SEMINORMALITY OF OPERATORS
FROM THEIR TENSOR PRODUCT

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ABSTRACT. The question of seminormality of tensor products of nonzero bounded linear operators on Hilbert spaces is investigated. It is shown that $A \otimes B$ is subnormal if and only if so are A and B.

INTRODUCTION

It is well known that some properties of Hilbert space operators are preserved by the operation of taking the two–fold tensor product. However, it may happen that $A \otimes B$ possesses a property P, though A and B do not (mainly because $A \otimes B = (cA) \otimes (c^{-1}B)$ for any nonzero $c \in \mathbb{C}$). The aim of this paper is to show that if P is related to seminormality, then the tensor product $A \otimes B$ of nonzero bounded linear operators possesses the property P if and only if so do $c\,A$ and $c^{-1}\,B$ for some nonzero $c \in \mathbb{C}$. Since the same can be done for operators whose adjoints have the property P, we can concentrate on a more narrow class of properties. Let us recall some of them (cf. [2, 3, 7]).

We say that a bounded linear operator A on a complex Hilbert space is quasi-normal (resp. subnormal, hyponormal) if $A\,A^* A = A^* A^2$ (resp. A is a restriction of a bounded normal operator to its closed invariant subspace, $A\,A^* \leq A^*\,A$).

1. PRELIMINARIES

A sequence $\{\alpha_n\}_{n=0}^{\infty}$ of real numbers is said to be a *Stieltjes moment sequence* if there exists a positive Borel measure μ on the closed half-line $[0, +\infty)$ such that

$$(1.1) \qquad \alpha_n = \int_0^{+\infty} t^n d\mu(t), \qquad n \geq 0.$$

The measure μ is called a *representing measure* of $\{\alpha_n\}_{n=0}^{\infty}$.

Recall two basic properties of Stieltjes moment sequences (cf. [1] and [9]).

(S1) If $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are Stieltjes moment sequences, then so is their product $\{\alpha_n\,\beta_n\}_{n=0}^{\infty}$.

(S2) The class of all Stieltjes moment sequences is closed with respect to the topology of pointwise convergence.

The following two lemmas play the crucial role in this paper.

Lemma 1.1. *If $\{\alpha_n\}_{n=0}^{\infty}$ is a Stieltjes moment sequence, then*
  (i) $\alpha_{n+1}^2 \leq \alpha_n\,\alpha_{n+2}$, $n \geq 0$,
  (ii) $\alpha_n = 0$ *for every* $n \geq 1$ *if and only if there exists* $k \geq 1$ *such that* $\alpha_k = 0$.

Proof. Applying the Cauchy–Schwarz inequality we get (i). (ii) follows easily from the integral representation (1.1). $\qquad\square$

Lemma 1.2. *Let $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ be two sequences of positive real numbers. If $p = \limsup \alpha_{n+1}\,\alpha_n^{-1} < +\infty$ and $q = \limsup \beta_{n+1}\,\beta_n^{-1} < +\infty$, then the following conditions are equivalent:*
  (i) $\{\alpha_n\}_{n=0}^{\infty}$ *and* $\{\beta_n\}_{n=0}^{\infty}$ *are Stieltjes moment sequences.*
  (ii) *For every integer* $s \geq 0$, $\{\alpha_n\,\beta_{n+s}\}_{n=0}^{\infty}$ *and* $\{\alpha_{n+s}\,\beta_n\}_{n=0}^{\infty}$ *are Stieltjes moment sequences.*

Proof. The implication (i)$\Rightarrow$(ii) follows from (S1).

(ii)$\Rightarrow$(i). Put $p_n = \alpha_{n+1}\,\alpha_n^{-1}$ and $q_n = \beta_{n+1}\,\beta_n^{-1}$, $n \geq 0$. Applying Lemma 1.1(i) to the Stieltjes moment sequence $\{\alpha_n\,\beta_{n+s}\}_{n=0}^{\infty}$ ($s \geq 0$), we get

$$(1.2) \qquad p_n\,q_{n+s} \leq p_{n+1}\,q_{n+s+1}, \qquad n \geq 0, s \geq 0.$$

Thus the sequence $\{p_n\,q_n\}_{n=0}^{\infty}$ is weakly increasing, which in turn implies

$$0 < \lim_{n\to\infty} p_n\,q_n \leq \limsup_{n\to\infty} p_n \limsup_{n\to\infty} q_n = p\,q.$$

This means that $0 < p, q < +\infty$. It follows from (1.2) that

$$p_n = q^{-1} \limsup_{s\to\infty} p_n\,q_{n+s} \leq q^{-1} \limsup_{s\to\infty} p_{n+1}\,q_{n+s+1} = p_{n+1}, \qquad n \geq 0.$$

Since the sequence $\{p_n\}_{n=0}^{\infty}$ is weakly increasing, there exists $\lim_{n\to\infty} p_n = p$. Thus

$$\lim_{s\to\infty} \frac{\alpha_{n+s}}{\alpha_s} = \lim_{s\to\infty} (p_s\,p_{s+1} \cdot \ldots \cdot p_{s+n-1}) = p^n, \qquad n \geq 0,$$

which gives us

$$p^n\,\beta_n = \lim_{s\to\infty} \left(\frac{\alpha_{n+s}}{\alpha_s}\,\beta_n \right), \qquad n \geq 0.$$

It follows from (S2) that $\{p^n\,\beta_n\}_{n=0}^{\infty}$ and consequently $\{\beta_n\}_{n=0}^{\infty}$ are Stieltjes moment sequences. Similar arguments apply to the sequence $\{\alpha_n\}_{n=0}^{\infty}$. This completes the proof. $\qquad\square$

2. Main results

From now on, H and K are fixed complex Hilbert spaces and $\mathcal{B}(H)$ stands for the C^*–algebra of all bounded linear operators on H.

Let us begin with proving two results that will be of use later.

Proposition 2.1. *If $A_1, A_2 \in \mathcal{B}(H)$ and $B_1, B_2 \in \mathcal{B}(K)$ are nonzero operators, then $A_1 \otimes B_1 = A_2 \otimes B_2$ if and only if there exists $c \in \mathbb{C} \setminus \{0\}$ such that $A_2 = c A_1$ and $B_2 = c^{-1} B_1$.*

Proof. First we show that if $f_1 \otimes g_1 = f_2 \otimes g_2 \neq 0$, then there exists $c \in \mathbb{C} \setminus \{0\}$ such that $f_2 = c f_1$ and $g_2 = c^{-1} g_1$. Indeed, due to our assumption $(f_1, \cdot) g_1 = (f_2, \cdot) g_2$ and $(g_1, \cdot) f_1 = (g_2, \cdot) f_2$ and consequently there are $c, d \neq 0$ such that $f_2 = c f_1$ and $g_2 = d g_1$. Substituting these vectors into the original equality we get $d = c^{-1}$.

Assume that $A_1 f \otimes B_1 g = A_2 f \otimes B_2 g$ for all $f \in H, g \in K$. Then one can show that $\ker A_1 = \ker A_2$ and $\ker B_1 = \ker B_2$. In virtue of the previous paragraph, we can attach to each pair of nonzero vectors $f \in H \ominus \ker A_1$ and $g \in K \ominus \ker B_1$ a unique nonzero complex scalar c such that $A_2 f = c A_1 f$ and $B_2 g = c^{-1} B_1 g$. It is easily seen that c does not depend on f's and g's, which completes the proof. $\square$

Proposition 2.2. *Let $A_1, A_2 \in \mathcal{B}(H)$ and $B_1, B_2 \in \mathcal{B}(K)$ be nonnegative operators. If $A_1 \neq 0$ and $B_1 \neq 0$, then the following conditions are equivalent:*
  (i) *$A_1 \otimes B_1 \leq A_2 \otimes B_2$.*
  (ii) *There exists $c > 0$ such that $A_1 \leq c A_2$ and $B_1 \leq c^{-1} B_2$.*

Proof. Denote by C_j and D_j the square roots of A_j and B_j, respectively ($j = 1, 2$).

(ii)$\Rightarrow$(i). Since $\|C_1 f\|^2 \leq c \|C_2 f\|^2$ for $f \in H$, there exists $S \in \mathcal{B}(H)$ such that $\|S\| \leq c^{1/2}$ and $C_1 = S C_2$ (cf. [2], Lemma 1.1). Likewise there exists $T \in \mathcal{B}(K)$ such that $\|T\| \leq c^{-1/2}$ and $D_1 = T D_2$. Consequently

$$(A_1 \otimes B_1 \, h, h) = \|C_1 \otimes D_1 \, h\|^2 = \|(S \otimes T)(C_2 \otimes D_2) \, h\|^2$$
$$\leq \|S\|^2 \|T\|^2 \|(C_2 \otimes D_2) \, h\|^2 \leq (A_2 \otimes B_2 \, h, h), \qquad h \in H \otimes K.$$

(i)$\Rightarrow$(ii). It follows from (i) that

(2.1) $$\|C_1 f\| \, \|D_1 g\| \leq \|C_2 f\| \, \|D_2 g\|, \qquad f \in H, \, g \in K.$$

First notice that $\ker C_2 \subseteq \ker C_1$ and $\ker D_2 \subseteq \ker D_1$. Indeed, if $C_2 f = 0$, then by (2.1) we have $\|C_1 f\| \, \|D_1 g\| = 0$ for every $g \in K$. However $D_1 \neq 0$, so $C_1 f = 0$ (similarly we show the other inclusion). The above remark and (2.1) imply that there exists $0 < c < +\infty$ such that

$$\sup_{C_1 f \neq 0} \|C_1 f\| \, \|C_2 f\|^{-1} \leq c^{1/2} \leq \inf_{D_1 g \neq 0} \|D_2 g\| \, \|D_1 g\|^{-1}.$$

Hence

$$(A_1 f, f) = \|C_1 f\|^2 \leq c \|C_2 f\|^2 = c (A_2 f, f), \qquad f \in H.$$

Similarly we show that $(B_1 g, g) \leq c^{-1} (B_2 g, g)$ for $g \in K$, which completes the proof. $\square$

In the proof of the main result of this paper we will make use of the following version of Lambert's criterion for subnormality of bounded operators.

Proposition 2.3. *If $A \in \mathcal{B}(H)$, then the following conditions are equivalent:*
  (i) *A is subnormal.*
  (ii) *For every $f \in H$, $\{\|A^n f\|^2\}_{n=0}^{\infty}$ is a Stieltjes moment sequence.*
  (iii) *For every $f \in H \ominus \ker A$, $\{\|A^n f\|^2\}_{n=0}^{\infty}$ is a Stieltjes moment sequence.*

Proof. The equivalence (i)⇔(ii) is essentially due to Lambert (cf. [6, 10]).

(iii)⇒(ii). If $f \in H$, then $f = g \oplus h$, where $g \in \ker A$, $h \in H \ominus \ker A$. Therefore $\|A^n f\|^2 = \int_0^\infty t^n d(\mu + \|g\|^2 \delta_0)(t)$ for $n \geq 0$, where μ is a representing measure of $\{\|A^n h\|^2\}_{n=0}^\infty$ and δ_0 is the probability measure concentrated on $\{0\}$. This completes the proof. $\qquad\square$

We are now in a position to prove the main result of the paper.

Theorem 2.4. *If $A \in \mathcal{B}(H)$ and $B \in \mathcal{B}(K)$ are nonzero operators, then*
  (i) *$A \otimes B$ is isometric (resp. unitary, selfadjoint, nonnegative) if and only if so are cA and $c^{-1}B$ for some $c \in \mathbb{C} \setminus \{0\}$,*
  (ii) *$A \otimes B$ is normal (resp. quasinormal, subnormal, hyponormal) if and only if so are A and B.*

Proof. (i) Among the properties mentioned in (i), only the positivity needs some explanation, as the other ones can be easily deduced from Proposition 2.1 (see also *Normality* below).

Positivity. If $A \geq 0$ and $B \geq 0$, then $A \otimes B = (A^{1/2} \otimes B^{1/2})^2 \geq 0$. Conversely, if $A \otimes B \geq 0$, then by (i) there exists $c \neq 0$ such that cA and $c^{-1}B$ are selfadjoint. Therefore we can assume without loss of generality that A and B are selfadjoint and

$$(2.2) \qquad\qquad (Af, f)\,(Bg, g) \geq 0, \qquad f \in H,\, g \in K.$$

Since $A \neq 0$, there exists f_0 such that $(Af_0, f_0) \in \mathbb{R} \setminus \{0\}$. If $(Af_0, f_0) > 0$, then by (2.2) we have $B \geq 0$. However, $B \neq 0$, so there exists g_0 such that $(Bg_0, g_0) > 0$. Applying once more (2.2) we obtain $A \geq 0$. The other case $(Af_0, f_0) < 0$ leads to $-A \geq 0$ and $-B \geq 0$.

(ii) *Normality.* If $A \otimes B$ is normal, then by Proposition 2.1 there exists $c \neq 0$ such that $A^*A = cAA^*$ and $B^*B = c^{-1}BB^*$. Consequently $c > 0$, $\|A\|^2 = c\|A\|^2$ and $\|B\|^2 = c^{-1}\|B\|^2$, which is possible only if $c = 1$. Thus A and B are normal. The converse implication is obvious.

Quasinormality. If $A \otimes B$ is quasinormal, then by Proposition 2.1 there exists $c \neq 0$ such that $AA^*A = cA^*A^2$ and $BB^*B = c^{-1}B^*B^2$. This in turn implies that $(A^*A)^2 = c(A^2)^*A^2$ and $(B^*B)^2 = c^{-1}(B^2)^*B^2$. Consequently $c > 0$, $\|A\|^4 = c\|A^2\|^2 \leq c\|A\|^4$ and $\|B\|^4 = c^{-1}\|B^2\|^2 \leq c^{-1}\|B\|^4$, which yields $c = 1$. Therefore A and B are quasinormal. The converse implication is obvious.

Subnormality. Assume that $A \otimes B$ is subnormal. Put $\alpha_n(f) := \|A^n f\|^2$ and $\beta_n(g) := \|B^n g\|^2$ for $f \in H$, $g \in K$ and $n \geq 0$. Then $\alpha_{n+i}(f)\,\beta_{n+j}(g) = \|(A \otimes B)^n (A^i f \otimes B^j g)\|^2$ for $i, j \geq 0$, so by Proposition 2.3 we have

$$(2.3) \qquad \{\alpha_{n+i}(f)\,\beta_{n+j}(g)\}_{n=0}^\infty \text{ is a Stieltjes moment sequence}$$
$$\text{for all } f \in H,\, g \in K \text{ and } i, j \geq 0.$$

Note that if $f \notin \ker A$ and $g \notin \ker B$, then $\alpha_n(f) > 0$ and $\beta_n(g) > 0$ for every $n \geq 0$. Indeed, if $\alpha_k(f) = 0$ for some $k \geq 1$ ($\alpha_0(f) \neq 0$!), then by $B \neq 0$ there exists $h \notin \ker B$. It follows from (2.3) and Lemma 1.1(ii) that $\|Af\|^2 \|Bh\|^2 = \alpha_1(f)\,\beta_1(h) = 0$, which contradicts $Af \neq 0$ and $Bh \neq 0$. Similar arguments apply to $\{\beta_n(g)\}_{n=0}^\infty$.

Take arbitrary nonzero vectors $f \in H \ominus \ker A$ and $g \in K \ominus \ker B$. Then $\limsup_{n \to \infty} \alpha_{n+1}(f) \, \alpha_n^{-1}(f) \leq \|A\|^2 < +\infty$ and $\limsup_{n \to \infty} \beta_{n+1}(g) \, \beta_n^{-1}(g) \leq \|B\|^2 < +\infty$. This and condition (2.3) imply—via Lemma 1.2—that $\{\alpha_n(f)\}_{n=0}^\infty$ and $\{\beta_n(g)\}_{n=0}^\infty$ are Stieltjes moment sequences. Hence, by Proposition 2.3, A and B are subnormal. The converse implication is obvious.

Hyponormality. The sufficiency can be directly derived from Proposition 2.2. If $A \otimes B$ is hyponormal, then $(A \, A^* \otimes B \, B^*) \leq A^* A \otimes B^* B$ and consequently—by Proposition 2.2—there exists $c > 0$ such that $A \, A^* \leq c \, A^* A$ and $B \, B^* \leq c^{-1} B^* B$. Hence $\|A\|^2 \leq c \, \|A\|^2$ and $\|B\|^2 \leq c^{-1} \|B\|^2$. Since both A and B are nonzero, we come to the conclusion that $c = 1$. This means that A and B are hyponormal. $\square$

3. Comments and concluding remarks

The following example shows that the condition (ii) of Lemma 1.2 cannot be weakened.

Example 3.1. For every integer $s \geq 1$, there exist two sequences $\{\alpha_n\}_{n=0}^\infty$ and $\{\beta_n\}_{n=0}^\infty$ of positive real numbers such that
  (i) $\limsup \alpha_{n+1} \, \alpha_n^{-1} < +\infty$ and $\limsup \beta_{n+1} \, \beta_n^{-1} < +\infty$,
  (ii) $\{\alpha_n\}_{n=0}^\infty$ is not a Stieltjes moment sequence,
  (iii) for every $j = 0, 1, ..., s$, $\{\alpha_n \, \beta_{n+j}\}_{n=0}^\infty$ and $\{\alpha_{n+j} \, \beta_n\}_{n=0}^\infty$ are Stieltjes moment sequences.

Take an arbitrary Stieltjes moment sequence $\{p_n\}_{n=0}^\infty$ of positive real numbers. Choose α_0 and β_0 so that $\alpha_0 \, \beta_0 = p_0 \cdots p_{s-1}$. Put $\alpha_n = \beta_0^{-1} \, p_0 \cdots p_{n+s-1}$ and $\beta_n = \beta_0 \, (p_0 \cdots p_{n-1})^{-1}$ for $n \geq 1$. Then $\{\alpha_n \, \beta_{n+j}\}_{n=0}^\infty$ is the constant sequence for $j = s$ and the finite product $\prod_{k=j}^{s-1} p_{n+k}$ of Stieltjes moment sequences for $j < s$. It follows from (S1) that $\{\alpha_n \, \beta_{n+j}\}_{n=0}^\infty$, $j = 0, 1, ..., s$, are Stieltjes moment sequences. The same reasoning applies to the sequences $\{\alpha_{n+j} \, \beta_n\}_{n=0}^\infty$, $j = 0, 1, ..., s$. If additionally there exist $\delta > 0$ and $M > 0$ such that $\delta \leq p_n \leq M$ for all $n \geq 0$ (or equivalently $\{p_n\}_{n=0}^\infty$ is a Hausdorff moment sequence whose representing measure has 1 as an atom), then (i) holds. To get (ii) it is sufficient to assume that $\{p_n\}_{n=0}^\infty$ is strictly decreasing. Indeed, if $\{\alpha_n\}_{n=0}^\infty$ were a Stieltjes moment sequence, the sequence $\{\alpha_{n+1} \, \alpha_n^{-1}\}_{n=0}^\infty$ would be weakly increasing (apply Lemma 1.1(i)), which would contradict the equality $\alpha_{n+1} \, \alpha_n^{-1} = p_{n+s}$, $n \geq 0$. The sequence $p_n := 1 + (n+1)^{-1}$ meets all the requirements.

A part of Theorem 2.4(ii) concerning subnormality of $A \otimes B$ can be directly inferred from the definition of subnormality, provided A and B have nonempty point spectrum. Among operators with empty point spectrum there are injective weighted shifts. This particular case is precisely covered by Lemma 1.2. Indeed, since $A \otimes B$ is unitarily equivalent to $(\sum_{k=0}^\infty \bigoplus A_k) \oplus (\sum_{k=1}^\infty \bigoplus B_k)$, where A, B, A_k and B_k are weighted shifts with weights $\{\lambda_n\}_{n=0}^\infty$, $\{\mu_n\}_{n=0}^\infty$, $\{\lambda_{n+k} \mu_n\}_{n=0}^\infty$ and $\{\lambda_n \mu_{n+k}\}_{n=0}^\infty$, respectively, we can apply the Berger-Gellar-Wallen criterion for subnormality of weighted shifts (cf. [4, 5, 8]).

We conclude the paper with the following

Open Question. Is it true that if the two-fold symmetric tensor product of a nonzero bounded linear operator A by itself is subnormal, then so is A ?

We know the affirmative answer for injective weighted shifts. Indeed, in this particular case the symmetric tensor product of A by itself is unitarily equivalent

to the orthogonal sum $\sum_{k=0}^{\infty} \bigoplus S_k$ of weighted shifts S_k with weights $\{\lambda_n\,\lambda_{n+k}\}_{n=0}^{\infty}$, $k \geq 0$, where $\{\lambda_n\}_{n=0}^{\infty}$ are weights of A. Applying the Berger-Gellar-Wallen theorem and Lemma 1.2 we get the conclusion.

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