ON CHARACTERISTICS
OF CIRCLE-ININVARIANT PRESYMPLECTIC FORMS

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ABSTRACT. We prove that a circle-invariant exact 2-form of rank $2n$ on a compact $(2n+1)$-dimensional manifold admits two closed characteristics. This solves a particular case of a generalized Weinstein conjecture.

1. Introduction

A contact manifold is a couple $(M, \alpha)$, where $M$ is a smooth manifold of dimension $2n+1 \geq 3$ and $\alpha$ is a contact form on $M$, that is, $\alpha$ is a 1-form such that $\alpha \wedge (d\alpha)^n$ is everywhere nonzero. The characteristic vector field of $\alpha$ is the unique vector field $\xi$ on $M$ such that $i_{\xi} \alpha = 1$ and $i_{\xi} d\alpha = 0$. Here, $i_{(\cdot)}$ denotes the interior product operation. An integral curve of the vector field $\xi$ is called a characteristic of the contact manifold $(M, \alpha)$.

In [WEI], Weinstein conjectured that there exists at least one closed characteristic on a compact (simply connected) contact manifold. Viterbo [VIT] proved this conjecture for compact contact manifolds $(M, \alpha)$ which are hypersurfaces of contact type in $\mathbb{R}^{2n+2}$; that is, if $\gamma: M \to \mathbb{R}^{2n+2}$ is the embedding of $M$ into $\mathbb{R}^{2n+2}$ and $\Omega = \sum_{i=1}^{n+1} dx_i \wedge dy_i$ is the standard symplectic form, then $\gamma^* \Omega = d\alpha$. Several generalizations of the results were subsequently obtained by Hofer-Viterbo [HV1], [HOV]. Their methods are analytical: Viterbo first treated this problem as a problem of finding periodic solutions of Hamilton equations on an energy surface and translated it into finding critical points of some action functional on some loop space. The subsequent approach of Hofer and Viterbo uses, in addition, the theory of pseudo-holomorphic curves [HOV], [HV1], [HOF].

Recently, Hofer [HOF] proved the Weinstein conjecture for $S^3$ and for any compact 3-manifold $M$ with $\pi_2(M) \neq 0$ or the given contact form is over-twisted (see Eliashberg [ELI] for the definition). Observe that the closed characteristic Hofer finds must be a contractible loop.
In [BAN], one of us proved that if the contact manifold \((M, \alpha)\) is K-contact, then there are at least two closed characteristics and these are not necessarily contractible.

In this note, we show that this result is a consequence of a more general one on the existence of closed characteristics for exact circle-invariant 2-forms of maximum rank. In particular, the exterior derivative of a circle-invariant contact form falls into this category and therefore, we will obtain that any circle-invariant contact form on a compact manifold admits at least two closed characteristics. Moreover, our theorem guarantees the existence of two periodic solutions of the Hamilton equations on other hypersurfaces in \(\mathbb{R}^{2n}\) which are not necessarily of contact type.

2. Characteristics of presymplectic forms and generalization of the Weinstein conjecture

Let \(M\) be a \((2n+1)\)-dimensional manifold. A presymplectic form \(\omega\) on \(M\) is a closed 2-form of constant rank \(2n\). The kernel of \(\omega\) is a completely integrable \(p\)-dimensional distribution and hence defines a \(p\)-dimensional foliation \(F_\omega\), called the characteristic foliation of \(\omega\).

Question. When does the foliation \(F_\omega\) admit a compact leaf?

In this note we focus our attention on the case \(\dim(F_\omega) = 1\). If \(\omega\) is not exact, it is easy to construct, following Zehnder, a presymplectic form \(\omega\) such that \(F_\omega\) admits no compact leaf [ZEH].

We prove the following result:

**Theorem.** Let \(M\) be a \((2n+1)\)-dimensional oriented compact manifold with a 1-form \(\alpha\) such that the 2-form \(\omega = d\alpha\) has rank \(2n\) everywhere. If there exists a locally free circle action on \(M\) which preserves \(\omega\), then the characteristic flow of \(\omega\) has at least two closed leaves.

**Proof.** For \(s \in S^1\), we denote by \(s: M \to M\) the diffeomorphism given by the \(S^1\) action: \(s(x) = s \cdot x\). The 1-form

\[
\alpha_0 = \int_{S^1} (s^*\alpha)\, d\sigma
\]

is \(S^1\)-invariant. Here \(\sigma\) is the Haar measure on \(S^1\). Let \(Z\) be the infinitesimal generator of the \(S^1\) action. Then we have

\[
L_Z \alpha_0 = 0,
\]

where \(L_Z\) is the Lie derivative \(L_Z(.) = i_Z d(.) + d i_Z(.)\). We define a smooth function \(S: M \to R\) by

\[
S(x) = -\alpha_0(x)(Z(x)) = -\int_{S^1} (s^*\alpha)(Z(x))\, d\sigma
\]

that is, \(S = -i_Z \alpha_0\). We have then \(dS = i_Z d\alpha\). Indeed, observe first the fact that

\[
\alpha_1 = \alpha - \alpha_0 = \int_{S^1} (\alpha - s^*\alpha)\, d\sigma
\]
is a closed 1-form because $s^*d\alpha = d\alpha$ and

$$d\alpha_1 = \int_{S^1} (d\alpha - s^*d\alpha) d\sigma = 0.$$  

Hence

$$L_Z\alpha = L_Z(\alpha_0 + \alpha_1) = L_Z\alpha_1 \quad \text{since} \quad L_Z\alpha_0 = 0$$

$$= di_Z\alpha_1 + i_Z d\alpha_1 = di_Z\alpha_1 \quad \text{since} \quad d\alpha_1 = 0.$$  

Let $\xi$ be a nonsingular section of the characteristic distribution $F = F_\omega$ of $\omega$. The vector field $\xi$ verifies

$$i_\xi \omega = 0.$$  

Hence, $L_\xi \omega = di_\xi \omega + i_\xi d\omega = di_\xi \omega = 0$ since $d\omega = 0$. We see that

$$(2.1) \quad i_{[\xi, Z]} \omega = L_\xi i_Z \omega - i_Z L_\xi \omega = 0$$  

since $L_\xi \omega = 0$ and

$$L_\xi i_Z \omega = i_\xi di_Z \omega + di_\xi i_Z \omega$$

$$= i_\xi di_Z \omega = i_\xi di_Z d\alpha = i_\xi d^2 S = 0.$$  

The identity (2.1) means that $[\xi, Z]$ is proportional to $\xi$, that is

$$[\xi, Z] = \lambda \xi$$  

for some function $\lambda$. In other words, $Z$ is a foliate vector field for the characteristic flow of $\omega = d\alpha$.

Let $U$ be a flow box (for the flow of $\xi$) and

$$\psi: U \to D^{2n} \times (-\epsilon, \epsilon)$$

be a chart around $p \in U$ with $\psi(p) = ((0, 0), 0)$. For $q \in U$ we write

$$\psi(q) = ((x, y), z), \quad x = (x_1, \ldots, x_n), \quad y = (y_1, \ldots, y_n).$$

In this chart, $\xi|_U = \frac{\partial}{\partial z}$. The identity $[\xi, Z] = \lambda \xi$ then means that in our chosen coordinates, we have

$$(2.2) \quad Z(q) = \sum_{i=1}^n X_i(x, y) \frac{\partial}{\partial x_i} + \sum_{i=1}^n Y_i(x, y) \frac{\partial}{\partial y_i} + u(x, y, z) \frac{\partial}{\partial z}.$$  

Now let $p \in M$ be a critical point of $S$. Since $i_Z d\alpha = i_Z \omega = dS$, one has

$$0 = (dS)(p) = \omega(p)(Z(p)).$$  

This implies that

$$Z(p) = u(p)\xi(p)$$  

since $\xi(p)$ spans the kernel of $\omega(p)$. Hence within our chart centered at $p \in U$, $\psi(p) = ((0, 0), 0)$, we have, by (2.2)

$$Z(p) = \sum_{i=1}^n X_i(0, 0) \frac{\partial}{\partial x_i} + \sum_{i=1}^n Y_i(0, 0) \frac{\partial}{\partial y_i} + u((0, 0), 0) \frac{\partial}{\partial z} = u((0, 0), 0) \frac{\partial}{\partial z}.$$  

Let $q \in \mathcal{O}_p(\xi) \cap U$ where $\mathcal{O}_p(\xi)$ denotes the orbit of $\xi$ through $p$. Then within our coordinates

$$\psi(q) = ((0, 0), t),$$  

and

\[
Z(q) = \sum_{i=1}^{n} X_i(0, 0) \frac{\partial}{\partial x_i} + \sum_{i=1}^{n} Y_i(0, 0) \frac{\partial}{\partial y_i} + u((0, 0), t) \frac{\partial}{\partial z} = u(0, 0, t) \frac{\partial}{\partial z}.
\]

Hence \( Z \) is proportional to \( \xi \) at any point along the orbit of \( \xi \) in the flow box \( U \). Covering \( \mathcal{O}_p(\xi) \) by flow boxes, one sees that \( Z \) is proportional to \( \xi \) all along \( \mathcal{O}_p(\xi) \), that is, \( Z \) is tangent to \( \mathcal{O}_p(\xi) \). Therefore \( \mathcal{O}_p(\xi) \) is an orbit of \( Z \) as well. Since all orbits of \( Z \) are closed, so is \( \mathcal{O}_p(\xi) \).

We have just established that at a critical point \( p \) of \( S \), the orbit of \( \xi \) passing through \( p \) is periodic. If \( S \) is not constant, it has at least two distinguished critical points, a maximum and a minimum since \( M \) is compact. These two points are on different orbits of \( \xi \) since \( S \) is constant along those orbits. To see this, we write

\[
L_\xi S = i_\xi dS = i_\xi i_Z d\alpha = -i_Z i_\xi d\alpha = 0.
\]

Finally, if \( S \) is constant, then \( dS = 0 \) so that

\[
i_Z d\alpha = 0,
\]

which means that \( Z \) is everywhere proportional to \( \xi \), hence all orbits of \( \xi \) are periodic. In any case, we see that there are at least two periodic orbits for \( \xi \).

**Example 1.** Let \( (P, d\lambda) \) be an exact symplectic manifold with a circle action preserving the symplectic form \( \Omega = d\lambda \). Let \( H: P \to \mathbb{R} \) be a circle-invariant smooth function on \( P \). Therefore any regular energy level \( M = H^{-1}(a) \) (where \( a \in \mathbb{R} \) is a regular value of \( H \)) is a \( C^\infty \) manifold equipped with a circle action which preserves the exact symplectic form \( \omega = \gamma^* \Omega \), where \( \gamma: M \to P \) is the natural inclusion. Our theorem asserts that the hamiltonian vector field \( X_H \) admits two periodic orbits on the energy level \( M \). For instance on \( \mathbb{R}^{2n} \), the symplectic form \( \Omega = \sum d\xi_i \wedge dy_i \) is invariant by \( S^1 \). If \( F: \mathbb{R}^{2n} \to \mathbb{R} \) is any smooth function and \( H: \mathbb{R}^{2n} \to \mathbb{R} \) is defined as

\[
H(x_1, \ldots, x_n, y_1, \ldots, y_n) = F(t_1, \ldots, t_n)
\]

where \( t_i = x_i^2 + y_i^2 \), then \( H \) is \( S^1 \)-invariant. Therefore the hamiltonian equations

\[
\dot{x}_i = -\frac{\partial H}{\partial y_i}, \quad \dot{y}_i = \frac{\partial H}{\partial x_i}
\]

admit at least two periodic solutions on each regular energy level.

**Remark.** We insist that in the example above, the hypersurface \( M \) is not necessarily of contact type.

**3. Circle-invariant contact forms**

As an immediate consequence of our theorem, we have the following

**Corollary.** The characteristic vector field \( \xi_\alpha \) of a compact contact manifold \( (M, \alpha) \) where \( \alpha \) is an \( S^1 \)-invariant contact form has at least two periodic orbits.

**Example 2.** \( S^1 \)-invariant contact forms are well known in dimension 3, by Lutz's work [LUT]. On \( S^3 \) for example, the exotic contact forms [BEN]

\[
\alpha_n = \cos\left(\frac{\pi}{4} + n(x_1^2 + x_2^2)\right)(x_1 dx_2 - x_2 dx_1) + \sin\left(\frac{\pi}{4} + n(x_1^2 + x_2^2)\right)(x_3 dx_4 - x_4 dx_3)
\]
are $S^1$-invariant. As an immediate consequence of our theorem, all of those forms have at least two closed characteristics.

Also, the existence of at least two closed characteristics for a K-contact manifold [BAN] is a consequence of the following:

**Proposition 1. K-contact forms are circle-invariant.**

*Proof.* Let $\alpha$ be a K-contact form with characteristic vector field $\xi$ and K-contact metric $g$. In particular, K-contactness requires that $L_\xi g = 0$ and $\alpha(X) = g(\xi, X)$ for all vector fields $X$ [BLA]. By the classical theorem of Meyer-Steenrod, the group $I(M)$ of isometries of the compact Riemannian manifold $(M, g)$ is a compact Lie group. Let $G$ be the closure in $I(M)$ of the 1-parameter group $\phi_t$, $t \in \mathbb{R}$, generated by $\xi$. Then $G$ is a compact abelian Lie group, hence a torus $T^k = S^1 \times \cdots \times S^1$. Therefore $S^1 \subset T^k$ acts smoothly on $M$. For any $s \in S^1$, $s \circ \phi_t = \phi_t \circ s$ since both $s$ and $\phi_t$ belong to the abelian group $G$. Hence $(s_* \xi) = \xi(s(x))$. Therefore

$$s^* \alpha(x)(X) = \alpha(s(x))(s_* X) = g(s(x))(\xi(s(x)), s_* X(x))$$

$$= g(s(x))(s_* \xi(x), (s_* X(x)) = s^* g(\xi, X)$$

$$= g(x)(\xi, X) = \alpha(X).$$

**Remark.** We insist on the fact that an $S^1$-invariant contact form is not necessarily K-contact. For example, the standard contact form

$$\alpha = \cos x_3 dx_1 + \sin x_3 dx_2$$

on the 3-dimensional torus is $S^1$-invariant but cannot be K-contact [RUK].

For a given circle action on a contact manifold $M$, the class of circle-invariant contact forms is stable by multiplication by nowhere zero circle-invariant functions. Therefore our theorem leads to:

**Proposition 2.** Let $\alpha_0$ be a circle-invariant contact form on a compact manifold $M$. Let $u: M \to \mathbb{R}$ be a smooth positive function which is also invariant under the circle action. Then the characteristic vector field $\xi_u$ of $\alpha = u\alpha_0$ has at least two periodic orbits.

**Example 3.** Let $\alpha_0$ be the standard contact form on $S^{2n+1}$ with characteristic vector field $\xi_0$, and let $u: S^{2n+1} \to \mathbb{R}$ be a positive function. Then

$$M = \{ z.\sqrt{u(z)} \}; \quad z \in S^{2n+1}$$

is a star-shaped hypersurface. Let $X_M$ be its characteristic vector field (that is, the characteristic vector field of the restriction of $\sum dx_i \wedge dy_i$ to $M$). Bennequin [BEN1] has observed that the diffeomorphism

$$\psi: M \to S^{2n+1}, \quad x \mapsto \frac{x}{\|x\|}$$

takes $X_M$ to the characteristic vector field $\xi_u$ of the contact form $\alpha = u\alpha_0$. If the function $u$ is constant along the characteristics of $\alpha_0$, then $X_M$ has at least two periodic orbits. We already knew by Rabinowitz's theorem that $X_M$ has one periodic orbit [RAB], for any smooth positive function $u$. 
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