A NOTE ON THE KERNEL OF A LOCALLY NILPOTENT DERIVATION

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Abstract. This note concerns locally nilpotent derivations $D$ of the polynomial ring $\mathbb{C}[X_1, \ldots, X_n]$. It is shown that if $D$ annihilates a polynomial in two variables, then $D$ annihilates a variable.

Let $k$ be a field of characteristic 0, and let $A$ be a commutative $k$-algebra. A $k$-derivation $D$ on $A$ is said to be locally nilpotent if, for every $f \in A$, there is an integer $s \geq 0$ such that $D^s f = 0$. This paper is motivated by the following open question.

Question. For $n \geq 2$, does every locally nilpotent derivation on the polynomial ring $k[X_1, \ldots, X_n]$ annihilate a variable?

(Recall that $f \in k[X_1, \ldots, X_n]$ is a variable if there exist $f_2, \ldots, f_n \in k[X_1, \ldots, X_n]$ such that $k[f, f_2, \ldots, f_n] = k[X_1, \ldots, X_n]$.) Geometrically, this question is equivalent to asking whether, with respect to some choice of coordinates, a given algebraic action of the additive group of $k$ on $\mathbb{A}^n$ (affine $n$-space over $k$) fixes a hyperplane (cf. [3]). When $n = 2$, an affirmative answer to the question is given by the following result, due to Rentschler [1].

Theorem 1. If $D$ is a locally nilpotent derivation on $k[X, Y]$, then there exists a tame $k$-algebra automorphism $\gamma$ of $k[X, Y]$ and a polynomial $f \in k[X]$ such that $\gamma D \gamma^{-1} = f(X) \cdot \frac{\partial}{\partial Y}$.

Using Rentschler’s Theorem, it is shown below that an affirmative answer can also be given in the following case.

Theorem 2. Let $D$ be a locally nilpotent derivation on $R_n(\mathbb{C}) = \mathbb{C}[X_1, \ldots, X_n]$, and suppose that the set

$$T = (\mathbb{C}[X_1, X_2] \cap \ker(D)) - \mathbb{C}$$

is non-empty (where $\ker(D)$ denotes the kernel of $D$). Then there exists a variable $\rho \in R_n(\mathbb{C})$ such that $\rho \in \mathbb{C}[X_1, X_2]$ and $D\rho = 0$.

In other words, if $D$ annihilates a polynomial in two variables over $\mathbb{C}$, then $D$ annihilates a variable. Before giving the proof, some preliminary results are needed.
Let $R_n(k)$ denote $k[X_1,\ldots,X_n]$. Given any locally nilpotent derivation $D$ on $R_n(k)$, and given $f \in R_n(k)$, $f \neq 0$, define 
\[ \nu_D(f) = \max\{s \mid D^s f \neq 0\}. \]
Also, set $\nu_D(0) = -\infty$. (Note that $\nu_D(f) = \deg_s \exp(tD)(f)$.) The following properties are immediate (the first as a corollary to the Leibniz rule):
(i) $\nu_D(fg) = \nu_D(f) + \nu_D(g)$.
(ii) $\nu_D(f + g) \leq \max\{\nu_D(f), \nu_D(g)\}$.
(iii) $\nu_D(Df) = \nu_D(f) - 1$ if $Df \neq 0$.

Next, a multiplicatively closed subset $S \subset R_n(k)$ is said to be saturated if the following property holds:
\[ fg \in S \iff f \in S \text{ and } g \in S. \]
It is well known that, for any locally nilpotent derivation $D$ on $R_n(k)$, the set $(\ker(D) - 0)$ is saturated, since
\[ fg \in (\ker D - 0) \iff D(fg) = 0 \quad (f, g \neq 0) \]
\[ \iff 0 = \nu_D(fg) = \nu_D(f) + \nu_D(g) \]
\[ \iff \nu_D(f) = \nu_D(g) = 0 \]
\[ \iff f, g \in (\ker D - 0). \]

A noteworthy consequence of saturation is the following.

**Proposition.** Let $D$ be a locally nilpotent derivation of $R_n(k)$, let $S \subset R_n(k)$ be a multiplicative subset, and let $\tilde{D}$ be the extension of $D$ to the localization $S^{-1}R_n(k)$ (via the quotient rule). The following are equivalent:
(i) $D$ is locally nilpotent.
(ii) $S \subset (\ker(D) - 0)$.

**Proof.** If $\tilde{D}$ is locally nilpotent and $f \in S$ is given, then $(\ker \tilde{D} - 0)$ is saturated, and
\[ 0 = \tilde{D}(1) = \tilde{D}\left(\frac{1}{f}\right) \implies f \in \ker \tilde{D} \implies 0 = \tilde{D}f = Df. \]
Conversely, if $S \subset (\ker(D) - 0)$, let $h = (f/g)$ for $f \in R_n(k)$ and $g \in S$; then $\tilde{D}^s h = \tilde{D}^s (f/g) = g^{-1} \tilde{D}^s f = g^{-1} D^s f = 0$ for $s \gg 0$. \hfill $\square$

**Lemma.** Given a polynomial $q \in \mathbb{C}[X,Y]$, suppose $f \in \mathbb{C}[X,Y]$ is an irreducible non-constant divisor of both $\frac{\partial q}{\partial X}$ and $\frac{\partial q}{\partial Y}$. Then there exists $c \in \mathbb{C}$ such that $f$ divides $(q + c)$.

**Proof.** Let $Z \subset \mathbb{C}^2$ be the curve defined by $f$; by hypothesis, $\frac{\partial q}{\partial X}$ and $\frac{\partial q}{\partial Y}$ evaluated along $Z$ are zero. Given a non-singular point $P \in Z$, let $\alpha(t) = (x(t),y(t))$ be a local parametrization of $Z$ at $P$. Define $Q : \mathbb{C} \to \mathbb{C}$ to be the evaluation of $q$ along $\alpha$, i.e., $Q = q \circ \alpha$. Then
\[ \frac{dQ}{dt} = \frac{\partial q}{\partial X}(\alpha(t)) \cdot \frac{dx}{dt} + \frac{\partial q}{\partial Y}(\alpha(t)) \cdot \frac{dy}{dt} = 0. \]
Therefore $Q \equiv c$ for some $c \in \mathbb{C}$, which implies that $q \equiv c$ along the entire connected component of $Z$ containing $P$. But $f$ being irreducible implies $Z$ is connected (in the complex topology); see, for example, [2, Chapter 7, §2]. Hence, $(q - c)$ vanishes along $Z$, and $f$ divides $(q - c)$. \hfill $\square$
Proof of Theorem 2. To simplify notation, let $X = X_1$ and $Y = X_2$. If $q \in \mathcal{T}$ is of minimal degree, then $q$ has the following property:

(1) \((q + c)\) is irreducible for all \(c \in \mathbb{C}\).

To see this, note first that \((q + c) \in \ker(D)\) for all \(c \in \mathbb{C}\). If \((q + c)\) is reducible for some \(c\), let \(q_0\) be one of its irreducible factors. Since \(\ker(D)\) is saturated, \(q_0 \in \ker(D)\); hence \(q_0 \in \mathcal{T}\) as well. But \(q_0\) would then be of smaller degree than \(q\), which is impossible. So \((q + c)\) must be irreducible, as claimed.

Next, since \(q \in \mathbb{C}[X,Y]\), we may write

(2) \(0 = Dq = \frac{\partial q}{\partial X}DX + \frac{\partial q}{\partial Y}DY\).

By the preceding lemma, condition (1) above implies that \(\frac{\partial q}{\partial X}\) and \(\frac{\partial q}{\partial Y}\) have no common factor. It follows from (2) that \(\frac{\partial q}{\partial Y}\) divides \(DX\) and \(\frac{\partial q}{\partial X}\) divides \(DY\). So there exist \(l, m \in R_n(\mathbb{C})\) such that \(DX = l \cdot \frac{\partial q}{\partial Y}\) and \(DY = m \cdot \frac{\partial q}{\partial X}\). Substitution in (2) then shows that \(m = -l\). We thus have

\[
DX = l \cdot \frac{\partial q}{\partial Y} \quad \text{and} \quad DY = -l \cdot \frac{\partial q}{\partial X}.
\]

If \(l = 0\), the theorem is proved, so assume \(l \neq 0\).

Define a derivation \(\Delta\) on \(\mathbb{C}[X,Y]\), by setting \(\Delta X = \frac{\partial q}{\partial Y}\) and \(\Delta Y = -\frac{\partial q}{\partial X}\). Observe the following:

(3) \(Df = l \cdot \Delta f\) for all \(f \in \mathbb{C}[X,Y]\).

Claim. \(\Delta\) is locally nilpotent.

Proof of Claim. Suppose, to the contrary, that there is an element \(p \in \mathbb{C}[X,Y]\) such that, for all \(n \geq 0\), \(\Delta^n p \neq 0\). Then \(\nu_D(\Delta^n p) \geq 0\) for all \(n \geq 0\). Set \(L = \nu_D(l)\); since \(l \neq 0\), \(L \geq 0\). Using (3) above, we see that, for all \(n \geq 1\),

\[
D(\Delta^{(n-1)} p) = l \cdot \Delta(\Delta^{(n-1)} p) = l \cdot \Delta^n p.
\]

Now apply \(\nu_D\) to each side of this equation:

\[
\nu_D(\Delta^{(n-1)} p) - 1 = L + \nu_D(\Delta^n p) \Rightarrow \nu_D(\Delta^n p) = \nu_D(\Delta^{(n-1)} p) - (L + 1).
\]

Recursive application of this rule yields \(\nu_D(\Delta^n p) = \nu_D(p) - n(L + 1)\). But this implies \(-\infty < \nu_D(\Delta^n p) < 0\) for \(n\) sufficiently large, a contradiction. Therefore \(\Delta\) is locally nilpotent, and the Claim is proved.

We can now apply Rentschler’s Theorem to \(\Delta\): there exist \(X'\) and \(Y'\) in \(\mathbb{C}[X,Y]\) and \(f \in \mathbb{C}[X']\) such that \(\mathbb{C}[X',Y'] = \mathbb{C}[X,Y]\), and \(\Delta = f(X') \frac{\partial}{\partial Y'}\). Consequently, \(DX' = l \cdot \Delta X' = 0\). Since \(X'\) is clearly a variable in \(R_n(\mathbb{C})\), we may take \(\rho = X'\). \(\square\)

References


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