

## ALMOST LINEARITY OF $\epsilon$ -BI-LIPSCHITZ MAPS BETWEEN REAL BANACH SPACES

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ABSTRACT. Let  $X$  and  $Y$  be real Banach spaces. A map  $f$  between  $X$  and  $Y$  is called an  $\epsilon$ -bi-Lipschitz map if  $(1 - \epsilon)\|x - y\| \leq \|f(x) - f(y)\| \leq (1 + \epsilon)\|x - y\|$  for all  $x, y \in X$ . In this note we show that if  $f$  is an  $\epsilon$ -bi-Lipschitz map with  $f(0) = 0$  from  $X$  onto  $Y$ , then  $f$  is almost linear. We also show that if  $f : X \rightarrow Y$  is a surjective  $\epsilon$ -bi-Lipschitz map with  $f(0) = 0$ , then there exists a linear isomorphism  $I : X \rightarrow Y$  such that

$$\|I(x) - f(x)\| \leq E(\epsilon, \alpha)(\|x\|^\alpha + \|x\|^{2-\alpha})$$

where  $E(\epsilon, \alpha) \rightarrow 0$  as  $\epsilon \rightarrow 0$  and  $0 < \alpha < 1$ .

### 1. INTRODUCTION

It is a well-known classical result of Mazur and Ulam [4] that an isometry  $f$  from a real Banach space  $X$  onto a real Banach space  $Y$  with  $f(0) = 0$  is automatically linear. A map  $f$  between Banach spaces  $X$  and  $Y$  is called an  $(m, M)$ -rigid map if

$$m\|x - y\| \leq \|f(x) - f(y)\| \leq M\|x - y\|$$

for all  $x, y \in X$ . We denote a  $(1 - \epsilon, 1 + \epsilon)$ -rigid map by  $\epsilon$ -bi-Lipschitz map. The following theorem follows from [1, Proposition 2].

**Theorem.** *Let  $f$  be an  $\epsilon$ -bi-Lipschitz map from a real Banach space  $X$  onto a real Banach space  $Y$ , with  $f(0) = 0$  and  $0 \leq \epsilon < \frac{1}{3}$ . Then*

$$\|f(x + y) - f(x) - f(y)\| \leq c_1(3\epsilon)^{c_2}(\|x\| + \|y\|), \text{ for } x, y \in X$$

where

$$c_1 = (8/3)((\alpha/8)(\alpha^2 + 4\alpha - 1)/(\alpha - 1) + 1)$$

and

$$c_2 = (\log(2/\alpha))(\log((\alpha + 7)/(\alpha - 1)) + \log(2/\alpha))^{-1}$$

for  $1 < \alpha < 2$ .

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Note that  $c_2$  attains its maximum for  $\alpha \in (1, 2)$  at  $\alpha = 1.1572\dots$  and for this value of  $\alpha$ ,  $c_1$  and  $c_2$  have values  $5.5704\dots$  and  $0.1216\dots$ , respectively.

Jarosz [2] conjectured that if  $f$  is an  $\epsilon$ -bi-Lipschitz map from a real Banach space  $X$  onto a real Banach space  $Y$ , then  $X$  and  $Y$  are linearly isomorphic for sufficiently small  $\epsilon$ . Throughout,  $R$  represents the real number field. There is an  $\epsilon$ -bi-Lipschitz map  $f$  on  $R$  which is not a linear map on  $R$ .

**Example.** We define an  $\epsilon$ -bi-Lipschitz map  $f : R \rightarrow R$  such that

$$f(x) = \begin{cases} (1 + \epsilon)x, & \text{if } x \leq 2, \\ (1 - \epsilon)(x - 2) + 2(1 + \epsilon), & \text{if } x > 2. \end{cases}$$

Throughout this paper,  $X$  and  $Y$  shall denote real Banach spaces. The  $r$ -neighborhood of a set  $A$  is denoted by  $B(A, r)$  and we abbreviate  $B(\{x\}, r)$  and  $B(\{0\}, r)$  by  $B(x, r)$  and  $B(r)$ , respectively. The corresponding closed balls are denoted by  $\overline{B}(x, r)$  and  $\overline{B}(r)$ . As usual,  $[x, y]$  denotes the closed segment determined by  $x$  and  $y$ . A bound subset  $A$  of  $X$  is said to be symmetric with respect to a point  $a$  if  $x$  in  $A$  implies that  $2a - x$  is also in  $A$ . If  $A$  is bounded and symmetric with respect to  $a$ , we define  $\text{rad}A$  to be  $\inf\{r > 0 : A \subset B(a, r)\}$ . We denote the Blaschke distance function on sets by  $D$ ; that is, for  $A_1, A_2 \subset X$ ,  $D(A_1, A_2) = \inf\{r > 0 : A_1 \subset B(A_2, r) \text{ and } A_2 \subset B(A_1, r)\}$ . For  $A \subset X$  and  $\alpha > 0$  we define  $T(A, \alpha) = \{x \in X : A \subset \overline{B}(x, \alpha)\} = \bigcap \{\overline{B}(y, \alpha) : y \in A\}$ . For  $x, y \in X$  we define  $S(x, y, \alpha) = T([x, y], \alpha) = T(\{x, y\}, \alpha) = \overline{B}(x, \alpha) \cap \overline{B}(y, \alpha)$ . We also define  $C(A, \alpha) = A \cap T(A, \alpha)$ . The following lemmas are due to Gevirtz [1] and John [3].

**Lemma 1** [1, Lemma 2]. *Let  $x, y \in X$  and let  $\beta, \gamma > |x - y|/2$ . Then*

$$D(S(x, y, \beta), S(x, y, \gamma)) \leq \frac{|\beta - \gamma|}{1 - |x - y|/2\beta}.$$

**Lemma 2** [1, Lemma 3]. *Let  $\delta > 0$ . For every bounded convex symmetric subset  $A \neq \emptyset$  of  $X$  with center  $a$  and all  $\beta, \gamma$  for which  $B(a, \delta) \subset C(A, \beta)$  and  $\gamma > \text{rad}A$  there holds*

$$D(C(A, \beta), C(A, \gamma)) \leq (1 + \frac{\text{rad}A}{\delta})|\beta - \gamma|.$$

**Lemma 3** [1, Lemma 4]. *Let  $\delta > 0$ . For every bounded convex symmetric subset  $A \neq \emptyset$  of  $X$  with center  $a$  and  $F \subset X$  and  $\beta$  for which  $C(F, \beta) \neq \emptyset$  and  $B(a, \delta) \subset C(A, \beta)$  there holds*

$$D(C(A, \beta), C(F, \beta)) \leq (1 + \frac{4\text{rad}A}{\delta})D(A, F).$$

**Lemma 4** [1, Lemma 5]. *Let  $A \subset U \subset X$  and let  $f : U \rightarrow Y$  be  $(m, M)$ -rigid. Then for all  $\alpha > 0$*

$$f(U \cap T(A, \alpha)) \subset f(U) \cap T(f(A), M\alpha) \subset f(U \cap T(A, \frac{M}{m}\alpha)).$$

**Lemma 5** [3, Theorem II]. *Let  $f : X \rightarrow Y$  be  $(m, M)$ -rigid. Then  $f(B(a, r)) \supset B(f(a), mr)$ .*

2. THE RESULTS

**Definition 6.** A map  $f$  from a real Banach space  $X$  into a real Banach space  $Y$  is an almost linear map if it satisfies

$$\|f(x + y) - f(x) - f(y)\| \leq D(\epsilon)(\|x\| + \|y\|), \text{ for } x, y \in X$$

and

$$\|f(\lambda x) - \lambda f(x)\| \leq D(\epsilon)E(\lambda)\|x\|, \text{ for } \lambda \in R, x \in X$$

where  $D(\epsilon) \rightarrow 0$  as  $\epsilon \rightarrow 0$  and  $E(\lambda) \rightarrow 0$  as  $\lambda \rightarrow 0$ .

Let  $x \neq y$  be points in  $X$  and let  $d = \|x - y\|$ . For  $\alpha > 1$  we define  $S_n = S_n(x, y, d, \alpha)$  recursively as follows :  $S_1 = S(x, y, \frac{\alpha d}{2})$  and, for  $n \geq 1$ ,  $S_{n+1} = C(S_n, \frac{\alpha d}{2^n} + \sum_{i=1}^n \frac{1}{2^i}(\alpha - 1)d)$ .

**Lemma 7.** Let  $x \neq y$  be points in  $X$ ,  $a = \frac{x+y}{2}$  and  $\alpha > 1$ . Then for  $n \geq 1$ ,  $S_n$  is convex and symmetric with respect to  $a$  and

$$B(a, \frac{\alpha - 1}{2}d) \subset S_n \subset B(a, \frac{\alpha d}{2^n} + \sum_{i=2}^n \frac{1}{2^i}(\alpha - 1)d).$$

*Proof.* It is easy to verify that the assertion is true for  $n = 1$ . Assume inductively that it is true for a given  $n \geq 1$ . The inductive hypothesis implies that  $S_{n+1}$  is convex and symmetric with respect to  $a$ . If  $z \in B(a, \frac{\alpha - 1}{2}d)$  and  $u \in S_n$ , then

$$\begin{aligned} \|z - u\| &\leq \|z - a\| + \|a - u\| \\ &< \frac{\alpha - 1}{2}d + \frac{\alpha}{2^n}d + \sum_{i=2}^n \frac{1}{2^i}(\alpha - 1)d \\ &= \frac{\alpha}{2^n}d + \sum_{i=1}^n \frac{1}{2^i}(\alpha - 1)d. \end{aligned}$$

This implies that  $z \in S_{n+1}$ . If  $u \in S_{n+1}$ , then  $\|u - z\| \leq \frac{\alpha d}{2^n} + \sum_{i=1}^n \frac{1}{2^i}(\alpha - 1)d$  for all  $z \in S_n$ . Since  $S_{n+1}$  is symmetric with respect to  $a$ ,  $2a - u \in S_{n+1} \subset S_n$ . Putting  $2a - u$  instead of  $z$  in the above formula, we have  $\|u - a\| \leq \frac{\alpha d}{2^{n+1}} + \sum_{i=2}^{n+1} \frac{1}{2^i}(\alpha - 1)d$ . That is,  $rad S_{n+1} \leq \frac{\alpha d}{2^{n+1}} + \sum_{i=2}^{n+1} \frac{1}{2^i}(\alpha - 1)d$ .

**Proposition 8.** Let  $x, y \in X$ ,  $\|x - y\| = d, \alpha > 1$  and  $0 < m \leq M$ . Let  $f : X \rightarrow Y$  be an  $(m, M)$ -rigid map. Writing  $S_n, S'_n$  and  $\mu$  for  $S_n(x, y, d, \alpha), S_n(f(x), f(y), Md, \alpha)$  and  $\frac{M}{m}$ , respectively, there holds

$$D(f(S_n), S'_n) \leq K(m, M, \alpha)\|x - y\|$$

where  $K(m, M, \alpha) = M(\mu - 1)(\frac{7\alpha - 1}{\alpha - 1})^{n-1}(\frac{31\alpha^3 - 18\alpha^2 + 10\alpha - 2}{(\alpha - 1)(6\alpha - 4)})$ .

*Proof.* Replacing  $U, A$  by  $X, \{x, y\}$ , respectively, in Lemma 4, we obtain

$$(1) \quad f(S_1) \subset S'_1 \subset f(S(x, y, \alpha\mu d/2)).$$

Since  $S_{n+1} = C(S_n, \frac{\alpha d}{2^n} + \sum_{i=1}^n \frac{1}{2^i}(\alpha - 1)d)$ , Lemma 4 with  $A = U = S_n$  implies that

$$(2) \quad \begin{aligned} f(S_{n+1}) &\subset C(f(S_n), Md[\frac{\alpha}{2^n} + \sum_{i=1}^n \frac{1}{2^i}(\alpha - 1)]) \\ &\subset f(C(S_n, \mu d[\frac{\alpha}{2^n} + \sum_{i=1}^n \frac{1}{2^i}(\alpha - 1)])). \end{aligned}$$

Formula (1) gives

$$\begin{aligned} D(f(S_1), S'_1) &\leq D(f(S(x, y, \frac{\alpha d}{2})), f(S(x, y, \frac{\alpha \mu d}{2}))) \\ &\leq MD(S(x, y, \frac{\alpha d}{2}), S(x, y, \frac{\alpha \mu d}{2})). \end{aligned}$$

Lemma 1 with  $\beta = \frac{\alpha d}{2}$  and  $\gamma = \frac{\alpha \mu d}{2}$  implies that we have  $D(f(S_1), S'_1) \leq M\alpha^2 d \frac{(\mu-1)}{2(\alpha-1)}$ . Put  $D_n = D(f(S_n), S'_n)$ . Let  $n \geq 1$  and

$$W = C(f(S_n), Md[\frac{\alpha}{2^n} + \sum_{i=1}^n \frac{1}{2^i}(\alpha - 1)]).$$

Then (2) implies  $W \neq \emptyset$ , and so

$$(3) \quad D_{n+1} \leq D(f(S_{n+1}), W) + D(S'_{n+1}, W).$$

Formula (2) gives

$$\begin{aligned} D(f(S_{n+1}), W) &\leq D(f(S_{n+1}), f(C(S_n, \mu d[\frac{\alpha}{2^n} + \sum_{i=1}^n \frac{1}{2^i}(\alpha - 1)]))) \\ &= D(f(C(S_n, d[\frac{\alpha}{2^n} + \sum_{i=1}^n \frac{1}{2^i}(\alpha - 1)])), \\ &\quad f(C(S_n, \mu d[\frac{\alpha}{2^n} + \sum_{i=1}^n \frac{1}{2^i}(\alpha - 1)]))). \end{aligned}$$

By Lemma 2 and Lemma 7,

$$(4) \quad \begin{aligned} D(f(S_{n+1}), W) &\leq d(\mu - 1)M \left( 1 + \frac{\alpha d/2^n + \sum_{i=2}^n (\alpha - 1)d/2^i}{(\alpha - 1)d/2} \right) \\ &\quad \times \left[ \frac{\alpha}{2^n} + \sum_{i=1}^n \frac{1}{2^i}(\alpha - 1) \right] \\ &\leq d(\mu - 1)M \left( 1 + \frac{\alpha}{\alpha - 1} \right) \left( \frac{\alpha}{2} + \frac{\alpha - 1}{2} \right) \\ &\leq d(\mu - 1)M \frac{(2\alpha - 1)^2}{2(\alpha - 1)}. \end{aligned}$$

Similarly, we may apply Lemma 3 with  $a = \frac{f(x)+f(y)}{2}$ ,  $A = S'_n$ ,  $F = f(S_n)$ ,  $\beta = M[\frac{\alpha d}{2^n} + \sum_{i=1}^n \frac{1}{2^i}(\alpha - 1)d]$  and  $\delta = M\frac{\alpha-1}{2}d$ .

$$\begin{aligned}
 D(S'_{n+1}, W) &= D(C(S'_n, Md[\frac{\alpha}{2^n} + \sum_{i=1}^n \frac{1}{2^i}(\alpha - 1)]), \\
 &\quad C(f(S_n), Md[\frac{\alpha}{2^n} + \sum_{i=1}^n \frac{1}{2^i}(\alpha - 1)])) \\
 (5) \quad &\leq \left(1 + \frac{4(\alpha d/2^n + \sum_{i=2}^n (\alpha - 1)d/2^i)}{(\alpha - 1)d/2}\right) D_n \\
 &\leq \left(\frac{5\alpha - 1}{\alpha - 1}\right) D_n.
 \end{aligned}$$

From (4) and (5) we get

$$D_{n+1} \leq d(\mu - 1)M\frac{(2\alpha - 1)^2}{2(\alpha - 1)} + \left(\frac{5\alpha - 1}{\alpha - 1}\right) D_n.$$

Let  $G = d(\mu - 1)M\frac{(2\alpha-1)^2}{2(\alpha-1)}$ ,  $H = \frac{5\alpha-1}{\alpha-1}$ . By induction we then get

$$\begin{aligned}
 D_n &\leq G(1 + H + \dots + H^{n-2}) + H^{n-1}D_1 \\
 &= G\left(\frac{H^{n-1} - 1}{H - 1}\right) + H^{n-1}D_1 \\
 &\leq (\mu - 1)M\left(\frac{5\alpha - 1}{\alpha - 1}\right)^{n-1} \left(\frac{8\alpha^3 - 8\alpha^2 + 5\alpha - 1}{8\alpha(\alpha - 1)}\right) \|x - y\|.
 \end{aligned}$$

**Proposition 9.** *Let  $x, y \in X$ ,  $0 < m \leq M$ ,  $\mu = \frac{M}{m} < 2$ . Let  $f : X \rightarrow Y$  be  $(m, M)$ -rigid. Then*

$$\left\| \frac{f(x) + f(y)}{2} - f\left(\frac{x + y}{2}\right) \right\| \leq c_1(\mu - 1)^{c_2} M \|x - y\|$$

where  $c_1 = \frac{3\alpha-1}{2} + \frac{(9\alpha-5)(17\alpha^3-25\alpha^2+16\alpha-4)}{8(2\alpha-1)(\alpha-1)^2}$  and  $c_2 = \frac{\log 2}{\log \frac{2(9\alpha-5)}{\alpha-1}}$  for  $\alpha > 1$ .

*Proof.* Put  $a = \frac{x+y}{2}$  and  $p = \frac{f(x)+f(y)}{2}$ . By Lemma 5 and Lemma 7, we obtain  $B(f(a), m\frac{\alpha-1}{2}d) \subset f(S_n)$  and  $S'_n \subset B(p, M\frac{\alpha d}{2^n} + M\frac{\alpha-1}{2}d)$ . Since  $D_n = D(f(S_n), S'_n)$ ,  $f(S_n) \subset B(S'_n, D_n)$ . Thus we have

$$\begin{aligned}
 B(f(a), m\frac{\alpha-1}{2}d) &\subset B(B(p, M\frac{\alpha d}{2^n} + M\frac{\alpha-1}{2}d), D_n) \\
 &= B(p, M\frac{\alpha d}{2^n} + M\frac{\alpha-1}{2}d + D_n).
 \end{aligned}$$

Hence Proposition 8 implies

$$\begin{aligned}
 (6) \quad \|f(a) - p\| &\leq (M - m)\frac{\alpha - 1}{2}d + M\frac{\alpha d}{2^n} \\
 &\quad + d(\mu - 1)M\left(\frac{5\alpha - 1}{\alpha - 1}\right)^{n-1} \frac{8\alpha^3 - 8\alpha^2 + 5\alpha - 1}{8\alpha(\alpha - 1)}.
 \end{aligned}$$

Let  $E = \frac{1}{2}$  and  $F = \frac{5\alpha-1}{\alpha-1}$ . For given  $\alpha, \mu$  we use an integer  $n$  which is chosen in such a way that the last expression takes the form  $c_1(\mu - 1)^{c_2}M\|x - y\|$ , apart from negligible differences. Explicitly, we write  $n$  in the form

$$n = -\beta(\log(\mu - 1))/\log F + \xi + 1$$

where  $0 < \beta < 1$  and  $0 \leq \xi < 1$ . Since  $\mu < 2$ , we have that  $n \geq 1$ ,  $(\mu - 1)F^{n-1} = (\mu - 1)^{1-\beta}F^\xi \leq (\mu - 1)^{1-\beta}F$  and  $E^n = (\mu - 1)^{-\beta \log \frac{1}{2} / \log F} E^\xi \leq (\mu - 1)^{\beta \log 2 / \log F}$ . If  $\beta$  is determined so that  $1 - \beta = \beta \log 2 / \log F$ , then  $\beta = \frac{\log F}{\log 2 + \log F}$ . Put  $c_2 = 1 - \beta$ . Then

$$\begin{aligned} dM[\alpha \left(\frac{1}{2}\right)^n + (\mu - 1) \left(\frac{5\alpha - 1}{\alpha - 1}\right)^{n-1} \frac{8\alpha^3 - 8\alpha^2 + 5\alpha - 1}{8\alpha(\alpha - 1)}] \\ \leq dM(\mu - 1)^{c_2} \left(\alpha + \frac{(5\alpha - 1)(8\alpha^3 - 8\alpha^2 + 5\alpha - 1)}{8\alpha(\alpha - 1)^2}\right). \end{aligned}$$

Since  $0 < c_2 < 1$  and  $\mu \leq 2$

$$(M - m) \frac{\alpha - 1}{2} d \leq M(\mu - 1)^{c_2} \frac{\alpha - 1}{2} d.$$

Thus

$$\|f(a) - p\| \leq M(\mu - 1)^{c_2} \left(\frac{3\alpha - 1}{2} + \frac{(5\alpha - 1)(8\alpha^3 - 8\alpha^2 + 5\alpha - 1)}{8\alpha(\alpha - 1)^2}\right) \|x - y\|.$$

*Remark.* It is easy to show that  $c_2$  converges to  $\log 2$  as  $\alpha$  converges to infinity. If we choose  $\alpha = 10$ , then  $c_1 = 6.9314\dots$  and  $c_2 = 0.2902\dots$ . We denote these values by  $K_1$  and  $K_2$ , respectively. The later value is greater than  $0.1216\dots$  which is the maximum value of  $c_2$  in [1, Proposition 2]. If  $\mu \geq 2$ , there exists  $\delta > 0$  such that  $\|\frac{1}{2}(f(x) + f(y)) - f(\frac{x+y}{2})\| \leq \delta\|x - y\|$ , for  $x, y \in X$ .

**Lemma 10.** *Let  $f : X \rightarrow Y$  be a continuous map such that  $\|\frac{1}{2}(f(x) + f(y)) - f(\frac{x+y}{2})\| \leq K\|x - y\|$ , for some  $K > 0$  and for all  $x, y \in X$ , with  $f(0) = 0$ . Then*

$$\|f(\lambda x) - \lambda f(x)\| \leq 2K\|x\|, \quad 0 \leq \lambda \leq 1, x \in X.$$

*Proof.* Let  $Q_n = \{\frac{p}{2^n} | p = 1, 2, \dots, 2^n\}$  and  $Q = \bigcup_{n=1}^\infty Q_n$ . Then  $Q$  is dense in  $\{\lambda | 0 \leq \lambda \leq 1\}$ . Since  $f$  is continuous, it is sufficient to show that  $\|\lambda f(x) - f(\lambda x)\| \leq \sum_{i=0}^{n-1} \frac{1}{2^i} K\|x\|$  for all  $x \in X, \lambda \in Q_n$ . We prove the above formula by induction. It is clearly true for  $n = 1$ , and we assume it is true if we have  $Q_{n-1}$ , for  $n \geq 2$ . We take  $\lambda \in Q_n, \lambda = \frac{p}{2^n}$ . If  $p$  is divided by 2, then  $\lambda \in Q_{n-1}$ . Otherwise there exists an integer  $r$  such that  $p = 2r + 1$  and so  $r \leq 2^{n-1} - 1$ . Thus  $\frac{r}{2^{n-1}}, \frac{r+1}{2^{n-1}} \in Q_{n-1}$ . Hence we obtain

$$\begin{aligned} \|f(\frac{p}{2^n}x) - \frac{p}{2^n}f(x)\| &\leq \frac{K}{2^{n-1}}\|x\| + \frac{1}{2}\|f(\frac{r}{2^{n-1}}x) - \frac{r}{2^{n-1}}f(x) \\ &\quad + f(\frac{r+1}{2^{n-1}}x) - \frac{r+1}{2^{n-1}}f(x)\| \\ &\leq \frac{K}{2^{n-1}}\|x\| + K \sum_{i=0}^{n-2} \frac{1}{2^i}\|x\| \\ &= K \sum_{i=0}^{n-1} \frac{1}{2^i}\|x\|. \end{aligned}$$

That is,  $\|f(\lambda x) - \lambda f(x)\| \leq 2K\|x\|$  for  $0 \leq \lambda \leq 1, x \in X$ .

**Theorem 11.** *Let  $f : X \rightarrow Y$  be an  $\epsilon$ -bi-Lipschitz map with  $f(0) = 0$ , and with  $\epsilon < \frac{1}{3}$ . Then  $f$  is an almost linear map.*

*Proof.* Let  $x, y \in X$ . By Proposition 9 and the above Remark

$$(7) \quad \left\| \frac{f(x) + f(y)}{2} - f\left(\frac{x+y}{2}\right) \right\| \leq \frac{4}{3} K_1 (3\epsilon)^{K_2} \|x - y\|.$$

We put  $C(\epsilon) = \frac{4}{3} K_1 (3\epsilon)^{K_2}$ . Then

$$\begin{aligned} \left\| \frac{1}{2}(f(x) + f(y)) - \frac{1}{2}f(x+y) \right\| &\leq \left\| \frac{1}{2}(f(x) + f(y)) - f\left(\frac{x+y}{2}\right) \right\| \\ &\quad + \left\| f\left(\frac{x+y}{2}\right) - \frac{1}{2}f(x+y) \right\| \\ &\leq 2C(\epsilon)(\|x\| + \|y\|). \end{aligned}$$

That is,

$$(8) \quad \|f(x) + f(y) - f(x+y)\| \leq 4C(\epsilon)(\|x\| + \|y\|).$$

(7) and Lemma 10 imply

$$(9) \quad \|\lambda f(x) - f(\lambda x)\| \leq 2C(\epsilon)\|x\|, \text{ for } 0 \leq \lambda \leq 1, x \in X.$$

For each  $x \in X$ , we define  $g_x(\lambda) = \|\lambda f(x) - f(\lambda x)\|$  on  $R$ . Then for  $\lambda_1, \lambda_2 \in R$

$$\begin{aligned} |g_x(\lambda_1) - g_x(\lambda_2)| &= | \|\lambda_1 f(x) - f(\lambda_1 x)\| - \|\lambda_2 f(x) - f(\lambda_2 x)\| | \\ &\leq \|\lambda_1 f(x) - \lambda_2 f(x)\| + \|f(\lambda_1 x) - f(\lambda_2 x)\| \\ &\leq 2|\lambda_1 - \lambda_2|(1 + \epsilon)\|x\|. \end{aligned}$$

Putting  $\lambda_2 = 0$ , we obtain

$$(10) \quad \|\lambda f(x) - f(\lambda x)\| \leq 2|\lambda|(1 + \epsilon)\|x\|.$$

We define  $g_1(\lambda) = 2\lambda(1 + \epsilon)\|x\|$  and  $g_2(\lambda) = 2C(\epsilon)\|x\|$ , for  $\lambda \geq 0$ . Then  $g_1$  and  $g_2$  have the common value at  $\lambda = \frac{C(\epsilon)}{1+\epsilon}$ . By simple calculation for all  $0 < \alpha < 1$ ,

$$\min\{g_1(\lambda), g_2(\lambda)\} \leq \frac{8}{3} C(\epsilon)^{1-\alpha} \lambda^\alpha \|x\|.$$

Then (9) and (10) imply

$$\|\lambda f(x) - f(\lambda x)\| \leq \frac{8}{3} C(\epsilon)^{1-\alpha} \lambda^\alpha \|x\|, \text{ for } 0 \leq \lambda \leq 1.$$

We replace  $x$  by  $\frac{1}{\lambda}x$  and multiply  $\frac{1}{\lambda}$  ( $0 < \lambda \leq 1$ ) in the above formula and we get

$$\left\| \frac{1}{\lambda} f(x) - f\left(\frac{1}{\lambda}x\right) \right\| \leq \frac{8}{3} C(\epsilon)^{1-\alpha} \left(\frac{1}{\lambda}\right)^{2-\alpha} \|x\|.$$

Thus

$$\|\lambda f(x) - f(\lambda x)\| \leq \frac{8}{3}C(\epsilon)^{1-\alpha}(\lambda^\alpha + \lambda^{2-\alpha})\|x\| \quad \text{for } \lambda \geq 0.$$

Using (8),

$$\|f(x) + f(-x)\| \leq 4C(\epsilon)\|x\|.$$

For  $\lambda < 0$ , put  $-\lambda = \beta$ . Then

$$\begin{aligned} \|\lambda f(x) - f(\lambda x)\| &= \|f(-\beta x) + \beta f(x)\| \\ &\leq 4C(\epsilon)\beta\|x\| + \frac{8}{3}C(\epsilon)^{1-\alpha}(\beta^\alpha + \beta^{2-\alpha})\|x\| \\ &\leq \frac{40}{3}(C(\epsilon) + C(\epsilon)^{1-\alpha})(|\lambda|^\alpha + |\lambda|^{2-\alpha})\|x\|. \end{aligned}$$

That is,

$$\|f(\lambda x) - \lambda f(x)\| \leq D(\epsilon, \alpha)(|\lambda|^\alpha + |\lambda|^{2-\alpha})\|x\|$$

where  $D(\epsilon, \alpha) = \frac{40}{3}(C(\epsilon) + C(\epsilon)^{1-\alpha})$ ,  $0 < \alpha < 1$ . This completes the proof of the theorem.

Let  $f$  be an  $\epsilon$ -bi-Lipschitz map from a finite dimensional Banach space  $X$  onto a finite dimensional real Banach space  $Y$  with  $f(0) = 0$ . Then there exists a linear map  $I$  near  $f$ .

**Theorem 12.** *Let  $X$  and  $Y$  be finite dimensional real Banach spaces. If  $f : X \rightarrow Y$  is a surjective  $\epsilon$ -bi-Lipschitz map with  $f(0) = 0$  and  $\epsilon < \frac{1}{3}$ , then there exists a linear isomorphism  $I : X \rightarrow Y$  such that*

$$\|I(x) - f(x)\| \leq E(\epsilon, \alpha)(\|x\|^\alpha + \|x\|^{2-\alpha})$$

where  $E(\epsilon, \alpha) \rightarrow 0$  as  $\epsilon \rightarrow 0$  and  $0 < \alpha < 1$ .

*Proof.* Since  $f$  is a homeomorphism from  $X$  onto  $Y$ , there exists a basis

$$\{e_1, e_2, \dots, e_n\}$$

for  $X$  such that  $\{f(e_1), f(e_2), \dots, f(e_n)\}$  are linearly independent in  $Y$  and  $\|e_i\| = 1$ ,  $i = 1, 2, \dots, n$ . We define a surjective linear map  $I : X \rightarrow Y$  by

$$I\left(\sum_{i=1}^n \alpha_i e_i\right) = \sum_{i=1}^n \alpha_i f(e_i), \quad \alpha_i \in R, i = 1, 2, \dots, n.$$

If  $x \in X$ , there exist  $\alpha_1, \alpha_2, \dots, \alpha_n$  in  $R$  such that  $x = \sum_{i=1}^n \alpha_i e_i$ . By Theorem 11,

$$\begin{aligned}
\|f(x) - I(x)\| &= \left\| f\left(\sum_{i=1}^n \alpha_i e_i\right) - \sum_{i=1}^n \alpha_i f(e_i) \right\| \\
&\leq \left\| f\left(\sum_{i=1}^n \alpha_i e_i\right) - f\left(\sum_{i=1}^{n-1} \alpha_i e_i\right) - f(\alpha_n e_n) \right\| \\
&\quad + \left\| f\left(\sum_{i=1}^{n-1} \alpha_i e_i\right) - f\left(\sum_{i=1}^{n-2} \alpha_i e_i\right) - f(\alpha_{n-1} e_{n-1}) \right\| \\
&\quad \dots \\
&\quad + \|f(\alpha_1 e_1 + \alpha_2 e_2) - f(\alpha_1 e_1) - f(\alpha_2 e_2)\| \\
&\quad + \|f(\alpha_1 e_1) - \alpha_1 f(e_1)\| + \|f(\alpha_2 e_2) - \alpha_2 f(e_2)\| + \dots \\
&\quad + \|f(\alpha_n e_n) - \alpha_n f(e_n)\| \\
&\leq (n-1)D(\epsilon, \alpha) \left(\sum_{i=1}^n |\alpha_i|\right) \\
&\quad + D(\epsilon, \alpha) \left(\sum_{i=1}^n (|\alpha_i|^\alpha + |\alpha_i|^{2-\alpha})\right) \\
&\leq n^2 D(\epsilon, \alpha) K (\|x\|^\alpha + \|x\|^{2-\alpha}), \text{ for some } K > 0.
\end{aligned}$$

Put  $E(\epsilon, \alpha) = n^2 D(\epsilon, \alpha) K$ . This completes the proof of the theorem.

#### REFERENCES

1. J. Gevirtz, *Injectivity in Banach spaces and the Mazur-Ulam Theorem on isometries*, Trans. Amer. Math. Soc. **274** (1982), 307-318. MR **84h**:46024
2. K. Jarosz, *Ultraproducts and small bound perturbations*, Pacific J. Math. **148** (1991), 81-88. MR **91m**:46022
3. F. John, *On quasi-isometric mappings*, I, Comm. Pure Appl. Math. **21** (1968), 77-110. MR **36**:5716
4. S. Mazur and S. Ulam, *Sur les transformations isométriques d'espaces vectoriels normés*, C.R. Acad. Sci. Paris Sér. **194** (1932), 946-948.

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