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ASYMPTOTIC BEHAVIOR OF NONEXPANSIVE SEQUENCES AND MEAN POINTS

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ABSTRACT. Let E be a real Banach space with norm $\|\cdot\|$ and let $\{x_n\}_{n\geq 0}$ be a nonexpansive sequence in E (i.e., $\|x_{i+1}-x_{j+1}\| \leq \|x_i - x_j\|$ for all $i,\ j \geq 0$). Let $K = \bigcap_{n=1}^{\infty} \overline{co}\{\{x_i - x_{i-1}\}_{i\geq n}\}$. We deal with the mean point of $\{\frac{x_n}{n}\}$ concerning a Banach limit. We show that if E is reflexive and $d = d(0, K)$, then $d = d(0, \overline{co}\{\frac{x_n - x_0}{n}\})$ and there exists a unique point z_0 with $\|z_0\| = d$ such that $z_0 \in \overline{co}\{\frac{x_n - x_0}{n}\}$. This result is applied to obtain the weak and strong convergence of $\{\frac{x_n}{n}\}$.

1. Introduction

Let E be a real Banach space with norm $\|\cdot\|$ and let $\{x_n\}_{n\geq 0}$ be a nonexpansive sequence in E (i.e., $\|x_{i+1}-x_{j+1}\| \leq \|x_i - x_j\|$ for all $i,\ j \geq 0$). Recently, Djafari Rouhani [2] obtained an interesting result on the weak convergence of $\{\frac{x_n}{n}\}$ under the assumption that E is reflexive and strictly convex.

In this paper, we deal with his result without the assumption of strict convexity of E. That is, instead of the weak limit of $\{\frac{x_n}{n}\}$, we deal with the mean point of $\{\frac{x_n}{n}\}$ concerning a Banach limit under the assumption that E is reflexive. Using the mean point, we obtain the weak convergence of $\{\frac{x_n}{n}\}$, in which case E is reflexive and strictly convex. Also we obtain the strong convergence of $\{\frac{x_n}{n}\}$, in which case E^* has a Fréchet differentiable norm. Our results improve and extend the corresponding results in [5, 6, 7, 8, 9, 10] as in [2].

2. Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and dual $(E^*, \|\cdot\|)$. The duality pairing between E and E^* will be denoted by $(\cdot,\cdot)$. The duality mapping J from E into the family of nonempty closed convex subsets of E^* is defined by

$$J(x) = \{x^* \in E^* : (x, x^*) = \|x\|^2 = \|x^*\|^2\}.$$

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Note that we have for $x,\ y \in E$ and $j \in J(x)$,

$$(x-y,j) = \|x\|^2 - (y,j) \geq \|x\|^2 - \frac{1}{2}(\|y\|^2 + \|j\|^2) = \frac{1}{2}(\|x\|^2 - \|y\|^2).$$

We recall that if E is reflexive and strictly convex and K is a nonempty closed convex subset of E, the nearest point projection mapping P_K of E onto K is well defined, i.e., K is a Chebyshev set (see [1, 4]).

We say that the sequence $\{x_n\}_{n\geq 0}$ is nonexpansive if $\|x_{i+1} - x_{j+1}\| \leq \|x_i - x_j\|$ for all $i,\ j \geq 0$.

Let μ be a mean on integers N, i.e., a continuous linear functional on ℓ^∞ satisfying $\|\mu\| = 1 = \mu(1)$. Then we know that μ is a mean on N if and only if

$$\inf\{a_n : n \in \mathrm{N}\} \leq \mu(a) \leq \sup\{a_n : n \in \mathrm{N}\}$$

for every $a = (a_1, a_2, \cdots) \in \ell^\infty$. According to time and circumstances, we use $\mu_n(a_n)$ instead of $\mu(a)$. A mean μ on N is called a Banach limit if

$$\mu_n(a_n) = \mu_n(a_{n+1})$$

for every $a = (a_1, a_2, \cdots) \in \ell^\infty$. Using the Hahn–Banach theorem, we can prove the existence of a Banach limit. We know that if μ is a Banach limit, then

$$\liminf_{n\to\infty} a_n \leq \mu_n(a_n) \leq \limsup_{n\to\infty} a_n$$

for every $a = (a_1, a_2, \cdots) \in \ell^\infty$.

Let E be a reflexive Banach space and let $\{x_n\}$ be a bounded sequence in E. Then, for a Banach limit μ, we can obtain a point x_0 in E such that

$$\mu_n(x_n, x^*) = (x_0, x^*)$$

for all $x^* \in E^*$. In fact, the function $\mu_n(x_n, x^*)$ is linear in x^*. Further, since

$$|\mu_n(x_n, x^*)| \leq (\sup_n \|x_n\|) \cdot \|x^*\|,$$

the function $\mu_n(x_n, x^*)$ is also bounded in x^*. So, we have $x_0^{**} \in E^{**}$ such that $\mu_n(x_n, x^*) = (x_0^{**}, x^*)$ for every $x^* \in E^*$. Since E is reflexive, we obtain $x_0 \in E$ such that $\mu_n(x_n, x^*) = (x_0, x^*)$ for all $x^* \in E^*$. This point x_0 is called a mean point of $\{x_n\}$ concerning μ. We also know that this mean point x_0 is contained in $\bigcap_{n\geq 1}^\infty \overline{co}\{x_n\}$. In fact, if not, there exists $n_0 \in \mathrm{N}$ such that $x_0 \notin \overline{co}\{x_n : n \geq n_0\}$. By separation theorem, we obtain a point $x^* \in E^*$ such that

$$(x_0, x^*) < \inf\{(z, x^*) : z \in \overline{co}\{x_n : n \geq n_0\}\}.$$

So, we have

$$\begin{aligned}\mu_n(x_n, x^*) = (x_0, x^*) &< \inf\{(x_n, x^*) : n \geq n_0\} \\ &\leq \mu_n\{(x_n, x^*) : n \geq n_0\} = \mu_n(x_n, x^*).\end{aligned}$$

This is a contradiction. For these facts, see [11].

Let $S = \{x \in E : \|x\| = 1\}$. Then the norm of E is Fréchet differentiable if for each $x \in S$, the limit

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t}$$

exists uniformly for each $y \in S$. The following lemma is well known (cf. [3]).

Lemma 2.1. *E^* has a Fréchet differentiable norm if and only if E is reflexive and strictly convex, and has the following property : if the weak $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} \|x_n\| = \|x\|$ for a sequence $\{x_n\}$ in E, then $\{x_n\}$ converges strongly to x.*

Finally, let D be a subset of E. Then we denote the closure of D by $\overline{D}$ and the closed convex hull of D by $\overline{co}D$, respectively. We also denote its distance from a point x in E by $d(x, D) = \inf_{y\in D} \|x - y\|$.

3. Main result

In this section, for a nonexpansive sequence $\{x_n\}$ in E, we study the mean point of $\{\frac{x_n}{n}\}$ concerning a Banach limit.

We begin with the known result which will play a crucial role in our result.

Lemma 3.1 [2]**.** *Let E be a Banach space and let $\{x_n\}$ be a nonexpansive sequence in E. Then*

$$\lim_{n\to\infty} \|\frac{x_n}{n}\| \quad \textit{exists and} \lim_{n\to\infty} \|\frac{x_n}{n}\| = \inf_{n\geq 1} \|\frac{x_n - x_0}{n}\|.$$

The following result is essentially in spirit of Djafari Rouhani [2]. For completeness, we give the proof.

Lemma 3.2. *Let E be a reflexive Banach space and let $\{x_n\}$ be a nonexpansive sequence in E. Let*

$$K = \bigcap_{n=1}^{\infty} \overline{co}\{\{x_i - x_{i-1}\}_{i\geq n}\}.$$

Then $\lim_{n\to\infty} \|\frac{x_n}{n}\| = d(0, K) = \inf_{n\geq 1} \|\frac{x_n - x_0}{n}\|$.

Proof. Let $k \geq 1$ be fixed and $j_n \in J(x_n - x_{k-1})$ for $n \geq k$. Then we have for $n \geq k \geq 1$

$$\begin{aligned}(x_k - x_{k-1}, j_n) &\geq \frac{1}{2}\|x_n - x_{k-1}\|^2 - \frac{1}{2}\|x_n - x_k\|^2 \\ &\geq \frac{1}{2}\|x_n - x_{k-1}\|^2 - \frac{1}{2}\|x_{n-1} - x_{k-1}\|^2.\end{aligned}$$

Hence we obtain

$$\frac{2}{n^2}(x_k - x_{k-1}, \sum_{i=k}^{n} j_i) \geq \|\frac{x_n - x_{k-1}}{n}\|^2. \tag{3.1}$$

Let $S_n = \frac{2}{n^2}\sum_{i=k}^{n} j_i$ for $n \geq k$. Then we have

$$\|S_n\| \leq \frac{2}{n^2}\sum_{i=k}^{n} \|x_i - x_{k-1}\| = \frac{2}{n^2}\sum_{i=k}^{n} i\|\frac{x_i - x_{k-1}}{i}\|$$

and so $\{S_n\}$ is bounded since $\{\frac{x_n}{n}\}$ is bounded by Lemma 3.1. Thus from the weak–star compactness of the closed unit ball of E^*, it follows that the sequence

$\{S_n\}$ has a weak–star cluster point $j \in E^*$ (obviously independent of $k \geq 1$). Now by Lemma 3.1 and (3.1), we obtain

$$(x_k - x_{k-1}, j) \geq \lim_{n\to\infty} \|\frac{x_n}{n}\|^2$$

for $k \geq 1$ and hence

$$(\frac{x_n - x_0}{n}, j) \geq \lim_{n\to\infty} \|\frac{x_n}{n}\|^2 \tag{3.2}$$

for all $n \geq 1$. We also have

$$\begin{aligned}\|j\| &\leq \liminf_{n\to\infty} \|S_n\| \\ &\leq \limsup_{n\to\infty} \frac{2}{n^2}\sum_{i=k}^{n} i\|\frac{x_i - x_{k-1}}{i}\| = \lim_{n\to\infty}\|\frac{x_n}{n}\|\end{aligned}$$

and so

$$(x_k - x_{k-1}, j) \geq \lim_{n\to\infty} \|\frac{x_n}{n}\|^2 \geq \|j\|^2$$

for all $k \geq 1$. So, for any $z \in \overline{co}\{\{x_{i+1} - x_i\}_{i\geq 0}\}$,

$$\begin{aligned}\frac{1}{2}\lim_{n\to\infty}\|\frac{x_n}{n}\|^2 + \frac{1}{2}\|z\|^2 &\geq \frac{1}{2}\|j\|^2 + \frac{1}{2}\|z\|^2 \\ &\geq (z, j) \geq \lim_{n\to\infty}\|\frac{x_n}{n}\|^2 \geq \|j\|^2.\end{aligned} \tag{3.3}$$

Since $K \subset \overline{co}\{\{x_{i+1} - x_i\}_{i\geq 0}\}$, it follows from (3.3) that

$$\|j\| \leq \lim_{n\to\infty}\|\frac{x_n}{n}\| \leq \inf_{z\in K}\|z\| = d(0, K).$$

On the other hand, since $\{\frac{x_n}{n}\}$ is bounded and E is reflexive, $\{\frac{x_n - x_0}{n}\}$ contains a weakly convergent subsequence $\{\frac{x_{n_l} - x_0}{n_l}\}$. Let $\{\frac{x_{n_l} - x_0}{n_l}\}$ converge weakly to $q \in K$. Then we have

$$\|q\| \leq \liminf_{l\to\infty}\|\frac{x_{n_l} - x_0}{n_l}\| = \lim_{n\to\infty}\|\frac{x_n}{n}\|$$

and hence $\lim_{n\to\infty}\|\frac{x_n}{n}\| = d(0, K)$. This completes the proof.

Now, using Lemmas 3.1 and 3.2, we obtain the main result.

Theorem 3.3. *Let E be a reflexive Banach space and let $\{x_n\}$ be a nonexpansive sequence in E. Let*

$$K = \bigcap_{n=1}^{\infty} \overline{co}\{\{x_i - x_{i-1}\}_{i\geq n}\}$$

and $d = d(0, K)$. Then $d = d(0, \overline{co}\{\frac{x_n - x_0}{n}\})$ and there exists a point z_0 with $\|z_0\| = d$ such that $z_0 \in \overline{co}\{\frac{x_n - x_0}{n}\}$.

Proof. Since $\lim_{n\to\infty}\|\frac{x_n}{n}\| = \lim_{n\to\infty}\|\frac{x_n - x_0}{n}\| = d$ by Lemma 3.2, we may assume that $\{\frac{x_n - x_0}{n}\}$ is bounded. So, it follows from reflexivity of E that for a Banach limit μ, there exists $z_0 \in \overline{co}\{\frac{x_n - x_0}{n}\}$ such that

$$\mu_n(\frac{x_n - x_0}{n}, x^*) = (z_0, x^*) \tag{3.4}$$

for every $x^* \in E^*$. For $j_0 \in J(z_0)$, where J is the duality mapping of E, we have

$$\begin{aligned}\|z_0\|^2 = (z_0, j_0) &= \mu_n(\frac{x_n - x_0}{n}, j_0) \\ &\le \mu_n(\|\frac{x_n - x_0}{n}\|) \cdot \|j_0\| = d \cdot \|j_0\| = d \cdot \|z_0\|,\end{aligned}$$

and hence $\|z_0\| \le d$. On the other hand, by the proof of Lemma 3.2 and (3.2), there exists a functional $j \in E^*$ with $\|j\| \le d$ such that

$$(\frac{x_n - x_0}{n}, j) \ge d^2 \tag{3.5}$$

for all $n \ge 1$. Hence we have $(z_0, j) \ge d^2$. Since $\|j\| \le d$, we obtain

$$d^2 \ge \|z_0\| \cdot \|j\| \ge (z_0, j) \ge d^2$$

and hence $\|z_0\| = \|j\| = d$. It also follows from (3.5) that $(z, j) \ge d^2$ for every $z \in \overline{co}\{\frac{x_n - x_0}{n}\}$ and so

$$\|z\| \cdot d = \|z\| \cdot \|j\| \ge (z, j) \ge d^2.$$

Hence we have $\|z\| \ge d$ for every $z \in \overline{co}\{\frac{x_n - x_0}{n}\}$. Then we obtain

$$d = d(0, \overline{co}\{\frac{x_n - x_0}{n}\}).$$

Let w_0 be another point satisfying (3.4). Then for $j \in J(z_0 - w_0)$, we have

$$\|z_0 - w_0\|^2 = (z_0 - w_0, j) = \mu_n(\frac{x_n - x_0}{n} - \frac{x_n - x_0}{n}, j) = 0,$$

and hence $z_0 = w_0$. This completes the proof.

Corollary 3.4. *With the same assumptions as in Theorem* 3.3*, we have the following :*

(i) *If E is strictly convex, then the weak* $\lim_{n\to\infty} \frac{x_n}{n}$ *exists and coincides with* $P_K 0$ *with* $\|P_K 0\| = d$.

(ii) *If E^* has a Fréchet differentiable norm, then the strong* $\lim_{n\to\infty} \frac{x_n}{n}$ *exists and coincides with* $P_K 0$.

Proof. (i) Since a reflexive Banach space E is strictly convex, the set

$$\{z \in \overline{co}\{\frac{x_n - x_0}{n}\} : \|z\| = d\}$$

consists of exactly one point and $d(0, K) = \|P_K 0\|$. This point equals z_0 in Theorem 3.3. Let $\{\frac{x_{n_l}}{n_l}\}$ be a subsequence of $\{\frac{x_n}{n}\}$ such that $\{\frac{x_{n_l}}{n_l}\}$ converges weakly to $p \in K$. Then since

$$\|p\| \le \liminf_l \|\frac{x_{n_l}}{n_l}\| = \lim_{n\to\infty} \|\frac{x_n}{n}\| = \|P_K 0\|,$$

we have $p = z_0 = P_K 0$. This implies that $\{\frac{x_n}{n}\}$ converges weakly to $P_K 0$, which completes the proof.

(ii) It follows from Lemma 2.1 that $\{\frac{x_n}{n}\}$ converges strongly to $P_K 0$.

Remark. (1) Since our study is equivalent to the study of the asymptotic behavior of the sequence $\{\frac{T^n x}{n}\}$ in E, where T is a nonexpansive mapping from an arbitrary subset C of E into itself and $x \in C$, Theorem 3.3 is an improvement of Theorem 5 in [11].

(2) Corollary 3.4 is of interest in view of using the mean point. Compare this with Corollary 3.2 in [2].

(3) Corollary 3.4 also contains the previous corresponding results in [5, 6, 7, 8, 9, 10].

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