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ON FUNCTIONS THAT ARE TRIVIAL COCYCLES FOR A SET OF IRRATIONALS. II

LAWRENCE W. BAGGETT, HERBERT A. MEDINA, AND KATHY D. MERRILL

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ABSTRACT. Two results are obtained about the topological size of the set of irrationals for which a given function is a trivial cocycle. An example of a continuous function which is a coboundary with non-L^1 cobounding function is constructed.

A function $v : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ is called an (additive) coboundary for an irrational α if there is a measurable function $w : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ such that $v(x) = w(x) - w(x + \alpha)$ a.e. (where we parameterize $\mathbb{R}/\mathbb{Z}$ by the interval [0,1) with addition mod 1). It is called trivial if $v(x) - c$ is a coboundary for some $c \in \mathbb{R}$. In either case the function w, which is unique up to an additive constant, is called the cobounding function. The question of whether particular functions or classes of functions are coboundaries for a given α has applications in ergodic theory and the representation theory of non-Type I groups (see, for example, [BM1],[ILR]). Recent research has revealed an interesting interplay between classes of functions and the types of irrationals for which they can be coboundaries (e.g., [BM2], [M]). Thus it is natural to look at the coboundary question from an opposite point of view, fixing a function v, and asking for exactly which irrationals v is a coboundary. A simple Fourier series argument shows that a trigonometric polynomial must be a coboundary for every irrational α. For other types of functions, this question is much harder to answer.

In a 1988 paper in this journal [B], L. Baggett presented a proof that the set of irrationals for which a given continuous function is a coboundary must be of the first category, unless the function is a trigonometric polynomial. Shortly after this paper appeared, P. Liardet, A. Iwanik, and P. Hellekalek pointed out a gap in the proof. This gap remains unfilled. In this paper, we present an altered version of that proof in the case that the original function is real-analytic. We also present a parallel result for L^1 functions in which the cobounding function is also required to be L^1. Finally, we display a counterexample which shows that the requirement of an L^1 cobounding function can be a genuine restriction.

Theorem 1. *Let v be an integrable, real-analytic function on the open interval $(0, 1)$, which is not a trigonometric polynomial. Then the set S of all irrationals for which v is a trivial cocycle is of the first category.*

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Proof. Suppose S is not of the first category. Let $f(x) = e^{2\pi i v(x)}$. For each integer k and each positive integer j, let $A_{k,j}$ be the set of numbers (rational or irrational) α for which there exists a constant λ and a measurable function g such that

(1) $\|g\|_2 \leq 1$.
(2) $|\lambda| = 1$.
(3) $|\int_0^1 g(x)e^{2\pi ikx}\,dx| \geq \frac{1}{j}$.
(4) $f(x)g(x+\alpha) = \lambda g(x)$ for almost all $x \in [0,1)$.

We claim that each set $A_{k,j}$ is a closed set. Let $\{\alpha_n\}$ be a sequence in $A_{k,j}$ converging to α. For each α_n, there exists a constant λ_n and a measurable function g_n satisfying (1)–(4) above. By passing to a subsequence twice, we may assume that $\lambda_n \to \lambda$ and $g_n \to g$ weakly in L^2. λ and g satisfy conditions (1)–(3) above, and a computation shows that $f(x)g(x+\alpha) = \lambda g(x)$ for almost all $x \in [0,1)$. Thus $\alpha \in A_{k,j}$.

Clearly $S \subseteq \bigcup_{k,j} A_{k,j}$. By the Baire Theorem, some set A_{k_0,j_0} must contain an open interval. Therefore, there exists a positive integer Q such that for every $q \geq Q$ there is a rational number $p/q \in A_{k_0,j_0}$, and thus a constant λ_q and a function g_q satisfying

(1) $\|g_q\|_2 \leq 1$.
(2) $|\lambda_q| = 1$.
(3) $|\int_0^1 g_q(x)e^{2\pi ik_0x}\,dx| \geq \frac{1}{j_0}$.
(4) $f(x)g_q(x+\frac{p}{q}) = \lambda_q g_q(x)$ for almost all $x \in [0,1)$.

We also may assume that p is relatively prime to q. It follows by condition (4) that for each $q \geq Q$ there exists a p relatively prime to q such that

$$(*) \qquad f(x)f(x+\frac{p}{q})f(x+\frac{2p}{q})\ldots f(x+\frac{(q-1)p}{q}) = \lambda_q^q$$

for every x for which $g_q(x) \neq 0$, and, by condition (3), this is certainly a set of positive measure.

Now the function

$$f(x)f(x+\frac{p}{q})\ldots f(x+\frac{(q-1)p}{q})$$

has discontinuities at most at the multiples of $\frac{p}{q}$, and on each subinterval $(\frac{j}{q}, \frac{(j+1)}{q})$ it is real-analytic. By the identity theorem for real-analytic functions, it follows that

$$f(x)f(x+\frac{p}{q})\ldots f(x+\frac{p(q-1)}{q})$$

is identically λ_q^q on some one of these subintervals. By the invariance of $(*)$ under translation by $\frac{p}{q}$, it follows that

$$f(x)f(x+\frac{p}{q})\ldots f(x+\frac{p(q-1)}{q}) \equiv \lambda_q^q$$

for all x not of the form $\frac{pj}{q}$.

Now $f(x) = e^{2\pi i v(x)}$, so we have that

$$(**) \qquad v(x) + v(x+\frac{p}{q}) + \ldots + v(x+\frac{p(q-1)}{q}) = c_q + N_q(x)$$

where c_q is a constant and N_q is an integer-valued function. Because v is continuous, we have that N_q is constant on the subintervals $(\frac{j}{q}, \frac{(j+1)}{q})$.

Using $(**)$, we compute the nqth Fourier coefficient of v, $c_{nq}(v)$, and obtain

$$qc_{nq}(v) = 0$$

for every nonzero integer n. Since this computation holds for every $q \geq Q$, it follows immediately that v is a trigonometric polynomial. □

Remark. Michael Herman [H, Theorem 4.11] proved a similar result under the additional hypothesis that for all $n \neq 0$, $c_n(v) \neq 0$.

By requiring the cobounding functions to be integrable, we obtain the following stronger result.

Theorem 2. *Let v be a real-valued L^1 function on $\mathbb{R}/\mathbb{Z}$, which is not a trigonometric polynomial. Then the set of all irrationals for which v is a trivial cocycle with L^1 cobounding function is of the first category.*

Proof. By the Riemann-Lebesgue Lemma, if v is a coboundary for α with L^1 cobounding function w, then $|c_n(w)| = |c_n(v)|/|1 - e^{2\pi i n\alpha}| \to 0$ as $|n| \to \infty$. Thus it will suffice to find a dense G_δ set E of irrationals such that for $\alpha \in E$, $|c_n(v)|/|1 - e^{2\pi i n\alpha}| \not\to 0$. Since v is not a trigonometric polynomial, $\exists \{m_k\}_{k=1}^{\infty}$, $m_k \to \infty$, such that $|c_{m_k}(v)| = \epsilon_k \neq 0$. Choose a_k so that $a_k > \frac{1}{m_k \epsilon_k}$. Let

$$A_k = \bigcup_{j=1}^{m_k - 1} \Big(\frac{j}{m_k} - \frac{1}{a_k m_k^2}, \frac{j}{m_k} + \frac{1}{a_k m_k^2}\Big).$$

If $\alpha \in A_k$, then $\exists j$ such that $\left|\alpha - \frac{j}{m_k}\right| < \frac{1}{a_k m_k^2}$, which implies $|m_k\alpha - j| < \frac{1}{a_k m_k}$ and hence $|1 - e^{2\pi i m_k \alpha}| < \frac{1}{a_k m_k}$. Thus we see that for $\alpha \in A_k$, $|c_{m_k}(v)|/|1 - e^{2\pi i m_k\alpha}| > 1$. Let $E_n = \bigcup_{k=n}^{\infty} A_k$. E_n is open and dense for each n, and by the above we have that if $\alpha \in E_n$, $\exists m_k$, $k \geq n$, such that $|c_{m_k}(v)|/|1 - e^{2\pi i m_k\alpha}| > 1$. Set $E = \bigcap_{n=1}^{\infty} E_n$. □

The apparent advantage of the second theorem over the first raises the natural question of whether an L^1 coboundary, or even an analytic coboundary, must have an L^1 cobounding function.

Theorem 3. *Given any irrational α, there exists a continuous coboundary v for α, whose cobounding function is not L^1.*

Proof. Choose a sequence of rationals $\{\frac{p_n}{q_n}\}$ satisfying

$$\Big|\alpha - \frac{p_n}{q_n}\Big| \leq \frac{1}{n^3 2^{2n+1} q_n}.$$

(This can be done by choosing a subsequence of the convergents to α so that each element, $\frac{p_n}{q_n}$, of this subsequence has the property that $q_n \geq n^3 2^{2n+1}$.) For each $n \geq 1$, we define the function u_n by

$$u_n(x) = \begin{cases} 2^{n+1} + n2^{2n+1} q_n x & \text{if } x \in (-\frac{1}{n2^n q_n}, 0), \\ 2^{n+1} - n2^{2n+1} q_n x & \text{if } x \in (0, \frac{1}{n2^n q_n}), \end{cases}$$

and then define

$$w_n(x) = \sum_{p=0}^{q_n-1} u_n(x - \frac{p}{q_n}).$$

(The function w_n is triangular on $(\frac{p}{q_n} - \frac{1}{n2^n q_n}, \frac{p}{q_n} + \frac{1}{n2^n q_n})$ with $w_n(\frac{p}{q_n}) = 2^{n+1}$ for $p = 0, 1, \dots, q_n - 1$, and 0 everywhere else.) Finally, let

$$w(x) = \sum_{n=1}^{\infty} w_n(x).$$

To show that w is finite a.e., we show that $S_N = \sum_{n=1}^{N} w_n$ is Cauchy in measure. Indeed, for any $N > M$, $S_N - S_M = \sum_{n=M+1}^{N} w_n$ is supported on a set of measure $\sum_{n=M+1}^{N}(q_n)(\frac{1}{n2^n q_n})$, which goes to zero as N and M go to infinity. We see that w is not in L^1 by noting that $\int |w_n(x)|\, dx = \frac{2}{n}$ so that by the monotone convergence theorem we have

$$\int |w(x)|\, dx = \sum_{n=1}^{\infty} \int |w_n(x)|\, dx = \sum_{n=1}^{\infty} \frac{2}{n} = \infty.$$

Now we define

$$v(x) = w(x) - w(x + \alpha) = \sum_{n=1}^{\infty} w_n(x) - w_n(x + \alpha).$$

Since the $w_n(x) - w_n(x + \alpha)$ are continuous, it will follow from the M-test that v is continuous, if we can show that $|w_n(x) - w_n(x + \alpha)| < \frac{1}{n^2}$. By the periodicity of w_n, we have that

$$|w_n(x) - w_n(x + \alpha)| = |w_n(x + \frac{p_n}{q_n}) - w_n(x + \alpha)| \leq n2^{2n+1} q_n |\alpha - \frac{p_n}{q_n}| < \frac{1}{n^2},$$

since $n2^{2n+1} q_n$ is the maximum slope of a secant line of w_n. □

Remark. For certain α, we can modify the above construction to give C^r coboundaries with non-L^1 cobounding functions. In particular, if there is a sequence of rational approximations to α, $\{\frac{p_n}{q_n}\}$, such that $|\alpha - \frac{p_n}{q_n}| < \frac{1}{q_n^{r+2}}$, we can replace the continuous, piecewise linear functions w_n with C^r, piecewise $(r + 1)$st degree polynomials, with the same integral as before, and with the property that $\sum w_n^{(r)}(x) - w_n^{(r)}(x + \alpha)$ converges uniformly, thus giving v a continuous rth derivative. Y. Meyer [H, p. 187] has a related result in the $r = 1$ case, which implies that if α has bounded partial quotients in its continued fraction expansion, then there exists a C^1 function which is a coboundary for α with noncontinuous cobounding function. The question of whether there are analytic coboundaries with non-L^1 cobounding functions remains unanswered.

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Department of Mathematics, University of Colorado, Boulder, Colorado 80309

E-mail address: `baggett@euclid.colorado.edu`

Department of Mathematics, Loyola Marymount University, Los Angeles, California 90045

E-mail address: `hmedina@lmumail.lmu.edu`

Department of Mathematics, The Colorado College, Colorado Springs, Colorado 80903

E-mail address: `kmerrill@cc.colorado.edu`