(\beta)\gamma^n,
\]

where \( p_n(z) = (z + 4)^2q_n(z) \) and

\[
q_4(z) = 16z^2, \quad q_8(z) = 3z^3 - 46z + 24,
q_5(z) = -4(z^3 + 4z^2 - 5z + 8), \quad q_9(z) = -3z^2 - 8z + 17,
q_6(z) = -4(z^4 + z^3 - 14z^2 + 10z - 4), \quad q_{10}(z) = z + 4,
q_7(z) = -(z - 1)(z^3 - 7z^2 - 44z + 20),
\]
Consider the nonnegative subharmonic function \( w(z) = \sum_{n=4}^{10} |p_n(z)|d^n \) in the region \( D = \{ 1 - d \leq |z| \leq c, \ 0.8 \leq \arg z \leq 2\pi - 0.8 \} \). \( w(z) \) assumes its maximum on one of the four boundaries

\[
B_1 = \{ ce^{i\phi} : 0.8 \leq \phi \leq 2\pi - 0.8 \}, \quad B_3 = \{ xe^{i0.8} : 1 - d \leq x \leq c \},
\]
\[
B_2 = \{(1 - d)e^{i\phi} : 0.8 \leq \phi \leq 2\pi - 0.8 \}, \quad B_4 = \{ xe^{-i0.8} : 1 - d \leq x \leq c \}.
\]

It is easy to check that

\[
|p_{10}|d^{10} < 1.2 \times 10^{-5}, \quad |p_{6}|d^6 < 7.4 \times 10^{-2},
\]
\[
|p_{9}|d^9 < 1.9 \times 10^{-4}, \quad |p_{5}|d^5 < 2.2 \times 10^{-1},
\]
\[
|p_{8}|d^8 < 2.4 \times 10^{-3}, \quad |p_{4}|d^4 < 6.23 \times 10^{-1},
\]
\[
|p_{7}|d^7 < 1.7 \times 10^{-2},
\]

for \( z \in B_1 \); and

\[
|p_{10}|d^{10} < 9.2 \times 10^{-6}, \quad |p_{6}|d^6 < 6.1 \times 10^{-2},
\]
\[
|p_{9}|d^9 < 1.6 \times 10^{-4}, \quad |p_{5}|d^5 < 1.5 \times 10^{-1},
\]
\[
|p_{8}|d^8 < 1.9 \times 10^{-3}, \quad |p_{4}|d^4 < 3.4 \times 10^{-1},
\]
\[
|p_{7}|d^7 < 1.3 \times 10^{-2},
\]

for \( z \in B_2 \); and

\[
|p_{10}|d^{10} < 1.1 \times 10^{-5}, \quad |p_{6}|d^6 < 4.1 \times 10^{-2},
\]
\[
|p_{9}|d^9 < 1.6 \times 10^{-4}, \quad |p_{5}|d^5 < 1.2 \times 10^{-1},
\]
\[
|p_{8}|d^8 < 1.9 \times 10^{-3}, \quad |p_{4}|d^4 < 6.23 \times 10^{-1},
\]
\[
|p_{7}|d^7 < 9.4 \times 10^{-3},
\]

for \( z \in B_3 \cup B_4 \). Thus

\[
|p(\gamma, \beta)| \leq \max_{z \in D} \sum_{n=4}^{10} |p_n(z)|d^n < 0.94,
\]

and hence \( 0.94(m + \epsilon) > |\gamma p(\gamma, \beta)| \geq m \) by (3.4), a contradiction.

Therefore \( m \geq d \). Let \( \langle \phi, \psi \rangle \) denote the \( (2, 3, 7) \) triangle group with \( \phi^2 = \psi^3 = (\phi \psi)^7 = \text{id} \), and set \( f = [\phi, \psi] \), \( h = \phi \psi \), \( g = hfh^{-1} \). Then

\[
\beta(f) = \beta(g) = 2(\cos(2\pi/7) + \cos(\pi/7) - 1) = c,
\]
\[
\gamma(f, g) = \gamma(f, h)(\gamma(f, h) - \beta(f)) = 2 \cos(\pi/7) - 2 = -d. \quad \square
\]

3.8. Corollary. If \( \langle f, g \rangle \) is discrete with \( \gamma(f, g) \neq 0 \) and \( \gamma(f, g) \neq \beta(f) \) and if \( |\beta(f)| \leq 2(\cos(2\pi/7) + \cos(\pi/7) - 1) = 1.0489... \), then

\[
|\gamma(f, g)| \geq 2 - 2 \cos(\pi/7) = 0.198...
\]

Inequality (3.9) is sharp.

Proof. We observe that \( \gamma(f, g) = \gamma(f, fg) \). Thus if \( g \) or \( fg \) is not of order two, then (3.9) holds by Theorem 3.1. Suppose that \( g \) and \( fg \) are both of order two. Then \( \beta(fgfg^{-1}) = \beta(fg) = 0 \). Since

\[
\beta(fgfg^{-1}) = (\beta(f) - \gamma(f, g))(\beta(f) - \gamma(f, g) + 4),
\]

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we have $\beta(f) - \gamma(f, g) + 4 = 0$. Thus

$$|\gamma(f, g)| = |\beta(f) + 4| \geq 4 - c,$$

and hence (3.9) holds.

The example in Theorem 3.1 shows (3.9) is sharp. □

4. Equal trace problem

4.1. Theorem. Suppose that $\langle f, g \rangle$ is a discrete subgroup of $\mathbb{M}$ with $\gamma(f, g) \neq 0$ and $\beta(f) = \beta(g) \neq -4$. Then

$$|\gamma(f, g)| \geq 2 - 2\cos(\pi/7) = 0.198\ldots.$$  

Inequality (4.2) is sharp.

Proof. We use [4, Lemma 3.18]: For any discrete subgroup $\langle f, g \rangle$ with $\gamma(f, g) \neq 0$ and $\beta(f) = \beta(g) \neq -4$, if

$$\min\{|\beta(f)|, |\beta(fg)|, |\beta(fg^{-1})|\} \geq c = 2(\cos(2\pi/7) + \cos(\pi/7) - 1),$$

then $|\gamma(f, g)| \geq d = 2 - 2\cos(\pi/7)$.

If (4.3) holds, then (4.2) follows from [4, Lemma 3.18]. Otherwise, since

$$\gamma(f, g) = \gamma(fg, g) = \gamma(fg^{-1}, g),$$

we may assume by relabeling that $|\beta(f)| < c$. By assumption, $\beta(g) \neq -4$.

Hence $|\gamma(f, g)| \geq d$ by Theorem 3.1.

Finally the example in Theorem 3.1 shows (4.2) is sharp. □

5. Some consequences of Theorem 4.1

5.1. Theorem. Suppose that $\langle f, g \rangle$ is a discrete subgroup of $\mathbb{M}$ with $\gamma(f, g) \neq 0$ and $\gamma(f, g) \neq \beta(f)$. If $f$ is not of order two, then

$$|\gamma(f, g)(\beta(f) + 4)| \geq 2 - 2\cos(\pi/7) = 0.198\ldots,$$

$$|\gamma(f, g) - \beta(f) + 4| \geq 2 - 2\cos(\pi/7) = 0.198\ldots.$$

(5.4)

Each of these inequalities is sharp.

Proof. Consider the subgroup $\langle f, gfg^{-1} \rangle$.

$$\beta(gfg^{-1}) = \beta(f) \neq -4,$$

$$\gamma(f, gfg^{-1}) = \gamma(f, g)(\gamma(f, g) - \beta(f)) \neq 0.$$  

So, $|\gamma(f, gfg^{-1})| \geq d = 2 - 2\cos(\pi/7)$ by Theorem 4.1. The example in [4, p. 210] shows (5.2) is sharp.

By [7, Lemma 2.29], there exists an elliptic $h$ of order 2 such that $\langle f, h \rangle$ is discrete and $\gamma(f, h) = \gamma(f, g)$. Thus

$$\beta(hf) = \beta(fh) = \gamma(f, g) - \beta(f) - 4 \neq -4,$$

$$\gamma(hf, fh) = \gamma(f^2, h) = \gamma(f, h)(\beta(f) + 4) \neq 0.$$  

So, $|\gamma(hf, fh)| \geq d$ by Theorem 4.1. The example in [4, p. 210] and the fact that $\gamma(hf, h)(\beta(fh) + 4) = \gamma(f, h)(\gamma(f, g) - \beta(f))$ show that (5.3) is sharp.
By [7, Lemma 2.29], there exists an elliptic $h$ of order 2 such that $(f, h)$ is discrete and $\gamma(f, h) = \beta(f) - \gamma(f, g)$. Thus (5.4) follows from (5.3). The example for (5.3) and the property of $h$ show that (5.4) is sharp. □

5.5. Remark. Many universal constraints for a discrete Möbius group $G$ are obtained by studying the sequence $\{\text{tr}([f, g_n])\}$ where $f$ and $g_1$ are in $G$, $g_{n+1} = g_n f g_n^{-1}$. Among these results are the Shimizu-Leutbecher inequality, Jørgensen’s inequality, and some variants in [3], [5], [7], [14], and [15]. It follows from (5.2) that

$$\text{tr}[f, g_n] - 2 \geq d,$$

in the above process. One example is the $(2,3,7)$ triangle group $(f, g_1)$ for which $\gamma(f, g_1) = 2\cos(2\pi/7) - 1$, $\beta(f) = c$. Thus

$$\text{tr}[f, g_n] - 2 = 2\cos(\pi/7) - 2 = - .198 \ldots \quad \text{for } n = 2k,$$

$$\text{tr}[f, g_n] - 2 = 2\cos(2\pi/7) - 1 = .2469 \ldots \quad \text{for } n = 2k + 1.$$

See [6] for more about the iterated commutators.

If $f$ is the square of some element in a discrete group, then $|\gamma(f, g)| \geq d$ by (5.3). For example, let $(f, g)$ be a discrete group with $\gamma(f, g) \neq 0$. The Lie product of $f$ and $g$ defines a Möbius transformation $\phi$ which is elliptic of order two. The mapping $\phi f^{-1} g^{-1}$ is a square root of $fgf^{-1} g^{-1}$ [9, Section 4]. By (5.2), $|\gamma(f g f^{-1} g^{-1}, f) - \gamma(f, g)\gamma(f, g) - \beta(f)| \geq d$.

5.6. Theorem. Suppose that $(f, g)$ is discrete with $\gamma(f, g) \neq 0$.

- If $\beta(f) \neq -1$, then $|\gamma(f, g)| + |\beta(f) + 1| \geq c_1$, $\quad .426 < c_1 \leq .493 \ldots$.
- If $\beta(f) \neq -2$, then $|\gamma(f, g)| + |\beta(f) + 2| \geq c_2$, $\quad .806 < c_2 \leq 1$.
- If $\beta(f) \neq -3$, then $|\gamma(f, g)| + |\beta(f) + 3| \geq c_3$, $\quad .908 < c_3 \leq 1$.
- If $\beta(f) \neq -4$, then $|\gamma(f, g)| + |\beta(f) + 4| \geq c_4$, $\quad .890 < c_4 \leq 1.048 \ldots$.

Proof. Let $\gamma = \gamma(f, g)$, $\beta = \beta(f)$. It follows from (5.3) and the Arithmetic-Geometric Mean inequality that $c_4 \geq 2\sqrt{d} = .89 \ldots$. If we replace $f$ by $f^2$ in (5.3) and minimize $|\gamma| + |\beta + 2|$ subject to the constraint

$$|\gamma(f^2, g)(\beta(f^2) + 4)| = |\gamma(\beta + 4)(\beta + 2)^2| \geq d,$$

we get $c_2 > .806$. Next we replace $f$ by $f^3$ in (5.2) and (5.3). Minimizing $|\gamma| + |\beta + 3|$ subject to the constraint

$$|\gamma(f^3, g)(\gamma(f^3, g) - \beta(f^3))| = |\gamma(\beta + 3)^3(\gamma - \beta)| \geq d$$

gives $c_3 > .908$. Minimizing $|\gamma| + |\beta + 1|$ subject to the constraint

$$|\gamma(f^3, g)(\beta(f^3) + 4)| = |\gamma(\beta + 3)^2(\beta + 4)(\beta + 1)^2| \geq d$$

gives $c_1 > .426$.

We now give the upper bounds. Since $\gamma_1 = 2\cos(2\pi/7) - 1$, $\beta_1 = \gamma_1 - 1$, $\beta'_1 = -4$ are discrete parameters, $c_1 \leq 2(2\cos(2\pi/7) - 1) = .943 \ldots$. By [10], for $1 < a < \infty$, $\gamma_3 = 4(a - 1/a)^2$, $\beta_3 = -4$, $\beta'_3 = -(a + 1/a)^2$ are discrete parameters. Thus $|\gamma_3| + |\beta_3 + 3| \rightarrow 1$ as $a \rightarrow \infty$. Hence $c_3 \leq 1$. Since $\beta_2 = (\sqrt{5} - 5)/2$, $\gamma_2 = \beta_2 + 1$, $\beta'_2 = -4$ are discrete parameters, $c_2 \leq 1$. Note that $\gamma = 2\cos(2\pi/7) - 1$, $\beta = c$, $\beta' = c$ are discrete parameters. So
\[ \gamma_4 = 2 \cos(2\pi/7) - 1, \quad \beta_4 = \gamma - \beta - 4, \quad \beta'_4 = -4 \] are discrete parameters. Thus \[ c_4 \leq |\gamma_4| + |\beta_4 + 4| = c = 1.048 \ldots \]

5.7. **Remark.** There are some similar results in [14] and [15]. Gehring and Martin have shown that \[ |\gamma(f, g)| + |\beta(f) + 1| \geq 1 \] if \[ \gamma(f, g) \neq 0, \quad \beta(f) \neq -1, \] and \[ \gamma(f, g) \neq \beta(f) + 1. \]

5.8. **Lemma.** Suppose that \( A \) and \( B \) generate a discrete nonelementary subgroup of \( \text{SL}(2, \mathbb{C}) \). Then

\[ \|A - I\| \|B - I\| > k, \quad 0.46 < k \leq 0.52 \ldots . \]

**Proof.** For any \( C \in \text{SL}(2, \mathbb{C}) \), we define \( m(C) = \|C - C^{-1}\| \) where \( \| \cdot \| \) is the usual Hilbert-Schmidt norm of a matrix (see [5]). Then

\[ 4\|C - I\|^2 = 2|\text{tr}(C) - 2|^2 + m^2(C), \]

\[ \|A - I\|^2 \|B - I\|^2 = \frac{1}{4}|\text{tr}(A) - 2|^2|\text{tr}(B) - 2|^2 + \frac{1}{8}|\text{tr}(A) - 2|^2 m^2(B) + \frac{1}{8}|\text{tr}(B) - 2|^2 m^2(A) + \frac{1}{16} m^2(A) m^2(B). \]

Let \( x = \min\{|\text{tr}^2(A) - 4|, |\text{tr}^2(B) - 4|\} \). By [5, Theorem 2.7],

\[ m^2(C) \geq 2|\text{tr}^2(C) - 4|, \quad m^2(A) m^2(B) \geq 16|\text{tr}(A, B) - 2|. \]

If \( x \leq 0.8 \), then \( \|A - I\| \|B - I\| > 0.46 \) by (5.9) and Jørgensen's inequality. If \( 0.8 \leq x \leq c \), then we replace \( m^2(A) m^2(B) \) by \( 16d \) and get \( \|A - I\| \|B - I\| > 0.46 \). If \( x \geq c \), then \( \|A - I\| \|B - I\| > 0.46 \) by (5.9).

Let \( (\phi, \psi) \) denote the \((2, 3, 7)\) triangle group with \( \phi^2 = \psi^3 = (\phi \psi)^7 = \text{id}. \) The transformations \( \phi \) and \( \psi \) can be represented by the matrices

\[ A = \frac{i}{\sin a} \begin{pmatrix} -\cos b & -p \\ p & \cos b \end{pmatrix}, \quad B = \begin{pmatrix} e^{ia} & 0 \\ 0 & e^{-ia} \end{pmatrix}, \]

where \( a = \pi/3, \quad b = \pi/7 \) and \( p = (\cos^2 b - \sin^2 a)^{1/2} \) [12, p. 88]. We set \( C = [A, B] \) and \( D = AB \). Then

\[ \gamma(C, D) = 2 \cos(2\pi/7) - 1, \quad \beta(C) = c, \quad \beta(D) = 2 \cos(2\pi/7) - 2. \]

We can find a Möbius transformation \( h \) which sends the fixed points of \( C \) to \( \{w, -w\} \) and sends the fixed points of \( D \) to \( \{1/w, -1/w\} \). By [5, Lemma 2.12], such a \( w \) satisfies the equation

\[ (w^2 - 1/w^2)^2 = 16 \frac{\gamma(C, D)}{\beta(C) \beta(D)}. \]

Let \( u = |w|^2 + 1/|w|^2 \) and \( v = 2 \cos(2\pi/7) - 1 \). Then \( m^2(hCh^{-1}) = u \beta(C) \) and \( m^2(hDh^{-1}) = -u \beta(D) \) by [5, Lemma 2.10]. Therefore,

\[ \|hCh^{-1} - I\| \|hDh^{-1} - I\| = \left( v + \frac{1}{4}v^2d^2 + \frac{1}{2}(v^2 - v^3 + cd^2)\sqrt{v/c(1-v)} \right)^{1/2} = 0.5214 \ldots . \]

5.10. **Remark.** Waterman has shown that \( \|A - I\| \|B - I\| > \sqrt{2} - 1 \) by means of Jørgensen's inequality [16].
Set $E = BCB^{-1}$. By [5, Lemma 2.27], there exists an $h$ in $M$ such that $m^2(hCh^{-1}) = 2\beta(C)$ and $m^2(hEh^{-1}) = 2\beta(E)$. So $||hCh^{-1} - I|| = ||hEh^{-1} - I|| = ((v^2 + c)/2)^{1/2} = .7449 \ldots$. Therefore, if $(A, B)$ is a discrete nonelementary subgroup of $SL(2, \mathbb{C})$, then
\[ \max\{||A - I||, ||B - I||\} > t, \quad .67 < t \leq .74 \ldots \]

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