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WEIGHTED INEQUALITIES FOR THE MAXIMAL GEOMETRIC MEAN OPERATOR

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ABSTRACT. For nonnegative Borel measures μ on R^1 and for the maximal geometric mean operator G_f, we characterize the weight pairs (w, v) for which G_f is of weak type (p, p) and of strong type (p, p), $0 < p < \infty$. No doubling conditions are needed. We also note that a previously published different characterization for the strong type inequality for G_f has an incorrect proof.

1. INTRODUCTION

Let μ be a nonnegative Borel measure on R^1. The maximal geometric mean operator is (see X. Shi [7])

$$G_f(x) = \sup \exp\left[\frac{1}{|I|}\int_I \log(|f(y)|)\,d\mu(y)\right],$$

where the sup is taken over all intervals I in R^1 containing x such that the integral is defined and $0 < |I| = \mu(I) < \infty$; if no such I exists we take $G_f(x) = 0$.

If $0 < p < \infty$, $w = v$ and $d\mu$ is Lebesgue measure on R^1, then the inequality

$$\int G_f(x)^p w(x) d\mu(x) \le C_1 \int |f(x)|^p v(x) d\mu(x) \tag{1}$$

for all f in $L^p(vd\mu)$ is equivalent to the requirement that

$$\left(\frac{1}{|I|}\int_I w d\mu\right)\left(\exp\left[\frac{1}{|I|}\int_I \log(1/v)\,d\mu\right]\right) \le C_2 \tag{2}$$

for all intervals I (see [7]) and is also equivalent to the weak type inequality

$$\int_{\{x|G_f(x)>\lambda\}} w\,d\mu \le \frac{C_3}{\lambda^p}\int |f(x)|^p v(x) d\mu(x) \tag{3}$$

for all f in $L^p(vd\mu)$ (see [3]). In [3] it is stated that in spaces of homogeneous type condition (2) implies inequality (1) for general not necessarily equal v and w provided that μ and $vd\mu$ satisfy doubling conditions. The proof given in [3], however, is incorrect. They state on page 71 that "it is easy to see that" if $f_0(x) = f(x)$ on the set where $|f(x)| \le \alpha/2$ and 0 elsewhere and $f_1(x) = f(x) - f_0(x)$, then $G_f(x) \le G_{f_1}(x) + \alpha/2$. A simple counterexample is $f(x) = \chi_{(0,1]}(x,x) + e^n\chi_{(1,\infty)}(x,x)$ and $\alpha = 2$; for $0 < x < 1$ we have $G_f(x) = e^n$ and $G_{f_1}(x) = 0$. We

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will show in Theorem 1 that in general a stronger condition than (2) is needed to imply (1).

It might seem that since condition (2) implies (3) for all p, the Marcinkiewicz interpolation theorem would show that (2) implies (1). The Marcinkiewicz theorem cannot be applied here, however, because the operator is not quasilinear. This is easily seen in R^1 with $d\mu = dx$ by taking E to be a set such that $E \cap I$ and $E^c \cap I$ both have positive measure for every interval I. Then with $f = \chi_E$ and $g = \chi_{E^c}$ we have G_{f+g} equal to 1 for all x while G_f and G_g are 0 for all x.

2. The main theorems

Theorem 1. *Let μ be a nonnegative Borel measure on R^1, $0 < p < \infty$, and let w and v be two μ measurable weight functions. Then inequality* (1) *holds for all f in $L^p(vd\mu)$ if and only if*

$$\int_I G_{(v^{-1}\chi_I)}(x)w(x)\,d\mu(x) \le C_4|I| \tag{4}$$

for every interval I.

Theorem 2. *Let w, v, μ and p be as in Theorem 1. Then inequality* (3) *holds for all f in $L^p(vd\mu)$ if and only if* (2) *holds for every interval I.*

From the proof of X. Shi [7] we can easily see that (4) is equivalent to (2) if $w = v$. This is also a simple consequence of Theorem 1 and well-known facts by the following reasoning. If $w = v$ satisfies (2), then by Theorem 1, page 254 of [2], w satisfies the condition A_∞. By the Theorem on page 104 of [4] there is a $p > 1$ such that w satisfies A_p, and since $G_f(x) \le Mf(x)$, where M denotes the Hardy-Littlewood maximal operator, we have (1) by Theorem 2, page 216 of [5]. Theorem 1 then shows that w satisfies (4).

Without the assumption $w = v$ conditions (2) and (4) are not equivalent; in section 6 we give an example of a pair of functions v, w that satisfy (2) but do not satisfy (4).

These theorems can be viewed as limiting cases of known results about the Hardy-Littlewood maximal operator M. This is based on the equality

$$\lim_{p\to\infty}\left(\frac{1}{|I|}\int_I v^{-1/p}d\mu\right)^p = \exp\left[\frac{1}{|I|}\int_I \log(1/v)\,d\mu\right],$$

which is valid if the left side is finite (see [1], [2]), and the consequence that $G_f(x) = \lim_{p\to\infty}[M(f^{1/p})]^p$ for suitably restricted f. Theorem 1 is a limiting case of the fact from Theorem B, p. 7 of [6] that $\int[M(f^{1/p})]^p w\,d\mu \le C\int|f|v\,d\mu$ if and only if for every interval I $\int_I[M(\chi_I v^{-1/(p-1)})]^p w\,d\mu \le C\int_I v^{-1/(p-1)}\,d\mu$. Theorem 2 is a limiting case of the fact from Theorem 6, p. 219 of [5] that

$$\int_{\{x|[M(f^{1/p})]^p>\lambda\}} w\,d\mu \le \frac{C}{\lambda}\int|f|v\,d\mu$$

if and only if for every interval I

$$\left(\frac{1}{|I|}\int_I w\,d\mu\right)\left(\frac{1}{|I|}\int_I v^{-1/(p-1)}\,d\mu\right)^{p-1} \le C.$$

3. Covering lemmas

Our proofs of the theorems are based on the following lemmas. Lemma 1 is the covering idea used in the proof of Theorem (a), p. 1231 of [8].

Lemma 1. *If ν is a Borel measure, E is a subset of R^1, B is a collection of intervals that cover E and there is a constant C such that $0 < \nu(I) \leq C$ for every I in B, then there is a disjoint sequence $\{I_j\}$ of intervals in B such that $\nu(E) \leq \sum 5\nu(I_j)$.*

To prove this each I_j is chosen to be disjoint from $I_1, \ldots, I_{j-1}$ and with $\nu(I_j) > (1/2)\sup \nu(I)$, where the sup is taken over all I's in B disjoint from $I_1, \ldots, I_{j-1}$. If $\sum \nu(I_j) = \infty$, there is nothing more to prove. Therefore, assume that $\sum \nu(I_j) < \infty$. For each I_j define I_j^* to be the union of all intervals H in B that intersect I_j and have $\nu(H) < 2\nu(I_j)$. Then $\nu(I_j^*) \leq 5\nu(I_j)$ is immediate. Furthermore, if x is in E, let J be a member of B that contains x. Since $\lim_{j\to\infty} \nu(I_j) = 0$, there is a first member I_j of the sequence that intersects J. Then by the selection procedure $\nu(I_j) > (1/2)\nu(J)$ and by its definition $J \subset I_j^*$. From this $E \subset \bigcup I_j^*$. Therefore, $\nu(E) \leq \sum \nu(I_j^*) \leq \sum 5\nu(I_j)$. This completes the proof of Lemma 1.

Lemma 2. *If E is a subset of R^1 and B is a collection of closed intervals of positive length that cover E, then E is covered by a countable subcollection of B.*

To prove this let A be the set of points in E that occur only as left ends of intervals in B. For every x in A there is an interval in B that contains it, and this interval contains no other points of A in its interior. Therefore, A is countable and is consequently covered by a countable subcollection from B. The same reasoning applies to the points that occur only as right end points. The other points of E are covered by the open interiors of the intervals in B, and there is a countable subcollection that covers them by Lindelöf's theorem.

4. Proof of Theorem 1

The proof that (1) implies (4) is trivial by taking $f(x) = v(x)^{-1/p}\chi_I(x)$.

For the proof that (4) implies (1) we need only consider the case $p = 2$ since $G_f(x)^p = G_{|f|^p}(x)$. It is enough to prove the result for bounded functions f with support E having finite μ measure and $\int |f|^2 v d\mu < \infty$. For these f's it is enough to consider v's with positive lower bound by the following reasoning. Given arbitrary $v(x)$, let $v_n(x)$ be the larger of $1/n$ and $v(x)$. Then v_n will still satisfy (4) with the same value of C_4. It will appear in the proof that the constant C_1 depends only on C_4. Therefore, (1) will be valid for all v_n's with fixed C_1, and the restriction on f insures that the right side of (1) converges properly as $n \to \infty$.

Having fixed such an f and v, we define a function F on intervals I in R^1 by

$$F(I) = \exp\left[\frac{1}{|I|}\int_I \log(|f(y)|)\, d\mu(y)\right]$$

for intervals with $0 < |I| < \infty$ for which the integral is defined and $F(I) = 0$ on other intervals. For each x for which $G_f(x) > 0$ we will now construct a closed interval I_x containing x such that $F(I_x) > G_f(x)/2$ and so that the end points of I_x are measurable functions of x.

To do this let E_k be the set of x for which $2^k < G_f(x) \leq 2^{k+1}$, and let C_k consist of all closed intervals I with $2^k < F(I) \leq 2^{k+1}$. Let A be the points a in E_k with $[a, a] \in C_k$. The set A is countable since if a is in A, $|[a, a]| > 0$ and a lies in the support of f which has been assumed to have finite measure. Therefore, there is a countable subcollection of C_k that covers A. If $x \in E_k \cap A^c$ there is an interval I containing x with$F(I) > 2^k$ and I must have positive length since $[x, x]$ is not in C_k. If one or both ends of I are open, a closed interval J of positive length can be found inside I that contains x and has $F(J) > 2^k$. Therefore, the intervals of positive length in C_k cover $E_k \cap A^c$, and by Lemma 2 there is a countable subcollection that covers this set. Let $\{J_n\}$ be a sequence of closed intervals in C_k that cover E_k. For each x in E_k define I_x to be the first J_n that contains x. Then for x in E_k we have $F(I_x) > 2^k \geq G_f(x)/2$ and the ends of I_x are measurable functions because the set where $I_x = J_n$ is the measurable set $J_n \cap E_k \cap \bigcap_{m=1}^{n-1} J_m^c$.

Then if B is the set where $G_f(x) > 0$,

$$\begin{aligned}
\int G_f(x)^2 w(x) d\mu(x) &< \int_B 4 \left[\exp\left(\frac{1}{|I_x|} \int_{I_x} \log(|f|)\, d\mu \right) \right]^2 w(x) d\mu(x) \\
&= 4 \int_B \left[\exp\left(\frac{1}{|I_x|} \int_{I_x} \log(|f| v^{1/2}) d\mu \right) \right]^2 \\
&\quad \times \exp\left(\frac{1}{|I_x|} \int_{I_x} \log\left(\frac{1}{v}\right) d\mu \right) w(x) d\mu(x).
\end{aligned} \tag{5}$$

Some justification is needed for this equality since in general an equality of the form

$$\exp\left[\int_I \log(g) d\mu\right] = \exp\left[\int_I \log(gh) d\mu\right] \exp\left[\int_I \log(1/h) d\mu\right]$$

can fail if the right side has the form $0 \cdot \infty$. This does not happen here because the restrictions on f and v insure that $\int_{I_x} \log(|f|)\, d\mu$, $\int_{I_x} \log(|f| v^{1/2}) d\mu$ and $\int_{I_x} \log(1/v) d\mu$ are all finite. By Jensen's inequality (5) is bounded by

$$4 \int_B \left[\frac{1}{|I_x|} \int_{I_x} |f| v^{1/2} d\mu \right]^2 \exp\left[\frac{1}{|I_x|} \int_{I_x} \log\left(\frac{1}{v}\right) d\mu \right] w(x) d\mu(x). \tag{6}$$

Define the measure U by $dU(x) = w(x) \exp(\frac{1}{|I_x|} \int_{I_x} \log(\frac{1}{v}) d\mu) d\mu(x)$, and let T be the linear operator defined by $Tg(x) = \frac{1}{|I_x|} \int_{I_x} g d\mu$. Then T is clearly bounded from $L^\infty(d\mu)$ to $L^\infty(dU)$. We shall show that T is bounded from $L^1(d\mu)$ to weak $L^1(dU)$. This will be sufficient since these imply that T is bounded from $L^2(d\mu)$ to $L^2(dU)$ and, therefore, that (6) equals

$$4 \int_B T(f v^{1/2})^2 dU \leq C \int |f v^{1/2}|^2 d\mu = C \int |f|^2 v d\mu.$$

To prove the weak type assertion for T fix g in $L^1(d\mu)$ and $\lambda > 0$, let $D = \{Tg(x) > \lambda\}$, and let $E = \bigcup_{x \in D} I_x$. Since the set of all I_x's is countable, E can be written as the countable union of its connected components $\{J_n\}$. Note that if x is in $D \cap J_n$, then $I_x \subset J_n$. We have, therefore,

$$\begin{aligned}
\int_D dU &= \sum_n \int_{D \cap J_n} \exp\left[\frac{1}{|I_x|} \int_{I_x} \log\left(\left| \frac{1}{v} \right| \right) d\mu \right] w(x)\, d\mu(x) \\
&\leq \sum_n \int_{J_n} \exp\left[\frac{1}{|I_x|} \int_{I_x} \log\left(\left| \frac{1}{v} \right| \chi_{J_n} \right) d\mu \right] w(x)\, d\mu(x).
\end{aligned}$$

By condition (4) this is bounded by

$$C_4 \sum_n |J_n|. \tag{7}$$

Now each J_n is the union of I_x's with x in D. From the fact that x is in D we have $0 < |I_x| < \frac{1}{\lambda}\int_{I_x} g\, d\mu \le \frac{1}{\lambda}\|g\|_1$. We can, therefore, apply Lemma 1 to each J_n. This will produce disjoint sequences of intervals $I_{n,k} \subset J_n$, and since the $I_{n,k}$'s are I_x's with $x \in D$,

$$\lambda < \frac{1}{|I_{n,k}|}\int_{I_{n,k}} |g|\, d\mu. \tag{8}$$

Furthermore, since the J_n's are disjoint, all the intervals $I_{n,k}$ are disjoint. Using these two facts and Lemma 1 shows that (7) has the bound

$$5C_4 \sum_{n,k} |I_{n,k}| < \sum_{n,k} \frac{5}{\lambda} C_4 \int_{I_{n,k}} |g| d\mu \le \frac{5}{\lambda} C_4 \|g\|_1.$$

This completes the proof of Theorem 1.

5. Proof of Theorem 2

The proof that (3) implies (2) is trivial by taking $f(x) = v(x)^{-1/p}\chi_I(x)$ and $\lambda = (1/2)\exp(\frac{1}{|I|}\int_I \log(v^{-1/p})\, d\mu)$.

For the proof that (2) implies (3) we need only consider the case $p = 1$. As in the case of the proof of Theorem 1 we need only consider functions f that are bounded with support of finite μ measure and $\int |f| v\, d\mu < \infty$ and weights v with a positive lower bound. Given such functions f and v and a $\lambda > 0$, define an operator M_f by

$$M_f(x) = \sup_I \frac{\int_I |f(y)| v(y)\, d\mu(y)}{\int_I w(t)\, d\mu(t)}$$

with the sup taken over all intervals I containing x with the quotient taken as 0 in ambiguous cases. By (2), Jensen's inequality and the restrictions on f and v

$$\begin{aligned} G_f(x) &= \sup_I \exp\left(\left[\frac{1}{|I|}\int_I \log(|f| v)\, d\mu\right]\left[\frac{1}{|I|}\int_I \log\left(\frac{1}{v}\right) d\mu\right]\right) \\ &\le \sup_I C_2 \frac{\frac{1}{|I|}\int_I |f| v\, d\mu}{\frac{1}{|I|}\int_I w\, d\mu} = C_2 M_f(x). \end{aligned} \tag{9}$$

Let E be the set where $M_f(x) > \lambda/C_2$, let D be the set where $M_f(x) = \infty$ and let C be the collection of intervals I such that $\int_I |f| v\, d\mu > 0$ and $\int_I w\, d\mu = 0$. Let A be the sets of points a such that $[a, a] \in C$. The assumption that $\int |f| v\, d\mu < \infty$ implies that A is countable. The set $D \cap A^c$ is covered by intervals in C with positive length. Therefore, by Lemma 2 and the countability of A, D has a countable covering $\{I_k\}$ of intervals in C. From this $\int_D w d\mu \le \sum_k \int_{I_k} w d\mu = 0$. This and (9) show that

$$\int_{\{G_f(x) > \lambda\}} w d\mu \le \int_E w\, d\mu = \int_{E \cap D^c} w\, d\mu. \tag{10}$$

Now let B consist of all intervals I with $\int_I |f| v\, d\mu > \frac{\lambda}{C_2}\int_I w d\mu > 0$. Then B covers $E \cap D^c$ and we can apply Lemma 1 with $d\nu = w d\mu$. The result is a disjoint sequence $\{J_n\}$ from B and

$$\int_{E\cap D^c} w\,d\mu \le 5\sum_n \int_{J_n} w\,d\mu < \frac{5C_2}{\lambda}\sum_n \int_{J_n} |f|v\,d\mu \le \frac{5C_2}{\lambda}\int |f|v\,d\mu. \tag{11}$$

Combining (10) and (11) completes the proof of Theorem 2.

6. An example

In this section we derive a pair (w, v) of functions that satisfy (2) but do not satisfy (1). It is also easily seen that they do not satisfy (4).

Let $d\mu(x) = dx$,

$$v(x) = \begin{cases} \exp\left[-\frac{1}{x(\log x)^2}\right], & x \in (0, e^{-2}], \\ \infty, & \text{elsewhere}, \end{cases}$$

and

$$w(x) = \chi_{(0,e^{-2}]}(x)\frac{d}{dx}\left[x\exp\left(\frac{1}{x\log x}\right)\right].$$

Note that

$$w(x) = \chi_{(0,e^{-2}]}(x)\left(1 - \frac{1}{x\log x} - \frac{1}{x(\log x)^2}\right)\exp\left(\frac{1}{x\log x}\right)$$

so that for x in $(0, e^{-2}]$

$$w(x) \le \frac{-2}{x\log x}\exp\left(\frac{1}{x\log x}\right) \tag{12}$$

and

$$w(x) \ge \frac{-1}{2x\log x}\exp\left(\frac{1}{x\log x}\right). \tag{13}$$

To prove (2) we need only consider intervals $I \subset [0, e^{-2}]$ since the second factor in (2) is 0 if $I \not\subset [0, e^{-2}]$. We will prove that (2) holds for $I = (a, b)$ in two cases: $a < 2b/3$ and $a \ge 2b/3$.

For $a < 2b/3$

$$\frac{1}{|I|}\int_I w(x)\,dx \le \frac{3}{b}\int_0^b w(x)dx = 3\exp\left(\frac{1}{b\log b}\right), \tag{14}$$

and since $\log(\frac{1}{v}) = \frac{1}{x(\log x)^2}$ is decreasing on $(0, e^{-2}]$,

$$\frac{1}{|I|}\int_I \log\left(\frac{1}{v(x)}\right)dx < \frac{1}{b}\int_0^b \log\left(\frac{1}{v(x)}\right)dx = \frac{-1}{b\log b}.$$

Therefore,

$$\exp\left[\frac{1}{|I|}\int_I \log\left(\frac{1}{v(x)}\right)dx\right] < \exp\left(\frac{-1}{b\log b}\right). \tag{15}$$

Combining (14) and (15) gives (2) with $C_2 = 3$ for this case.

For $a \ge 2b/3$ use (12) and the fact that the right side of (12) is increasing on $(0, e^{-2}]$ to see that

$$\frac{1}{|I|}\int_I w(x)dx < \frac{-2}{b\log b}\exp\left(\frac{1}{b\log b}\right). \tag{16}$$

Next, since $\frac{1}{x(\log x)^2}$ decreases on $(0, e^{-2}]$,

$$\frac{1}{|I|}\int_I \log\left(\frac{1}{v}\right)\,dx = \frac{1}{b-a}\int_a^b \frac{dx}{x(\log x)^2} < \frac{1}{a(\log a)^2}. \tag{17}$$

Now for $2b/3 \le a < b \le e^{-2}$

$$\frac{1}{a(\log a)^2} < \frac{-1}{2a\log a} < \frac{-1}{(4/3)b\log b}.$$

Using this, (16) and (17) gives

$$\left[\frac{1}{|I|}\int_I w(x)dx\right]\left[\exp\left(\frac{1}{|I|}\int_I \log\left(\frac{1}{v(x)}\right)dx\right)\right] < \frac{-2}{b\log b}\exp\left(\frac{1}{4b\log b}\right).$$

Since $8y\exp(-y) \le 8/e$ for y in $(0,\infty)$, the right side is bounded by $8/e$. This completes the proof of (2) with $C_2 = 3$ for this pair.

To show that (1) is false for this v and w take $f(x) = v(x)^{-1/p}\chi_{(0,e^{-2})}(x)$. The integral on the right side of (1) is e^{-2}. Now

$$G_f(x) \ge \exp\left(\frac{1}{x}\int_0^x \log f(t)\,dt\right) = \exp\left(\frac{-1}{px\log x}\right).$$

This and (13) show that the left side of (1) is bounded below by $\int_0^{e^{-2}} \frac{-dx}{2x\log x} = \infty$. This computation with $p = 1$ also shows directly that (4) fails for this pair.

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