PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 124, Number 1, January 1996

ON FACTOR STATES OF C^*-ALGEBRAS AND THEIR EXTENSIONS

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(Communicated by Palle E. T. Jorgensen)

ABSTRACT. We obtain some results on the unique extension of (factor) states of C^*-algebras which complement various existing results. Our results also lead to a class of C^*-algebras whose states are σ-convex sums of factor states.

1. INTRODUCTION

Recently Pfitzner [19] proved that every von Neumann algebra is a Grothendieck space, thus solving a long-standing open problem (cf. [9; p.104] and see also [1]). Using this interesting result, we obtain some new information concerning the unique extension of (factor) states of C^*-algebras which complements the existing results in [2, 4, 7, 14, 15]. Extensions of factor states have been investigated by many authors (e.g., [3, 6, 17, 20, 21, 23, 25]). Our approach to unique extension, however, is based on a simple observation (Lemma 1) that if states φ of certain type on a subalgebra extend uniquely to states $\overline{\varphi}$ of certain type on the containing algebra, then the natural map $\varphi \to \overline{\varphi}$ is weak*-continuous. This observation enables us to apply Pfitzner's result to study unique extension of states if the containing algebra is a von Neumann algebra. We show, for instance, that given a separable C^*-subalgebra B of a von Neumann algebra M, then B must be scattered if every factor state of B extends uniquely to a weak*-limit of factor states on M. Further, if B is abelian, then every pure state of B extends uniquely to a pure state of M if and only if $B = \mathbb{C}p_1 \oplus \mathbb{C}p_2 \oplus \cdots \oplus \mathbb{C}p_n \oplus \cdots$, where each p_n is a minimal projection in M and the sum is a c_0-sum and may be finite. This latter result complements the results of Kadison and Singer [14], Anderson [2], and Archbold, Bunce and Gregson [7]. Other extension result also leads us to study a class of C^*-algebras whose states are σ-convex sums of factor states.

We are very grateful to the referee for many helpful suggestions, including those which strengthen our earlier results in Theorem 7 and Corollary 8.

2. EXTENSION OF FACTOR STATES

Let A be a C^*-algebra and let $S(A)$ be its state space. Given a subset $K \subset A^*$, we will denote by $\overline{K}$ the weak* closure of K in A^*, and by $\overline{\overline{K}}$ the norm-closure of K in A^*. Let $P(A)$ be the set of pure states of A and let $F(A)$ be the set of factor states of A.

Received by the editors August 8, 1994.
1991 *Mathematics Subject Classification.* Primary 46L05; Secondary 46L10, 46L30.

The works of Anderson and Bunce [3], Longo [17] and Popa [20] show that if B is a *separable* C^*-subalgebra of A, then every $\varphi \in F(B)$ extends to a $\overline{\varphi} \in F(A)$. Also Archbold [5] has shown that for any C^*-subalgebra $B \subset A$, every $\varphi \in \overline{F(B)}$ extends to a $\overline{\varphi} \in \overline{F(A)}$. Naturally, one is interested in the question of when these extensions are unique. Of course, if B separates points of $F(A)$ (respectively $\overline{F(A)}$), then every $\varphi \in F(B)$ extends uniquely to a $\overline{\varphi} \in F(A)$ (respectively $\overline{F(A)}$); but in this case, the Stone-Weierstrass theorem of Longo [17] and Popa [20] (respectively Glimm [11]) implies $B = A$. In the following, we will show various necessary conditions for the unique extension of factor states. If B is abelian, the question of unique factor state extension is equivalent to that of unique pure state extension which has been considered in [2, 4, 7, 14]; nevertheless, we are able to describe B with such unique extension explicitly if it is separable and if A is a von Neumann algebra. We begin with a simple but useful lemma.

Lemma 1. *Let A be a C^*-algebra and B be a C^*-subalgebra of A. Let K_B be a subset of B^* and K_A be a weak*-compact set of A^*. Suppose that every $\varphi \in K_B$ extends uniquely to $\overline{\varphi} \in K_A$. Then the natural map $\varphi \in K_B \to \overline{\varphi} \in K_A$ is weak*-continuous.*

Proof. Let $\Phi : K_B \to K_A$ be the natural injection defined by $\Phi(\varphi) = \overline{\varphi}$. Let F be a weak*-closed subset of K_A. We show that $\Phi^{-1}(F)$ is weak*-closed in K_B. Let $\varphi = w^* - \lim_\alpha \varphi_\alpha$ with $\varphi_\alpha \in \Phi^{-1}(F)$. By weak* compactness of K_A, there exists a subnet $\{\varphi_\beta\}$ such that $\Phi(\varphi_\beta)$ converges to $\psi \in F$, say, in the weak* topology. We have

$$\psi|_B = w^* - \lim_\beta \Phi(\varphi_\beta)|_B = \lim_\beta \varphi_\beta = \varphi.$$

By unique extension, we have $\Phi(\varphi) = \overline{\varphi} = \psi \in F$. □

Let $K_B \subset B^*$ have the weak* topology as above, and let $C(K_B)$ denote the C^*-algebra of complex bounded continuous functions on K_B. Identifying K_B as a subset of $C(K_B)^*$, we have the following general criterion for unique extension.

Proposition 2. *Let A be a C^*-algebra and B a C^*-subalgebra of A. Let K_B be a subset of B^* and let K_A be a weak*-compact set in A^*. The following conditions are equivalent:*

(i) *Every $\varphi \in K_B$ extends uniquely to a $\overline{\varphi} \in K_A$;*

(ii) *There is a linear map $Q : A \to C(K_B)$ such that for every $\varphi \in K_B$ with extension $\overline{\varphi} \in K_A$, we have $\overline{\varphi} = \varphi \circ Q$.*

Proof. It is evident that (ii) $\Longrightarrow$ (i). Assume condition (i). By Lemma 1, the map $Q : A \to C(K_B)$ given by

$$Q(a)(\varphi) = \overline{\varphi}(a) \qquad (a \in A,\ \varphi \in K_B)$$

is well defined and satisfies condition (ii). □

Remark. In the above result, Q is continuous if K_B is a norm-bounded subset of B^*.

We have the following immediate corollary which has been obtained by Anderson [2] under the assumption that A is unital and with a different method.

Corollary 3. *Let B be an abelian C^*-subalgebra of a C^*-algebra A. The following conditions are equivalent:*

(i) *Every $\varphi \in P(B) \cup \{0\}$ extends uniquely to a $\overline{\varphi} \in P(A) \cup \{0\}$;*

(ii) *There is a contractive projection $Q : A \to B$ such that $\overline{\varphi} = \varphi \circ Q$ for every $\varphi \in P(B) \cup \{0\}$ with extension $\overline{\varphi}$ on A.*

Proof. We note that if a pure state has a unique pure state extension, then it has unique state extension. We identify B with $C_0(P(B)) = \{b \in C(P(B) \cup \{0\}) : b(0) = 0\}$. Letting $K_B = P(B) \cup \{0\}$ and $K_A = \overline{P(A) \cup \{0\}}$ in Proposition 2, we see that the map $Q : A \to C(K_B)$ has range $Q(A) \subset C_0(P(B)) = B \subset Q(A)$ and Q is a desired contractive projection. □

Now we are going to show how Lemma 1 together with the aforementioned result of Pfitzner can be used to obtain new information on the unique extension of states. Recall that a Banach space A is called a *Grothendieck space* [9] if every weak*-convergent sequence in A^* is $\sigma(A^*, A^{**})$-convergent. It is evident that a quotient space of a Grothendieck space is also a Grothendieck space and that a separable Grothendieck space must be reflexive by the weak compactness of the dual ball.

Theorem 4 (Pfitzner [19]). *Every von Neumann algebra is a Grothendieck space.*

It follows from Pfitzner's theorem that there is no surjective continuous linear map from a von Neumann algebra onto an infinite-dimensional *separable* C^*-algebra. This interesting fact will be used to deduce Theorem 6 below. We note that if M is an infinite-dimensional von Neumann algebra, then its predual M_* (and hence M^*) is not a Grothendieck space. Indeed, M contains a copy of c_0 and so M_* contains a complemented copy of l_1 [16; Proposition 2.e.8] which is not a Grothendieck space. Also $M_* \supset l_1$ implies that M^* contains a bounded sequence which has *no* weak*-convergent subsequence [16, Theorem 2.e.7]. Therefore, in order to apply Pfitzner's result, we need to assume the separability condition in some of the following arguments.

We note that there are C^*-algebras, other than von Neumann algebras, which are Grothendieck spaces. For instance, the Calkin algebra $B(H)/K(H)$ is such a space. If K is a compact Hausdorff space in which any two disjoint open F_σ sets have disjoint closures, then $C(K)$ is a Grothendieck space [22]. Also, given a C^*-algebra A which is a Grothendieck space, it is easy to verify that the matrix algebra $M_n(A)$ over A is also a Grothendieck space. In fact, one can deduce from [4] that a C^*-algebra A is a Grothendieck space if and only if there is no surjective continuous linear map from A onto c_0. In Theorem 6, Theorem 7 and Corollary 8 below, one can actually replace the von Neumann algebra there by a C^*-algebra which is a Grothendieck space.

Lemma 5. *Let φ and ψ be two states of a C^*-algebra A. Suppose there is a minimal projection p in A such that $\varphi(p) = \psi(p) = 1$. Then $\varphi = \psi$.*

Proof. Note that φ is a normal state of A^{**} which has identity 1. For $x \in A$, we have $|\varphi((1-p)x)|^2 \leq \varphi(1-p)\varphi(x^*x) = 0$, which gives $\varphi(px) = \varphi(x)$. Likewise $\varphi(xp) = \varphi(x)$. Therefore $\varphi(pxp) = \varphi(xp) = \varphi(x)$. Similarly we obtain that $\psi(pxp) = \psi(x)$. Hence, for $x \in A$ with $pxp = \lambda p$ where $\lambda \in \mathbb{C}$, we have $\varphi(x) = \varphi(pxp) = \lambda = \psi(pxp) = \psi(x)$. □

Given a sequence $\{A_n\}_{n=1}^{\infty}$ of C^*-algebras, their c_0-sum is denoted by $A_1 \oplus A_2 \oplus \cdots \oplus A_n \oplus \cdots$ which is the C^*-algebra consisting of all sequences $\{a_n\}$ such that $a_n \in A_n$ and for any $\varepsilon > 0$ the set $\{n : \|a_n\| \geq \varepsilon\}$ is finite, with the coordinatewise algebraic operations and the supremum norm. For a finite sequence, it is the usual C^*-direct sum.

Theorem 6. *Let B be a separable abelian C^*-subalgebra of a von Neumann algebra M. The following conditions are equivalent:*

(i) *Every pure state of B extends uniquely to a pure state of M.*

(ii) $B = \mathbb{C}p_1 \oplus \mathbb{C}p_2 \oplus \cdots \oplus \mathbb{C}p_n \oplus \cdots$, *where each p_n is a minimal projection in M and the sum may be finite.*

Proof. (i) $\Longrightarrow$ (ii). If B is unital, then $B = C(P(B))$ and the identity of B is a projection $p \in M$. Every pure state of B extends to a unique pure state of pMp. By Corollary 3 and the remarks after Theorem 4, we infer that B is finite-dimensional and hence $B = \mathbb{C}p_1 \oplus \mathbb{C}p_2 \oplus \cdots \oplus \mathbb{C}p_n$, where $p_1, p_2, ..., p_n$ are mutually orthogonal projections in M.

Suppose that B is nonunital. Then $B = C_0(P(B)) = \{b \in C(P(B) \cup \{0\}) : b(0) = 0\}$, where $X = P(B) \cup \{0\}$ is a compact metric space. We have only to show that $P(B)$ is countable and discrete. Let $\{V_n\}$ be a decreasing sequence of open neighbourhoods of 0 such that $\overline{V_{n+1}} \subset V_n$ and $\bigcap_{n=1}^{\infty} V_n = \{0\}$. Then $K_n = X \backslash V_n$ is a compact subset of $P(B)$. Define a linear map $Q : M \to C(K_n)$ by

$$Q(a)(\varphi) = \overline{\varphi}(a) \quad (a \in M,\ \varphi \in K_n),$$

where $\overline{\varphi} \in P(M)$ is the unique extension of φ. By Lemma 1, Q is well defined. Also Q is continuous since $\|Q(a)\| \leqq \|a\|$. Moreover Q is surjective. Indeed, given $h \in C(K_n)$, there exists $b \in C_0(P(B)) \subset M$ such that $b|_{K_n} = h$ which gives $Q(b)(\varphi) = \overline{\varphi}(b) = \varphi(b) = h(\varphi)$ for $\varphi \in K_n$, that is, $Q(b) = h$. Since $C(K_n)$ is separable, K_n must be finite by the remark after Theorem 4. Hence $X = \{0\} \cup \bigcup_{n=1}^{\infty} K_n$ is countable. For each n, $X \backslash \overline{V_n}$ is a finite open subset of $P(B)$ since it is contained in K_n. It follows that $P(B) = \bigcup_{n=1}^{\infty} (X \backslash \overline{V_n})$ is discrete. Thus we obtain that $B = \mathbb{C}p_1 \oplus \mathbb{C}p_2 \oplus \cdots \oplus \mathbb{C}p_n \oplus \cdots$, where $p_1, p_2, ..., p_n, ...$ are mutually orthogonal projections in M.

Assume, for contradiction, that dim $p_1 M p_1 \geq 2$. Then there are distinct pure states φ and ψ on $p_1 M p_1$. As $\varphi(p_1) = \psi(p_1) = 1$, it follows that the pure state ω on B with $\omega(p_1) = 1$ has distinct pure state extensions on M, which contradicts (i). So p_1, and likewise, $p_2, ..., p_n, ...$ are all minimal projections in M.

(ii) $\Longrightarrow$ (i). Let $\omega \in P(B)$. Then there exists i such that $\omega(p_i) = 1$ and $\omega(p_j) = 0$ for $j \neq i$. Let $\varphi, \psi \in P(M)$ be extensions of ω. Then $\varphi(p_i) = \omega(p_i) = \psi(p_i) = 1$. So $\varphi = \psi$ by Lemma 5. □

Remarks. 1. As already mentioned, Theorem 6 holds if M is replaced by, for instance, the Calkin algebra $B(H)/K(H)$. The referee has kindly pointed out an alternative approach to Theorem 6 which does not depend on M being a Grothendieck space but makes use of its ample supply of projections.

2. Theorem 6 is false for nonabelian subalgebras. In fact, it is even false for type I_0 subalgebras. Let M be the full operator algebra on an infinite-dimensional separable Hilbert space H and let B be the C^*-algebra of compact operators on H. Then B satisfies condition (i) in Theorem 6.

3. Kadison and Singer [14] have shown that if B is a maximal abelian subalgebra of $B(H)$ isomorphic to $L^\infty(0,1)$, then there exists a pure state on B with distinct extensions to $B(H)$.

4. Although every state of a C^*-algebra B extends uniquely to a normal state on B^{**}, Theorem 6 shows that a pure state on B need not extend uniquely to a pure state on B^{**}.

Let K be a subset of the state space of a C^*-algebra. We will denote by $\mathrm{co}(K)$ the convex hull of K throughout this paper.

Let A be a C^*-algebra, and let φ be a state of A. We denote by $(\pi_\varphi, H_\varphi, \xi_\varphi)$ the GNS representation of A associated with φ, that is, π_φ is a represenatation of A on the Hilbert space H_φ with the canonical cyclic vector ξ_φ and the inner product defined by $\langle \pi_\varphi(x)\xi_\varphi,\ \pi_\varphi(y)\xi_\varphi\rangle = \varphi(y^*x)$ for $x, y \in A$ ([10, 18]). A state φ is called a *type* I *state* if the GNS representation associated with φ is type I.

We recall that a C^*-algebra A is *dual* if and only if it is isomorphic to a C^*-subalgebra of the C^*-algebra $K(H)$ of compact operators, or equivalently, every maximal abelian subalgebra of A is generated by minimal projections [10, 4.7.20]. Note that dual C^*-algebras are scattered (cf. [12]).

Theorem 7. *Let B be a separable C^*-subalgebra of a von Neumann algebra M. If every $\varphi \in F(B)$ extends uniquely to a $\overline{\varphi} \in \overline{F(M)}$, then B is a scattered C^*-algebra.*

Proof. Let $\varphi \in F(B)$. We show that φ is type I. Let $F_I(B)$ be the set of all type I factor states of B. By [5, Corollary 3.4] and by separability of B, there is a sequence $\{\varphi_n\}$ in $F_I(B)$ such that $\varphi = w^* - \lim_{n\to\infty} \varphi_n$. By Lemma 1, we have $\overline{\varphi} = w^* - \lim_{n\to\infty} \overline{\varphi_n}$ in M^*. By Theorem 4, $\overline{\varphi} = \lim_{n\to\infty} \overline{\varphi_n}$ in the $\sigma(M^*, M^{**})$-topology and hence $\overline{\varphi} \in \overline{\overline{\mathrm{co}}}(\{\overline{\varphi_n} : n = 1, 2, ...\})$. It follows that $\varphi = \overline{\varphi}|_B \in \overline{\overline{\mathrm{co}}}(\{\varphi_n : n = 1, 2, ...\}) \subset \overline{\overline{\mathrm{co}}}(F_I(B))$. Therefore φ is type I since type I states form a norm-closed convex set by [13, Theorem 4.1].

Since B is type I, it contains a nonzero hereditary abelian C^*-subalgebra C (cf. [18, 6.1]) and by [18, 3.1.6], every pure state of C extends uniquely to one of M. By Theorem 6, C is generated by minimal projections of M and, in particular, the set P of all those minimal projections of M contained in B is nonempty. Let I be the norm-closed ideal in B generated by P and let J be the norm-closed ideal in M generated by P. Then I and J are dual C^*-algebras and $I = B \cap J$. Now pass to the inclusion $B/I \subset M/J$ and we can repeat the above argument since Theorem 6 is valid for the Grothendieck space M/J as remarked before. By transfinite induction, B has a composition series in which successive quotients are dual C^*-algebras. Hence B is scattered. □

Remark. The arguments in the last paragraph are due to the referee.

Corollary 8. *Let B be a separable hereditary C^*-subalgebra of a von Neumann algebra M. Then B is a dual C^*-algebra generated by minimal projections of M.*

Proof. By Theorem 7, B is scattered. Let C be any maximal abelian subalgebra of B. Then C is generated by projections. But if $p \in B$ is a projection, then pMp is contained in B and is therefore finite-dimensional by separability. So C is generated by minimal projections and it follows that B is a dual C^*-algebra generated by minimal projections of M. □

We recall that given a C^*-subalgebra B of a C^*-algebra A, then B is hereditary in A if and only if every $\varphi \in S(B)$ extends uniquely to a $\overline{\varphi} \in S(A)$ (cf. [15], [18]). Let B act on a Hilbert space H and let $V(B) = \{\omega_\xi : \xi \in H, \|\xi\| = 1\}$ be the set of vector states of B where $\omega_\xi(\cdot) = \langle \cdot \xi, \xi \rangle$ is defined on B. Then $\mathrm{co}(V(B)) \subset S(B) \subset \overline{\mathrm{co}}(V(B))$ [10, 3.4.1]. In the following, we will consider unique extension of states in $\mathrm{co}(V(B))$. Given $K \subset S(B)$, we denote by

$$\sigma(K) = \{\sum_{n=1}^{\infty} \lambda_n \varphi_n : \varphi_n \in K, \lambda_n \geq 0, \sum_{n=1}^{\infty} \lambda_n = 1\}$$

the σ-convex hull of K, where the infinite sum is norm-convergent.

We note that $\sigma(K) \subset \overline{\overline{\mathrm{co}}}(K)$. Indeed, let $\varphi = \sum_{n=1}^{\infty} \lambda_n \varphi_n \in \sigma(K)$ and let $\varepsilon > 0$. Then there exists j such that $\| \sum_{n=1}^{j} \lambda_n (\varphi - \varphi_n) \| < \varepsilon$ and $\lambda = \sum_{n=1}^{j} \lambda_n > \frac{1}{2}$. So $\|\varphi - \sum_{n=1}^{j} \frac{\lambda_n}{\lambda} \varphi_n \| < \frac{\varepsilon}{\lambda} < 2\varepsilon$ and hence $\varphi \in \overline{\overline{\mathrm{co}}}(K)$.

We also note that $\sigma(\sigma(K)) = \sigma(K)$. To see this, let $\mu = \sum_{n=1}^{\infty} \alpha_n \mu_n$ with $\alpha_n \geq 0$, $\sum_{n=1}^{\infty} \alpha_n = 1$ and $\mu_n \in \sigma(K)$. We show $\mu \in \sigma(K)$. Let $\mu_n = \sum_{k=1}^{\infty} \lambda_k^n \nu_k^n$ with $\lambda_k^n \geq 0$, $\sum_{k=1}^{\infty} \lambda_k^n = 1$ and $\nu_k^n \in K$. Since the norm is additive on $S(B)$, we have $\sum_{k=1}^{\infty} \lambda_k^n \|\nu_k^n\| = \|\mu_n\| = 1 = \sum_{n=1}^{\infty} \alpha_n \|\mu_n\|$. Let $\varphi_m = \sum_{n+k \leq m} \alpha_n \lambda_k^n \nu_k^n$. Then $\sum_{m=2}^{\infty} \|\varphi_m\| < \infty$ since it has bounded partial sums. By absolute convergence, we have, for $b \in B$,

$$\sum_{m=2}^{\infty} \varphi_m(b) = \sum_{m=2}^{\infty} \sum_{n+k \leq m} \alpha_n \lambda_k^n \nu_k^n(b) = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \alpha_n \lambda_k^n \nu_k^n(b) = \sum_{n=1}^{\infty} \alpha_n \mu_n(b) = \mu(b).$$

So $\mu = \sum_{m=2}^{\infty} \varphi_m \in \sigma(K)$.

Proposition 9. *For any C^*-algebra B, we have $\overline{\overline{\mathrm{co}}}(F(B)) = \sigma(F(B))$.*

Proof. By the above remarks, we need only show that $\sigma(F(B))$ is norm-closed. Let $\varphi = \lim_{n\to\infty} \varphi_n$ in the norm topology where $\varphi_n \in \sigma(F(B))$. Let $\psi = \sum_{n=1}^{\infty} \frac{1}{2^n} \varphi_n$. Then $\psi \in \sigma(\sigma(F(B))) = \sigma(F(B))$ and hence its GNS representation π_ψ is a (countable) direct sum of factor representations (cf. Theorem 11). Let $p \in B^{**}$ be the support of ψ and let $q \in B^{**}$ be the support of φ. Then $\varphi_n(1-p) = 0$ for all n implies $\varphi(1-p) = 0$. Hence $q \leq p$ and π_φ is equivalent to a subrepresentation of π_ψ. It follows that π_φ is a (countable) direct sum of factor representations and so $\varphi \in \sigma(F(B))$. □

Remark. One can show analogously that $\overline{\overline{\mathrm{co}}}(P(B)) = \sigma(P(B))$.

Theorem 10. *Let B be a separable C^*-algebra acting on a Hilbert space H such that its weak closure M is a direct sum of factors. Suppose that every $\varphi \in \mathrm{co}(V(B))$ extends uniquely to a $\overline{\varphi} \in S(M)$. Then $S(B) = \sigma(F(B))$.*

Proof. Let $K(H)$ be the C^*-algebra of compact operators on H. Let $\varphi \in S(B)$. Then $\varphi = \lambda\phi + (1-\lambda)\psi$ where $0 \le \lambda \le 1, \|\phi|_{B\cap K(H)}\| = 1$ and $\psi|_{B\cap K(H)} = 0$. Since $B \cap K(H)$ is a scattered C^*-algebra [12], $\phi|_{B\cap K(H)}$ is a σ-convex sum of pure states of $B \cap K(H)$. As $B \cap K(H)$ is an ideal in B, by unique extension, we conclude that $\phi \in \sigma(P(B))$. By [11, Theorem 2] and by separability of B, we have $\psi = w^* - \lim_{n\to\infty} \omega_n$ where $\omega_n \in V(B)$. By Lemma 1, the simultaneous extension map $\omega \in \mathrm{co}(V(B)) \to \overline{\omega} \in S(M)$ is weak*-continuous and unique extension also implies that the map $\omega - \omega' \in V(B) - V(B) \to \overline{\omega} - \overline{\omega'} \in S(M) - S(M)$ is well defined and weak*-continuous. It follows that $\overline{\omega_m} - \overline{\omega_n} \to 0$ in the weak* topology as $m, n, \to \infty$ and therefore $\omega = w^* - \lim_{n\to\infty} \overline{\omega_n}$ exists. By Theorem 4, we have $\omega = \lim_{n\to\infty} \overline{\omega_n}$ in the $\sigma(M^*, M^{**})$-topology which gives $\omega \in \overline{\overline{\mathrm{co}}}(\{\overline{\omega_n} : n = 1, 2, ...\})$. So

$$\psi = \omega|_B \in \overline{\overline{\mathrm{co}}}(\{\omega_n : n = 1, 2, ...\}).$$

By the hypothesis, the identity representation of B is a direct sum of factor representations. Since the GNS representation induced by ω_n is equivalent to a subrepresentation of the identity representation, it follows that $\omega_n \in \sigma(F(B))$. By Proposition 9, we have

$$\varphi = \lambda\phi + (1-\lambda)\psi \in \sigma(F(B)). \qquad \square$$

A special example of Theorem 10 is the C^*-algebra $B = K(H)$ in which case we even have $S(B) = \sigma(P(B))$ [10, 4.1.3]. We study below those algebras satisfying the conclusion of Theorem 10.

3. A class of C^*-algebras whose states are σ-convex sums of factor states

In Theorem 10, a class of C^*-algebras A occur for which $S(A) = \sigma(F(A))$. We derive some properties of these algebras in this section.

Let A be a C^*-algebra and let I be a closed two-sided ideal in A. We denote as before by $(\pi_\varphi, H_\varphi, \xi_\varphi)$ the GNS representation of A associated with a state φ. Given a state φ on I with unique extension $\overline{\varphi}$ to A, we have $\varphi \in F(I)$ if and only if $\overline{\varphi} \in F(A)$ since $\pi_\varphi(I)'' = \pi_{\overline{\varphi}}(A)''$ [8, Lemma 4.1.33]. Further, if $q : A \to A/I$ is the quotient map and if $\psi \in S(A/I)$, then by [24, Theorem 1.4] $\psi \in F(A/I)$ if and only if $\psi \circ q \in F(A)$.

Theorem 11. *Let A be a C^*-algebra and let I be a closed two-sided ideal of A. The following conditions are equivalent:*

(i) $S(A) = \sigma(F(A))$.

(ii) *Every nondegenerate representation of A is equivalent to a subrepresentation of a direct sum of factor representations.*

(iii) A^{**} *is a direct sum of factors.*

(iv) $S(I) = \sigma(F(A))$ *and* $S(A/I) = \sigma(F(A/I))$.

Proof. (i) $\Longrightarrow$ (ii). Given a cyclic representation $\pi_\varphi : A \to B(H_\varphi)$ with $\varphi \in S(A)$ and $\varphi = \sum_{n=1}^{\infty} \lambda_n \varphi_n \in \sigma(F(A))$ where $\varphi_n \in F(A)$ and $0 < \lambda_n < 1$, we have

$$\begin{aligned}\langle \pi_\varphi(\cdot)\xi_\varphi,\ \xi_\varphi\rangle = \varphi(\cdot) = \sum_{n=1}^{\infty} \lambda_n \varphi_n = \sum_{n=1}^{\infty} \langle \pi_{\varphi_n}(\cdot)\sqrt{\lambda_n}\xi_{\varphi_n},\ \sqrt{\lambda_n}\xi_{\varphi_n}\rangle \\ = \langle \bigoplus_n \pi_{\varphi_n}(\cdot)\xi,\ \xi\rangle\end{aligned}$$

where $\xi = \bigoplus_n \sqrt{\lambda_n}\xi_{\varphi_n} \in \bigoplus_n H_{\varphi_n}$ and π_{φ_n} is a factor representation.

(ii) $\Longrightarrow$ (iii). The universal representation is a direct sum of factor representations.

(iii) $\Longrightarrow$ (i). Let $\varphi \in S(A)$ and let $e \in A^{**}$ be its central support. Then $e = \sum_\alpha e_\alpha$ where e_α is a minimal central projection in A^{**}. Since $\sum_\alpha \varphi(e_\alpha) = \varphi(e) = 1$, $\sum_\alpha \varphi(e_\alpha)$ is a countable sum. Relabel $\{e_\alpha\}$ as $\{e_n\}$ so that $\lambda_n = \varphi(e_n) > 0$ with $\sum_n \lambda_n = 1$ and $\varphi(a) = \varphi(ae) = \sum_n \varphi(ae_n)$ for $a \in A$. Let $\varphi_n(\cdot) = \frac{1}{\lambda_n}\varphi(\cdot\ e_n)$. Then $\varphi_n \in S(A)$ and e_n is its central support. Indeed, let $p \in A^{**}$ be a central projection such that $\varphi_n(a) = \varphi_n(ap)$ for $a \in A$. Then $\varphi(ape_n) = \lambda_n \varphi_n(ap) = \varphi(ae_n)$. Let $e' = \sum_m e'_m$ where

$$e'_m = \begin{cases} e_m & \text{if } \ m \neq n, \\ pe_n & \text{if } \ m = n. \end{cases}$$

Then $\varphi(ae') = \sum_m \varphi(ae'_m) = \sum_m \varphi(ae_m) = \varphi(ae) = \varphi(a)$, which gives $e' \geq e$ and so $pe_n \geq e_n$. As e_n is a minimal central projection, we have $\varphi_n \in F(A)$. So $\varphi = \sum_n \lambda_n \varphi_n \in \sigma(F(A))$.

(i) $\Longleftrightarrow$ (iv). This follows from the remarks before Theorem 11 and the fact that given $\varphi \in S(A)$ and given a closed two-sided ideal $I \subset A$, we have $\varphi = \varphi_1 + \varphi_2$ with $\|\varphi_1\| = \|\varphi|_I\|$ and $\varphi_2(I) = \{0\}$. □

Remark. A C^*-algebra B is scattered if and only if it is type I and $S(B) = \sigma(F(B))$.

Acknowledgment

This research was carried out during the first author's visit at Kansai University. We gratefully acknowledge financial support from the Japan Association for Mathematical Sciences and the Nuffield Foundation. We are indebted to Dr. J. A. Erdos for his very useful comments.

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