POLYNOMIAL RINGS
OVER GOLDIE-KERR COMMUTATIVE RINGS II

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In memory of Pere Menal

Abstract. An overlooked corollary to the main result of the stated paper (Proc. Amer. Math. Soc. 120 (1994), 989–993) is that any Goldie ring $R$ of Goldie dimension 1 has Artinian classical quotient ring $Q$, hence is a Kerr ring in the sense that the polynomial ring $R[X]$ satisfies the acc on annihilators (= acc⊥).

More generally, we show that a Goldie ring $R$ has Artinian $Q$ when every zero divisor of $R$ has essential annihilator (in this case $Q$ is a local ring; see Theorem 1').

A corollary to the proof is Theorem 2: A commutative ring $R$ has Artinian $Q$ iff $R$ is a Goldie ring in which each element of the Jacobson radical of $Q$ has essential annihilator.

Applying a theorem of Beck we show that any acc⊥ ring $R$ that has Noetherian local ring $R_p$ for each associated prime $P$ is a Kerr ring and has Kerr polynomial ring $R[X]$ (Theorem 5).

Introduction

Throughout, $R$ denotes a commutative ring.

It is convenient to state the corollary in generalized form as follows:

1. Theorem. If $R$ is a Goldie ring in which each zero divisor $x$ has essential annihilator $x^\perp$, then $R$ has Artinian quotient ring $Q$.

In this case, it follows from Small’s theorem [S] that $R[X]$ has Artinian quotient ring.

A ring $R$ has finite Goldie (or uniform) dimension $n$ if $n$ is the maximal number of nonzero ideals in a direct sum contained in $R$. Furthermore, $R$ is Goldie if $R$ has acc⊥ and finite Goldie dimension. Uniform ring is another term for a ring with Goldie dimension 1, equivalently, 0 is an irreducible ideal. The singular ideal of $R$ is the set

$$Z(R) = \{ x \in R \mid x^\perp \text{ is essential} \}.$$ 

Then $Z(R)$ is contained in the set $z(R)$ of zero divisors. Obviously, when $R$ is uniform, we have that $z(R) = Z(R)$, and then $Z(Q)$ is the set of non-units of $Q$.

We can sharpen Theorem 1 in this terminology:

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1'. Theorem. If $R$ is a Goldie ring, and if $z(R) \subseteq Z(R)$, then $Q$ is an Artinian local ring (hence $R$ is Kerr). Conversely.$^1$

Proof. $Q$ also is a Goldie ring (see [F3]), and $Z(Q)$ is a nil ideal in any acc⊥ ring (see, e.g., [F1]). Moreover, every element $x$ in the Jacobson radical $J(Q)$ is a nonunit; hence $x = rs^{-1}$, where $r \in z(R) \subseteq Z(R)$ and $s \in R^*$. Then, $x$ has essential right annihilator $x^+$ in $Q$, since $r^+ \cap Q$ is essential in $R$, a fact that shows that $J(Q) \subseteq Z(Q)$. Thus, $J(Q)$ is nil, so $Q$ is Artinian by Theorem 1.1 of [F2]. Since $J(Q)$ is the set of non-units of $Q$, it follows that $Q$ is local.

The converse hinges on the fact that if $Q$ is Artinian, then $Q$ is Noetherian and hence Goldie, so $R$ is Goldie. Furthermore, $J(Q)$ is nilpotent, and every nilpotent element $x$ has essential annihilator. (Let $I$ be any nonzero ideal, and $x^n = 0$ where $x^{n-1} \neq 0$. If $x^+ \cap I = 0$, then $xI \neq 0$. Suppose $i$ is least such that $x^i I \neq 0$. Then $x^+ \cap I \supseteq x^i I \neq 0$, a contradiction which shows that $x^+$ is an essential ideal.) Since $Z(R)$ is nilpotent in an acc⊥ ring (loc. cit.), then $J(Q) = Z(Q)$. Since $Q$ is local and $J(Q)$ is nilpotent, then every zero divisor $x$ of $R$ lies in $J(Q) \cap R = Z(Q) \cap R = Z(R)$, so $z(R) \subseteq Z(R)$ as needed. □

The proof of Theorem 1' has the corollary.

2. Theorem. A ring $R$ has Artinian $Q$ if $R$ is a Goldie ring and the Jacobson radical of $Q$ coincides with its singular ideal, that is, $J(Q) = Z(Q)$.

Proof. If $J(Q) = Z(Q)$, then $R$ acc⊥ ⇒ $J(Q)$ is nil, so $R$ Goldie ⇒ $Q$ is Artinian by Theorem 1.1 of [F2]. Conversely, $Q$ Artinian ⇒ $J(Q)$ is nil, hence $J(Q) \subseteq Z(Q)$. But $Z(Q)$ is nil in an acc⊥ ring, hence $Z(Q) = J(Q)$. □

In a uniform ring every nonzero ideal is essential, so the theorems each imply that any uniform acc⊥ ring $R$ has Artinian $Q$, but $Q$ is in fact then quasi-Frobenius since $Q$ has simple socle. With this fact as a motivator, we next derive a more general theorem with the same conclusion (Theorem 2).

A ring $R$ is $F$-injective (= $\aleph_0$-injective) if every map $I \to R$ of a finitely generated ideal $I$ is extendable to $R \to R$. (Any $FP$-injective ring $R$ is $F$-injective; cf. [F3], p. 189.) Any $F$-injective ring $R$ coincides with its quotient ring $Q$. Any valuation ring $R$ has $FP$-injective $Q$ by a theorem of Facchini ([F-P], p. 96, Corollary 6-10; cf. [F-F]).

3. Theorem. If $R$ is an acc⊥ ring with $F$-injective (e.g., $FP$ or self-injective) quotient ring $Q$, then $R$ is Kerr, in fact $Q$ is quasi-Frobenius (= $QF$).

Proof. Every finitely generated ideal $I$ of an $F$-injective ring $R$ is an annihilator (see, e.g., [F3], p. 189, Prop. 23.21.2). The acc⊥ in $R$ implies the acc⊥ in $Q$, and hence $Q$ satisfies the acc on finitely generated ideals, so $Q$ is Noetherian. But a Noetherian $F$-injective ring is self-injective, hence $QF$. □

4. Corollary. Any uniform acc⊥ ring $R$, e.g. any acc⊥ valuation ring, is a Kerr ring. Furthermore $Q$ is Artinian in fact $QF$.

Proof. $Q$ is Artinian by Theorem 1, and has Goldie dim = 1, hence has simple socle, which by classical ideal theory (cf. Corollary 2 of [F1]) implies that $Q$ is $QF$. □

$^1$Classically, it is known that a ring $R$ has local $Q$ iff the set $z(R)$ is an ideal $P$. In this case, $P$ is a prime ideal and $Q = R_P$ is the local ring at $P$. Any Artinian ring $R$ is a finite product of local Artinian rings. See Theorem 2 in this connection.
When is $R[X]$ Kerr?

We raised the question in [F2]: If $R$ is Kerr, is $R[X]$ Kerr? We cited some obvious examples in [F2], e.g. any subring of a Noetherian ring, and mentioned the Camillo-Guralnick theorem which yields an affirmative answer for an algebra over an uncountable field. We next show that Beck’s theorem [B] yields another affirmative answer.

5. Theorem. If $R$ is an acc ring and if $R_p$ is Noetherian for every associated prime $P$, then the same is true of $R[X]$. Furthermore, both $R$ and $R[X]$ have (flat) embeddings into Noetherian rings, hence each is a Kerr ring.

Proof. In any acc ring $R$, the set $\text{Ass}R$ of associated prime ideals is finite (see [F4], Corollary 3.7 and Theorem 3.6), and obviously $\bigcup_{P \in \text{Ass}R} P$ is the set $z(R)$ of zero divisors (i.e., every $x \in z(R)$ is contained in some $P \in \text{Ass}R$). We can now apply Beck’s theorem ([B], Theorem 5.1) to conclude that $R$ has a flat embedding in a Noetherian ring $T$, and hence both $R$ and $R[X]$ are Kerr, since $R[X]$ is contained in a Noetherian ring $T[X]$.

Next, contraction induces a 1-1 correspondence $\text{Ass}R[X] \rightarrow \text{Ass}R$ under various conditions including acc in $R$ ([F4], Theorem 3.12 B; any acc ring is trivially a zip ring in the terminology employed there).

Thus, if $P \in \text{Ass}R[X]$, then

$$P_0 = P \cap R \in \text{Ass}R$$

and

$$R[X]_P = R_{P_0}[X]_{PR_0[X]} = R_M[X]_{PR_M[X]}$$

which holds for any prime ideal $P$ of $R[X]$, and $M$ any maximal ideal containing $P_0 = P \cap R$. (See, e.g., [H], p.73, Lemma 13.1.)

In particular, this shows that

$$R_{P_0} \text{ Noetherian } \Rightarrow R[X]_P \text{ Noetherian,}$$

so $R[X]$ has the stated property (this also follows from the theorem of Beck since $R[X] \hookrightarrow T[X]$ is a flat embedding in a Noetherian ring). \qed

Since a Kerr ring need not embed in a Noetherian ring, this shows that a Kerr ring $R$ does not in general localize to Noetherian rings at associated primes.

6. Corollary. If $R$ satisfies the hypothesis of Theorem 5, then so does the infinite polynomial ring $S = R[x_1, \ldots, x_n, \ldots]$, that is, $S$ has a flat embedding in a Noetherian ring, hence is Kerr.

Proof. By Theorem 5, $R$ has a flat embedding in a Noetherian ring $T$, and by a remark of D. D. Anderson (cited in [C], p. 75, Remark 2), $A = T[x_1, \ldots, x_n, \ldots]$ localized at the ideal $P$ consisting of all polynomials in $A$ of content 1 is a Noetherian ring. Thus, since $A \hookrightarrow A_P$ is a flat embedding in a Noetherian ring, $S \hookrightarrow A_P$ is also. \qed

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2In the meanwhile Cedó and Herbera have found a ring $R$ over which the polynomial ring in $n$ variables is Kerr but that in $n + 1$ variables is not (Fax of November 1994). See [C.H].
Note added in proof

The author has discovered an error in the proof of Theorem 2.2 in [F2]. The first sentence should read:

“If $I$ is an ideal of $R$, then $I$ is an annihilator of $R$ iff $IQ$ is an annihilator of $Q$ and $IQ \cap R = I$.”

The third sentence should read:

“This also implies that if $K$ is an ideal of $Q$, then $K \cap R \in \text{Ass } R$ iff $K \in \text{Ass } Q$.”

References

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