A RIGIDITY THEOREM FOR THE CLIFFORD TORI IN $S^3$

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Abstract. Let $S^3$ be the unit hypersphere in the 4-dimensional Euclidean space $\mathbb{R}^4$ defined by $\sum_{i=1}^4 x_i^2 = 1$. For each $\theta$ with $0 < \theta < \pi/2$, we denote by $M_\theta$ the Clifford torus in $S^3$ given by the equations $x_1^2 + x_2^2 = \cos^2 \theta$ and $x_3^2 + x_4^2 = \sin^2 \theta$. The Clifford torus $M_\theta$ is a flat Riemannian manifold equipped with the metric induced by the inclusion map $i_\theta: M_\theta \to S^3$. In this note we prove the following rigidity theorem: If $f: M_\theta \to S^3$ is an isometric embedding, then there exists an isometry $A$ of $S^3$ such that $f = A \circ i_\theta$. We also show no flat torus with the intrinsic diameter $\leq \pi$ is embeddable in $S^3$ except for a Clifford torus.

1. Introduction

Let $S^3$ be the unit hypersphere in the 4-dimensional Euclidean space $\mathbb{R}^4$ defined by $\sum_{i=1}^4 x_i^2 = 1$. For each $\theta$ with $0 < \theta < \pi/2$, we denote by $M_\theta$ the Clifford torus in $S^3$ given by

$$x_1^2 + x_2^2 = \cos^2 \theta, \quad x_3^2 + x_4^2 = \sin^2 \theta.$$ 

The Clifford torus $M_\theta$ is a flat Riemannian manifold equipped with the metric induced by the inclusion map $i_\theta: M_\theta \to S^3$. The authors are interested in the following question: For every isometric immersion $f: M_\theta \to S^3$, does there exist an isometry $A$ of $S^3$ such that $f = A \circ i_\theta$? Concerning this question, it is known that if $f_t: M_\theta \to S^3$, $-\infty < t < \infty$, is a smooth 1-parameter family of isometric immersions with $f_0 = i_\theta$, then for each $t$ there exists an isometry $A_t$ of $S^3$ such that $f_t = A_t \circ i_\theta$. However, the question above seems not to have been settled yet. In this note we give an affirmative answer to the question under the assumption that the immersion $f$ is an embedding. In other words, we prove the following rigidity theorem.

Theorem 1. If $f: M_\theta \to S^3$ is an isometric embedding, then there exists an isometry $A$ of $S^3$ such that $f = A \circ i_\theta$.

For each isometric immersion $f: M_\theta \to S^3$, we denote by $\text{Diam}(f)$ the diameter of the image $f(M_\theta)$ in $S^3$. Note that $\text{Diam}(i_\theta) = \pi$. The following theorem, which will be proved in §2, is a key ingredient in the proof of Theorem 1.

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Theorem 2. If $f: M_\theta \to S^3$ is an isometric immersion with $\text{Diam}(f) = \pi$, then there exists an isometry $A$ of $S^3$ such that $f = A \circ \theta$.

We now give the proof of Theorem 1. It follows from [2] that if $f$ is an isometric embedding of a flat torus $M$ into $S^3$, then the image $f(M)$ is invariant under the antipodal map of $S^3$. In particular, for each isometric embedding $f: M_\theta \to S^3$, we have $\text{Diam}(f) = \pi$. Therefore the assertion of Theorem 1 follows from Theorem 2.

In §3, we obtain Theorem 3, which generalizes Theorem 2 for an isometric immersion of a flat torus with intrinsic diameter less than or equal to $\pi$. An immediate consequence of Theorem 3 is that the only flat tori with intrinsic diameter $\leq \pi$ which can be embedded in $S^3$ are Clifford tori.

2. Proof of Theorem 2

We define a Riemannian covering map $T: \mathbb{R}^2 \to M_\theta$ of the 2-dimensional Euclidean space $\mathbb{R}^2$ into the Clifford torus $M_\theta$ by setting

$$T(u_1, u_2) = \left( R_1 \cos \left( \frac{u_1}{R_1} \right), R_1 \sin \left( \frac{u_1}{R_1} \right), R_2 \cos \left( \frac{u_2}{R_2} \right), R_2 \sin \left( \frac{u_2}{R_2} \right) \right),$$

where $R_1 = \cos \theta$ and $R_2 = \sin \theta$. Note that $T(u_1, u_2) = T(u_1 + l_1, u_2 + l_2)$ if and only if $l_i/2\pi R_i$ is an integer for each $i$. Let $V_1$ and $V_2$ be the vector fields on $M_\theta$ given by

$$\begin{align*}
V_1(T(u_1, u_2)) &= \left. \frac{d}{dt} \right|_{t=0} T(u_1 + R_1 t, u_2 + R_2 t), \\
V_2(T(u_1, u_2)) &= \left. \frac{d}{dt} \right|_{t=0} T(u_1 + R_1 t, u_2 + R_2 t).
\end{align*}$$

Then we have

$$g(V_1, V_1) = g(V_2, V_2) = 1, \quad g(V_1, V_2) = \cos 2\theta,$$

where $g$ denotes the Riemannian metric on $M_\theta$. For $i = 1, 2$, we denote by $\{\varphi^i_t\}$ the 1-parameter group of transformations of $M_\theta$ generated by the vector field $V_i$.

Lemma 1. Let $f: M_\theta \to S^3$ be an isometric immersion, and let $p$ be a point in $M_\theta$. If there exists a point $q \in M_\theta$ such that $f(p) = -f(q)$, then the curve $\gamma_i(t) = f(\varphi^i_t(p))$ is a unit speed geodesic in $S^3$.

Proof. Take a point $(a_1, a_2) \in \mathbb{R}^2$ such that $T(a_1, a_2) = p$. By (1) we obtain

$$\begin{align*}
\varphi^1_t(p) &= T(a_1 + R_1 t, a_2 + R_2 t), \\
\varphi^2_t(p) &= T(a_1 + R_1 t, a_2 - R_2 t).
\end{align*}$$

Let $d(p, q)$ denote the intrinsic distance between $p$ and $q$ in $M_\theta$. Then it follows from $f(p) = -f(q)$ that $d(p, q) \geq \pi$. Since the intrinsic diameter of $M_\theta$ is equal to $\pi$, we obtain $d(p, q) = \pi$. Hence

$$q = T(a_1 + R_1 \pi, a_2 + R_2 \pi) = T(a_1 + R_1 \pi, a_2 - R_2 \pi).$$

It follows from (3) and (4) that $\gamma_i(t)$ is a unit speed curve in $S^3$ such that $\gamma_i(0) = \gamma_i(2\pi) = f(p)$ and $\gamma_i(\pi) = f(q) = -f(p)$. This shows that $\gamma_i|_{[0, 2\pi]}$ is a geodesic in $S^3$. Since $\gamma_i(\pi + 2\pi) = \gamma_i(0)$, the curve $\gamma_i(t)$ is a unit speed geodesic in $S^3$.

Lemma 2. Let $f: M_\theta \to S^3$ be an isometric immersion with $\text{Diam}(f) = \pi$, and let $h$ denote the second fundamental form of the immersion $f$. Then $h(V_1, V_1) = h(V_2, V_2) = 0$ and $|h(V_1, V_2)| = \sin 2\theta$. 


Proof. We set $h_{ij} = h(V_i, V_j)$. By (2) and the equation of Gauss, we obtain
\[
\langle h_{12}, h_{12}\rangle - \langle h_{11}, h_{22}\rangle = \sin^2 2\theta,
\]
where $\langle \ , \ \rangle$ denotes the Riemannian metric on $S^3$. We now define $M^*_\theta$ to be the set of all $p \in M_\theta$ such that $f(p) = -f(q)$ for some $q \in M_\theta$. Using Lemma 1, we see that $h_{11} = h_{22} = 0$ on $M^*_\theta$. So it is sufficient to show that $M^*_\theta = M_\theta$. Since $\text{Diam}(f) = \pi$, there exists a point $p \in M^*_\theta$. Let $c(s)$ be the curve in $M_\theta$ given by $c(s) = \varphi^*_1(p)$. Then it follows from Lemma 1 that the curve $f(c(s))$ is a unit speed geodesic in $S^3$, and so $c(s) = -f(c(s + \pi))$. Hence $c(s) \in M^*_\theta$ for all $s$. For each $s \in \mathbb{R}$, let $c_s(t)$ be the curve in $M_\theta$ given by $c_s(t) = \varphi^*_2(c(s))$. By the same way as above we see that $c_s(t) \in M^*_\theta$. Hence $\varphi^*_2(\varphi^*_1(p)) \in M^*_\theta$ for all $(s, t) \in \mathbb{R}^2$. This implies $M^*_\theta = M_\theta$.

We now give the proof of Theorem 2. Let $f : M_\theta \to S^3$ be an isometric immersion with $\text{Diam}(f) = \pi$. We set $f_1 = i_\theta$ and $f_2 = f$. For $k = 1, 2$, let $h_k$ be the second fundamental form of the immersion $f_k$, and let $\xi_k = h_k(V_1, V_2) / \sin 2\theta$. Then it follows from Lemma 2 that $\xi_k$ defines a unit normal vector field along $f_k$, and $\langle h_1(V_i, V_j), \xi_1 \rangle = \langle h_2(V_i, V_j), \xi_2 \rangle$.

Hence the fundamental theorem of the theory of surfaces implies that there exists an isometry $A$ of $S^3$ such that $f_2 = A \circ f_1$. This completes the proof of Theorem 2.

3. A GENERALIZATION OF THEOREM 2

In this section we generalize Theorem 2 as follows.

**Theorem 3.** Let $M$ be a flat torus with intrinsic diameter less than or equal to $\pi$. If $f : M \to S^3$ is an isometric immersion with $\text{Diam}(f) = \pi$, then there exist an isometry $\varphi : M_\theta \to M$ for some $\theta \in (0, \pi/2)$ and an isometry $A$ of $S^3$ such that $f \circ \varphi = A \circ i_\theta$.

**Proof.** Since $\text{Diam}(f) = \pi$, there exist points $p, q$ in $M$ such that $f(p) = -f(q)$. Let $d(p, q)$ denote the intrinsic distance between $p$ and $q$ in $M$. It follows from $f(p) = -f(q)$ that $d(p, q) \geq \pi$. But our assumption on the intrinsic diameter of $M$ implies that $d(p, q) = \pi$. In addition, any geodesic of $M$ of length $\pi$ which connects $p$ and $q$ is mapped by $f$ to a geodesic $S^3$ which connects $f(p)$ to $f(q)$.

If $M$ is not isometric to $M_\theta$, we claim that $p$ and $q$ are connected by three geodesics in $M$ of length $\pi$ whose tangent vectors at $p$ lie in three distinct linear subspaces. To prove the claim consider the Riemannian covering map $k : \mathbb{R}^2 \to M$. We suppose that $0$, the origin, lies in $\Gamma = k^{-1}(q)$; of course, $M$ is isometric to $\mathbb{R}^2 / \Gamma$. Let $q_1$ and $q_2$ be the elements of smallest norm in $\Gamma \setminus \{0\}$ and $\Gamma \setminus \{nq_i : n \in \mathbb{Z}\}$, respectively. For notational reasons, let $0 = q_0$. Denote the triangle with vertices $q_i$, for $i = 0, 1, 2$, by $\Delta$. One may show that the circumcenter of $\Delta$, denote $p_0$, lies in $k^{-1}(p)$ and $d(p_0, q_i) = \pi$, for $i = 0, 1, 2$. Thus there are at least three geodesics in $M$ of length $\pi$ connecting $p$ to $q$. Assume that the tangent vectors to these geodesic arcs at $p$ must lie in a pair of linear subspaces of the tangent space to $M$ at $p$. Necessarily, two of the segments $\overline{p_0q_i}$ lie on the same line. Say, for example, that the segment $\overline{p_0q_1} = \overline{p_0q_0} \cup \overline{p_0q_2}$. Since the circumcenter of $\Delta$ is on the side $\overline{q_1q_2}$ of $\Delta$, one sees that $\Delta$ is a right triangle with right angle at $q_0$. It follows that $M$ is isometric to a Clifford torus. This contradiction proves the claim.

If $M$ is not isometric to $M_\theta$, then the images under $f$ of the geodesic segments mentioned in the previous paragraph are geodesics of $S^3$. This implies that the
immersion \( f \) is totally geodesic at \( p \), which is impossible. Hence \( M \) must be intrinsically isometric to \( M_\theta \). Now Theorem 3 follows from Theorem 2.

**Theorem 4.** It is impossible to embed a flat torus with intrinsic diameter \( \leq \pi \) in \( S^3 \) unless the flat torus is a Clifford torus.

*Proof.* Again, from [2], the image of any embedding must have antipodal symmetry. Thus the assertion of this theorem follows from Theorem 3.

**References**