A NOTE ON THE KERNEL
OF A LOCALLY NILPOTENT DERIVATION

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Abstract. This note concerns locally nilpotent derivations $D$ of the polynomial ring $\mathbb{C}[X_1,\ldots,X_n]$. It is shown that if $D$ annihilates a polynomial in two variables, then $D$ annihilates a variable.

Let $k$ be a field of characteristic 0, and let $A$ be a commutative $k$-algebra. A $k$-derivation $D$ on $A$ is said to be locally nilpotent if, for every $f \in A$, there is an integer $s \geq 0$ such that $D^s f = 0$. This paper is motivated by the following open question.

Question. For $n \geq 2$, does every locally nilpotent derivation on the polynomial ring $k[X_1,\ldots,X_n]$ annihilate a variable?

(Recall that $f \in k[X_1,\ldots,X_n]$ is a variable if there exist $f_2,\ldots,f_n \in k[X_1,\ldots,X_n]$ such that $k[f,f_2,\ldots,f_n] = k[X_1,\ldots,X_n]$.) Geometrically, this question is equivalent to asking whether, with respect to some choice of coordinates, a given algebraic action of the additive group of $k$ on $\mathbb{A}^n$ (affine $n$-space over $k$) fixes a hyperplane (cf. [3]). When $n = 2$, an affirmative answer to the question is given by the following result, due to Rentschler [1].

**Theorem 1.** If $D$ is a locally nilpotent derivation on $k[X,Y]$, then there exists a tame $k$-algebra automorphism $\gamma$ of $k[X,Y]$ and a polynomial $f \in k[X]$ such that $\gamma^{-1} D \gamma = f(X) \cdot \frac{\partial}{\partial Y}$.

Using Rentschler’s Theorem, it is shown below that an affirmative answer can also be given in the following case.

**Theorem 2.** Let $D$ be a locally nilpotent derivation on $R_n(\mathbb{C}) = \mathbb{C}[X_1,\ldots,X_n]$, and suppose that the set

$$T = (\mathbb{C}[X_1, X_2] \cap \ker(D)) - \mathbb{C}$$

is non-empty (where $\ker(D)$ denotes the kernel of $D$). Then there exists a variable $\rho \in R_n(\mathbb{C})$ such that $\rho \in \mathbb{C}[X_1, X_2]$ and $D\rho = 0$.

In other words, if $D$ annihilates a polynomial in two variables over $\mathbb{C}$, then $D$ annihilates a variable. Before giving the proof, some preliminary results are needed.
Let $R_n(k)$ denote $k[X_1, \ldots, X_n]$. Given any locally nilpotent derivation $D$ on $R_n(k)$, and given $f \in R_n(k), f \neq 0$, define
\[ \nu_D(f) = \max \{ s \mid D^s f \neq 0 \}. \]
Also, set $\nu_D(0) = -\infty$. (Note that $\nu_D(f) = \deg_t \exp(tD)(f)$.) The following properties are immediate (the first as a corollary to the Leibniz rule):
\begin{enumerate}
  \item $\nu_D(fg) = \nu_D(f) + \nu_D(g)$.
  \item $\nu_D(f + g) \leq \max \{ \nu_D(f), \nu_D(g) \}$.
  \item $\nu_D(Df) = \nu_D(f) - 1$ if $Df \neq 0$.
\end{enumerate}

Next, a multiplicatively closed subset $S \subset R_n(k)$ is said to be saturated if the following property holds:
\[ fg \in S \iff f \in S \text{ and } g \in S. \]
It is well known that, for any locally nilpotent derivation $D$ on $R_n(k)$, the set $(\ker(D) - 0)$ is saturated, since
\[ fg \in (\ker D - 0) \iff D(fg) = 0 \quad (f, g \neq 0) \]
\[ \iff 0 = \nu_D(fg) = \nu_D(f) + \nu_D(g) \]
\[ \iff \nu_D(f) = \nu_D(g) = 0 \]
\[ \iff f, g \in (\ker D - 0). \]

A noteworthy consequence of saturation is the following.

**Proposition.** Let $D$ be a locally nilpotent derivation of $R_n(k)$, let $S \subset R_n(k)$ be a multiplicative subset, and let $\overline{D}$ be the extension of $D$ to the localization $S^{-1}R_n(k)$ (via the quotient rule). The following are equivalent:
\begin{enumerate}
  \item $D$ is locally nilpotent.
  \item $S \subset (\ker(D) - 0)$.
\end{enumerate}

**Proof.** If $\overline{D}$ is locally nilpotent and $f \in S$ is given, then $(\ker \overline{D} - 0)$ is saturated, and
\[ 0 = \overline{D}(1) = \overline{D} \left( \frac{1}{f} \cdot f \right) \Rightarrow f \in \ker \overline{D} \Rightarrow 0 = \overline{D}f = Df. \]
Conversely, if $S \subset (\ker(D) - 0)$, let $h = (f/g)$ for $f \in R_n(k)$ and $g \in S$; then $\overline{D}^s h = \overline{D}^s(f/g) = g^{-1} \overline{D}^s f = g^{-1} \overline{D}^s f = 0$ for $s \gg 0$.

**Lemma.** Given a polynomial $q \in \mathbb{C}[X,Y]$, suppose $f \in \mathbb{C}[X,Y]$ is an irreducible non-constant divisor of both $\frac{\partial q}{\partial X}$ and $\frac{\partial q}{\partial Y}$. Then there exists $c \in \mathbb{C}$ such that $f$ divides $(q + c)$.

**Proof.** Let $Z \subset \mathbb{C}^2$ be the curve defined by $f$; by hypothesis, $\frac{\partial q}{\partial X}$ and $\frac{\partial q}{\partial Y}$ evaluated along $Z$ are zero. Given a non-singular point $P \in Z$, let $\alpha(t) = (x(t), y(t))$ be a local parametrization of $Z$ at $P$. Define $Q : \mathbb{C} \to \mathbb{C}$ to be the evaluation of $q$ along $\alpha$, i.e., $Q = q \circ \alpha$. Then
\[ \frac{dQ}{dt} = \frac{\partial q}{\partial X}(\alpha(t)) \cdot \frac{dx}{dt} + \frac{\partial q}{\partial Y}(\alpha(t)) \cdot \frac{dy}{dt} = 0. \]
Therefore $Q \equiv c$ for some $c \in \mathbb{C}$, which implies that $q \equiv c$ along the entire connected component of $Z$ containing $P$. But $f$ being irreducible implies $Z$ is connected (in the complex topology); see, for example, [2, Chapter 7, §2]. Hence, $(q - c)$ vanishes along $Z$, and $f$ divides $(q - c)$.
Proof of Theorem 2. To simplify notation, let $X=X_1$ and $Y=X_2$. If $q \in \mathcal{T}$ is of minimal degree, then $q$ has the following property:

(1) \((q + c)\) is irreducible for all $c \in \mathbb{C}$.

To see this, note first that $(q + c) \in \text{ker}(D)$ for all $c \in \mathbb{C}$. If $(q + c)$ is reducible for some $c$, let $q_0$ be one of its irreducible factors. Since $\text{ker}(D)$ is saturated, $q_0 \in \text{ker}(D)$; hence $q_0 \in \mathcal{T}$ as well. But $q_0$ would then be of smaller degree than $q$, which is impossible. So $(q + c)$ must be irreducible, as claimed.

Next, since $q \in \mathbb{C}[X,Y]$, we may write

(2) \[0 = Dq = \frac{\partial q}{\partial X} DX + \frac{\partial q}{\partial Y} DY.\]

By the preceding lemma, condition (1) above implies that $\frac{\partial q}{\partial X}$ and $\frac{\partial q}{\partial Y}$ have no common factor. It follows from (2) that $\frac{\partial q}{\partial X}$ divides $DX$ and $\frac{\partial q}{\partial Y}$ divides $DY$. So there exist $l, m \in R_n(\mathbb{C})$ such that $DX = l \cdot \frac{\partial q}{\partial X}$ and $DY = m \cdot \frac{\partial q}{\partial Y}$. Substitution in (2) then shows that $m = -l$. We thus have

$$DX = l \cdot \frac{\partial q}{\partial Y} \quad \text{and} \quad DY = -l \cdot \frac{\partial q}{\partial X}.$$ 

If $l = 0$, the theorem is proved, so assume $l \neq 0$.

Define a derivation $\Delta$ on $\mathbb{C}[X,Y]$, by setting $\Delta X = \frac{\partial q}{\partial Y}$ and $\Delta Y = -\frac{\partial q}{\partial X}$. Observe the following:

(3) \[Df = l \cdot \Delta f \quad \text{for all} \quad f \in \mathbb{C}[X,Y].\]

Claim. $\Delta$ is locally nilpotent.

Proof of Claim. Suppose, to the contrary, that there is an element $p \in \mathbb{C}[X,Y]$ such that, for all $n \geq 0$, $\Delta^n p \neq 0$. Then $\nu_D(\Delta^n p) \geq 0$ for all $n \geq 0$. Set $L = \nu_D(l)$; since $l \neq 0$, $L \geq 0$. Using (3) above, we see that, for all $n \geq 1$,

$$D(\Delta^{(n-1)} p) = l \cdot \Delta(\Delta^{(n-1)} p) = l \cdot \Delta^n p.$$ 

Now apply $\nu_D$ to each side of this equation:

$$\nu_D(\Delta^{(n-1)} p) - 1 = L + \nu_D(\Delta^n p) \Rightarrow \nu_D(\Delta^n p) = \nu_D(\Delta^{(n-1)} p) - (L + 1).$$

Recursive application of this rule yields $\nu_D(\Delta^n p) = \nu_D(p) - n(L + 1)$. But this implies $-\infty < \nu_D(\Delta^n p) < 0$ for $n$ sufficiently large, a contradiction. Therefore $\Delta$ is locally nilpotent, and the Claim is proved.

We can now apply Rentschler's Theorem to $\Delta$: there exist $X'$ and $Y'$ in $\mathbb{C}[X,Y]$ and $f \in \mathbb{C}[X']$ such that $\mathbb{C}[X',Y'] = \mathbb{C}[X,Y]$, and $\Delta = f(X') \frac{\partial}{\partial X'}$. Consequently, $DX' = l \cdot \Delta X' = 0$. Since $X'$ is clearly a variable in $R_n(\mathbb{C})$, we may take $p = X'$.

References