PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 124, Number 2, February 1996

EXTENSION OF ORDERINGS ON $*$-FIELDS

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(Communicated by Lance W. Small)

ABSTRACT. An analysis is made of the ways in which a total ordering of the set of symmetric elements of a skew field with involution can be extended to an ordering of a larger set of elements. This is done for several different types of orderings found in the literature.

1. Introduction

A $*$-field is a skew field D with an involution $*$ (anti-automorphism of order 2). Several different ways have been proposed for ordering subsets of nonskew elements in such a way that results from the theory of formally real fields can be extended to this setting. Typically, such orderings either give a total ordering of the symmetric elements of D (i.e., the set $S(D) = \{ d \in D \mid d^* = d \}$) or of a larger set, though it can never include skew elements ($Sk(D) = \{ d \in D \mid d^* = -d \}$). There have been very few results concerning the ways in which an ordering of the symmetric elements might extend to a corresponding type of ordering of a larger set (see [Ch3], [I]). In this paper we give a complete characterization of extensions in the case where the ordering is multiplicative. We demonstrate the basic situation in the case of various types of semiorderings (where the set is not multiplicative); in particular, we answer the open questions of Chacron [Ch3] regarding extension of c-orderings. The importance of the extensions of orderings to larger subsets of the $*$-field has primarily been that it facilitates proofs of the major results, though for c-orderings these have been the most studied. It is the symmetric form of orderings that corresponds best to what happens in formally real fields.

We shall begin by defining the orderings and semiorderings in which we shall be interested. For a more inclusive survey of these concepts, see [Cr4]. For the most part, we shall follow the terminology and notation of [Cr4]. In particular, for any subset $A \subseteq D$, we shall use the notation $A^\times$ to denote the set of nonzero elements in A. We write multiplicative commutators in the form $[a, b] = aba^{-1}b^{-1}$, and, more generally, for subsets $A,\ B \subseteq D^\times$, we write $[A, B] = \{ [a, b] \mid a \in A,\ b \in B \}$. We shall write $\prod_{nr}$ for the subgroup of all products of norms of $D^\times$.

This subject began with a definition given by Reinhold Baer of a semiordering which now bears his name.

Received by the editors July 1, 1993 and, in revised form, September 7, 1994.
1991 *Mathematics Subject Classification.* Primary 12E15, 12J15; Secondary 16K40.
Key words and phrases. Baer ordering, semiordering, $*$-ordering, valuation.

Definition 1.1. A *Baer ordering* on $(D,*)$ is a subset $P \subseteq S(D)^\times$ satisfying $P+P \subseteq P$, $dPd^* \subseteq P$ for all $d \in D^\times$, $1 \in P$, and $P \cup -P = S(D)^\times$.

These have been particularly studied by Holland [H1]. Baer orderings need not be restricted to the symmetric elements. We shall use the term *extended Baer ordering* to refer to any $*$-closed subset $P \subset D^\times$ satisfying $P+P \subseteq P$, $dPd^* \subseteq P$ for all $d \in D^\times$, $1 \in P$, and $P \cup -P \supseteq S(D)^\times$.

An alternative definition (equivalent for commutative fields) was given by Chacron in [Ch1] (see also [Ch2], [Ch3]).

Definition 1.2. A *c-ordering* on $(D,*)$ is a $*$-closed subset $P \subseteq D^\times$ satisfying $P+P \subseteq P$, $Pdd^* \subseteq P$ for all $d \in D^\times$, $1 \in P$, and $P \cup -P \supseteq S(D)^\times$.

The definition of symmetric c-ordering given in [Ch3] appears to give more general sets than are obtained by restricting c-orderings to the symmetric elements in the $*$-field. This is because Chacron only demands that the multiplicative condition in the next definition work when x is a norm, which does not seem to force the condition for products of norms (as happens for c-orderings). Consequently, we have slightly modified Chacron's definition and we answer the questions raised in [Ch3] in this context. Note that it is clear that, with the definition below, the intersection of a c-ordering with $S(D)$ is a symmetric c-ordering.

Definition 1.3. A *symmetric c-ordering* is a subset $P \subset S(D)^\times$ satisfying $1 \in P$, $P+P \subseteq P$, $P \cup -P = S(D)^\times$ and, for all $p \in P$, $x \in \prod_{nr}$, $px + x^*p \in P$.

Multiplicatively closed orderings were first introduced for $*$-fields in [H2] and the symmetric version of these orderings was introduced in [I]. Using different terminology we give equivalent definitions here.

Definition 1.4. An *extended $*$-ordering* on $(D,*)$ is a $*$-closed subset $P \subseteq D^\times$ satisfying $P+P \subseteq P$, $dPd^* \subseteq P$ for all $d \in D^\times$, $1 \in P$, $P \cup -P \supseteq S(D)^\times$, and $P \cdot P \subseteq P$. Equivalently, it is an extended Baer ordering closed under multiplication. A *$*$-ordering* on $(D,*)$ is a Baer ordering P satisfying $xy + yx \in P$ for all $x,\ y \in P$.

Much of the commutative theory of formally real fields can be extended to $*$-orderings. To do so requires a stronger form of semiordering introduced in [Cr2] and [Cr3]. To describe these, we use the notation $\Sigma(D)$ for the set of all sums of products of norms and elements of $[D^\times, S(D)^\times]$. This is precisely the intersection of all extended $*$-orderings [H2].

Definition 1.5. A *$*$-semiordering* on $(D,*)$ is a Baer ordering P satisfying $xy + y^*x \in P$ for all $x \in P$ and $y \in \Sigma(D)$. An *extended $*$-semiordering* on $(D,*)$ is an extended Baer ordering P satisfying $P \cdot \Sigma(D) \subseteq P$ and $P \cup -P$ contains all products of symmetric elements.

2. The valuation characterization of extended $*$-orderings

We shall write our valuations additively, with $v(0) = \infty$ being the maximum element of $\Gamma_v^+ = \{\, \gamma \in \Gamma_v \mid \gamma > 0 \,\} \cup \{\infty\}$. If v is an order valuation, then $v(s_1 s_2 s_1^{-1} s_2^{-1} - 1) > 0$ for $s_1, s_2 \in S(D)^\times$ by [H2, Theorem 5.6].

Proposition 2.1. *Let D be a $*$-field with a $*$-ordering P and let v be the associated order valuation with value group Γ_v. For each convex subset $A \subset \Gamma_v^+$ containing*

$\{ v(s_1 s_2 s_1^{-1} s_2^{-1} - 1) \mid s_1, s_2 \in S(D)^\times \}$, *there is an extended $*$-ordering Q_A containing P defined by* $Q_A = \{ s + k \mid s \in P,\ k^* = -k,\ v(k) - v(s) \in A \}$.

Note that A contains $v(0) = \infty$, so that for any $\gamma \in A$, the set also contains every element greater than γ. It sometimes happens that $A = \{\infty\}$ is an allowable set. This means that the $*$-ordering P is itself an extended $*$-ordering. For it to be closed under multiplication requires that $S(D) \subseteq Z_D$, and this happens only if D is either commutative or a standard quaternion algebra [D1, Lemma 1; D2, §14].

Proof. We have $Q_A = Q_A^*$ simply because $v(-k) = v(k)$. The conditions $0 \notin Q_A$, $Q_A \cup -Q_A \supset S(D)^\times$ and $dQ_A d^* \subset Q_A$ $(d \in D^\times)$ follow from the fact that P is a $*$-ordering. Let $s_i + k_i \in Q_A$, $i = 1, 2$. We have $v(s_1 + s_2) = \min[v(s_1), v(s_2)]$ by [Cr1, Lemma 2.4(a)], hence $v(k_1 + k_2) \geq \min[v(k_1), v(k_2)] \geq v(s_1 + s_2)$; if $\min[v(k_1), v(k_2)] = v(k_1)$, say, then the difference $v(k_1 + k_2) - v(s_1 + s_2) \geq v(k_1) - v(s_1) \in A$, whence the difference lies in A, an unbounded convex set. Thus $Q_A + Q_A \subset Q_A$. To check closure under multiplication, we write the product $(s_1 + k_1)(s_2 + k_2) = s + k$, where

$$s = (s_1 s_2 + s_2 s_1 + k_1 k_2 + k_2 k_1 + k_1 s_2 - s_2 k_1 + s_1 k_2 - k_2 s_1)/2$$

and

$$k = (s_1 s_2 - s_2 s_1 + k_1 k_2 - k_2 k_1 + k_1 s_2 + s_2 k_1 + s_1 k_2 + k_2 s_1)/2,$$

where $s^* = s$ and $k^* = -k$. We have $v(s_1 s_2 + s_2 s_1) = v(s_1) + v(s_2)$ by [Cr1, Lemma 2.4(b)], which, in turn, equals $v(s)$ since the remainder of s has larger value. Furthermore, we know $s_1 s_2 + s_2 s_1 \in P$, whence $s \in P$ since v and P are fully compatible. Now $v(k) \geq \min[v(s_1 s_2 - s_2 s_1), v(s_i) + v(k_j)\ (i \neq j)]$, where $v(s_i) + v(k_j) - v(s) \in A$ by hypothesis. And

$$v(s_1 s_2 - s_2 s_1) = v(s_1 s_2 s_1^{-1} s_2^{-1} - 1) + v(s_1) + v(s_2) = v(s_1 s_2 s_1^{-1} s_2^{-1} - 1) + v(s).$$

But $v(s_1 s_2 s_1^{-1} s_2^{-1} - 1) \in A$, so we obtain $v(k) - v(s) \in A$ as desired. □

Note that the proof shows that the condition that A contains $\{ v(s_1 s_2 s_1^{-1} s_2^{-1} - 1) \mid s_1, s_2 \in S(D)^\times \}$ is a necessary one for the set Q_A to be closed under multiplication. We write P_e for the $*$-ordering associated with $A = \Gamma_v^+$. The correspondence between P and P_e was first noted by Idris [I]. In fact, it follows from Holland [H2, Lemma 5.15], that every extended $*$-ordering containing P is contained in P_e.

Lemma 2.2. *Let D be a $*$-field, and let F be the subfield $Z_D \cap S(D)$. Let P be a $*$-ordering with associated order valuation v and let Q be an extension of P. Let k be a skew element with $1 + k \in Q$. Then $f(k) \in Q$ for any polynomial $f(x) \in A_v[x]$ with $f(0) \in P \cap U_v$.*

Proof. We first show that $1 + nk \in Q$ for all integers n. Note that it suffices to do this for $n \geq 2$ since negatives come from closure under $*$. Using induction on n and the fact that $-k^2 = kk^* \in P$, we have

$$1 + nk = (1 + (n-1)k)(1 + k) - (n-1)k^2 \in Q.$$

To see that $1 \pm k^n \in Q$ for $n > 1$, we induct on n. If n is even, we have $1 \pm k^n \in S(D) \cap P_e = P$. If $n \equiv 1 \pmod 4$, we use $1 + k^{4j+1} = k^{4j}(1+k) + (1 - k^{4j}) \in$

$P \cdot Q + P \subset Q$; if $n \equiv 3 \pmod 4$, we use $1 + k^{4j+3} = -k^{4j+2}(1-k) + (1 + k^{4j+2}) \in P \cdot Q + P \subset Q$; finally $1 - k^n = (1 + k^n)^* \in Q$.

Now let $a \in P$ with $v(a) \leq 0$. Then $a > \frac{1}{n}$ for some positive integer n, so we obtain $a \pm k^m = \frac{1}{n}(1 \pm nk^m) + (a - \frac{1}{n}) \in Q$.

Since we can factor out $f(0)^{-1}$, it suffices to assume that $f(0) = 1$. Now write $f(x) = 1 + g(x) + h(x)$, where $g(x)$ is an odd polynomial and $h(x)$ is an even polynomial. As above, we know that $\frac{1}{2} + h(k) \in P$. Let n be the degree of the highest power of x dividing $g(x)$. Then $k^{-n}g(k) \in S(D)$ and we may assume it lies in P; we can then write $\frac{1}{2} + g(k) = \left(\dfrac{k^n}{g(k)} + 2k^n\right)\dfrac{g(k)}{2k^n} \in Q \cdot P \subset Q$. □

Theorem 2.3. *Let D be a $*$-field with a $*$-ordering P and let v be the associated order valuation with value group Γ_v. Then there is a one-to-one correspondence between extended $*$-orderings Q containing P and convex subsets $A \subseteq \Gamma_v^+$ containing $\{\, v(s_1 s_2 s_1^{-1} s_2^{-1} - 1) \mid s_1, s_2 \in S(D)^\times \,\}$ defined by $Q_A = \{\, s + k \mid s \in P,\ k^* = -k,\ v(k) - v(s) \in A \,\}$ and $A_Q = \{\, v(k) \mid 1 + k \in Q \,\}$.*

Proof. Proposition 2.1 shows that any such set A can be used to define an extended $*$-ordering Q_A containing P. We must show that any extended $*$-ordering Q containing P arises as a set Q_A for some set A. Given Q, we define $A = \{\, v(k) - v(s) \mid s + k \in Q \,\}$. Since $Q \subseteq P_e$, we know that $A \subseteq \Gamma_v^+$. To check the commutator condition, let $s_1,\ s_2 \in S(D)^\times$. Replacing s_i by $-s_i$ if necessary, we may assume that $s_1,\ s_2 \in P$, so $s_1 s_2 \in Q$. Since $2s_1 s_2 = (s_1 s_2 + s_2 s_1) + (s_1 s_2 - s_2 s_1)$, as a sum of a symmetric and skew element, we have $v(s_1 s_2 - s_2 s_1) - v(s_1 s_2 + s_2 s_1) \in A$ by definition. But $v(s_1 s_2 s_1^{-1} s_2^{-1} - 1) + v(s_1) + v(s_2) = v(s_1 s_2 - s_2 s_1)$ and $v(s_1) + v(s_2) = v(s_1 s_2 + s_2 s_1)$ [Cr1, Lemma 2.6], hence $v(s_1 s_2 s_1^{-1} s_2^{-1} - 1) = (v(s_1 s_2 s_1^{-1} s_2^{-1} - 1) + v(s_1) + v(s_2)) - (v(s_1) + v(s_2)) \in A$.

Next we show that $A = \{\, v(k) \mid 1 + k \in Q \,\}$, a set it clearly contains. Let $s + k \in Q$. Then $s + k = s(1 + s^{-1}k)$, so $1 + s^{-1}k \in Q$. Similarly, $1 + ks^{-1} \in Q$, whence the sum $2 + s^{-1}k + ks^{-1}$ lies in Q. Since $s^{-1}k + ks^{-1}$ is skew, we will be done if we show $v(s^{-1}k + ks^{-1}) = v(k) - v(s)$. But, if not, then $v(s^{-1}k + ks^{-1}) > v(k) - v(s)$; also $v(s^{-1}k - ks^{-1}) > v(k) - v(s)$ [H2, Lemma 5.18], so adding these gives the contradiction $v(2s^{-1}k) > v(k) - v(s)$.

To complete the proof, we shall show that A is convex and $Q = Q_A$: i.e., that if $1 + k \in Q$, with k a nonzero skew element, and i is a skew element with $v(i) \geq v(k)$, then $1 + i \in Q$. For each rational number r, we have

$$\begin{aligned}(1 + r(k^{-1}i + ik^{-1}) + r^2 ik^{-2}i) + (k + 2ri + r^2 ik^{-1}i)\\ = (1 + rik^{-1})(1 + k)(1 + rik^{-1})^* \in Q\end{aligned}$$

and also

$$\begin{aligned}(1 - r(k^{-1}i + ik^{-1}) + r^2 ik^{-2}i) + (-k + 2ri - r^2 ik^{-1}i)\\ = (1 - rik^{-1})(1 - k)(1 - rik^{-1})^* \in Q.\end{aligned}$$

Adding these, we obtain $1 + r^2 ik^{-2}i + 2ri \in Q$ for all $r \in \mathbb{Q}$. If $v(i) > v(k)$, then $v(ik^{-2}i) > 0$; setting $r = \frac{1}{2}$, the conclusion follows from the compatibility of v and Q. Now assume that $v(i) = v(k)$. Since $v(ik^{-2}i) = 0$, we see that $ik^{-2}i$ must be positive in order that $1 + r^2 ik^{-1}i \in P$ for all r. Hence we can choose r so that $0 < r^2 ik^{-2}i < 1$, from which we obtain $1 + ri = \frac{1}{2}[(1 - r^2 ik^{-2}i) + (1 + r^2 ik^{-2}i + 2ri)] \in Q$. But then $1 + i \in Q$ by Lemma 2.2. □

3. Baer orderings

In some cases, extensions of Baer orderings from the symmetric elements to more general sets satisfying the same conditions can also be obtained with a method similar to that in Proposition 2.1. In general, the order valuation need not be a valuation in the ordinary sense. Holland [H1] has shown that the ring $A(P)$ is a total subring of D, but it need not be invariant [MW]. The set of values, Γ_v, is not a group in this case. When v is a $*$-valuation (i.e. $v(a) = v(a^*)$ for all $a \in D$), then Γ_v becomes a commutative group as usual [H1, 4.6].

In order to obtain closure under $p \mapsto dpd^*$, we shall need the order ring to be invariant. This, in fact, implies that we have a $*$-valuation by [H2, Lemma 5.4]. The other fact that we need is that v and P be fully compatible: $0 < a \leq b$ with respect to P implies $v(a) \geq v(b)$. This is easily seen to be equivalent to saying $v(a+b) = \min(v(a), v(b))$ whenever $a, b \in P$. This condition is not automatically true as it is for $*$-orderings and their order valuations, even in the case where the involution is the identity (and D is then necessarily commutative).

Proposition 3.1. *Let D be a $*$-field with a Baer ordering P and let v be the associated order valuation. We assume that v is a $*$-valuation fully compatible with P. For each convex subset $A \subset \Gamma_v^+$ (defined as in §2) containing ∞, there is an extended Baer ordering Q_A containing P defined by $Q_A = \{\, s + k \mid s \in P,\ k^* = -k,\ v(k) - v(s) \in A \,\}$.*

Proof. Let $s + k \in Q_A$ with $s \in P,\ k^* = -k$ and let $d \in D^\times$. Then

$$d(s+k)d^* = (dsd^*) + (dkd^*),$$

where $dsd^* \in P$ by definition of a Baer ordering and dkd^* is a skew element. Furthermore, $v(dkd^*) - v(dsd^*) = v(k) - v(s) \in A$, whence $d(s+k)d^* \in Q_A$. Closure of Q_A under addition and under $*$ follow as in the proof of Proposition 2.1; in this case the hypothesis of full compatibility is needed for the reference to [Cr1]. □

Unlike the situation for $*$-orderings, the converse of Proposition 3.1 is far from true.

Example 3.2. In the complex numbers, $\mathbb{C}$, all Baer orderings have the form $\{\, re^{i\theta} \mid r > 0,\ \theta \in I \,\}$, where I is either an open interval $(-\theta_0, \theta_0)$ with $\theta_0 \leq \pi/2$ or a closed interval $[-\theta_0, \theta_0]$ with $\theta_0 < \pi/2$. Since the order valuation on $\mathbb{C}$ is trivial, the only possibility for the set A of Proposition 3.1 is $A = \{\infty\}$; but there are infinitely many extended $*$-orderings containing $\mathbb{R}^+$.

Another way of viewing the closed sets in this example is as the sets $P \pm P\sigma = \{\, p_1 \pm p_2\sigma \in \mathbb{C} \mid p_1, p_2 \in P \,\}$, where $P = \mathbb{R}^+$ is a (symmetric) Baer ordering and σ is any complex number with positive real part. This suggests the following method of viewing an arbitrary Baer ordering.

Proposition 3.3. *Let P be a Baer ordering of $(D, *)$. Let B be any subset of $P + Sk(D)$ such that* (1) $B + B \subseteq B$, (2) $dBd^* \subseteq B$ *for any* $d \in D^\times$, (3) B *is $*$-closed, and* (4) $P \subseteq B$. *Then B is an extended Baer ordering and every extended Baer ordering arises in this way.*

Proof. We have $0 \notin B$ since $-P \cap Sk(D) \subset S(D)^\times \cap Sk(D) = \emptyset$. The remaining conditions are clear.

Conversely, let B be any extended Baer ordering and set $P = B \cap S(D)$. Any element $d \in B$ has the form $d = s + k$ with $s \in S(D)$ and $k \in Sk(D)$, where $s = \frac{1}{2}(d + d^*) \in B \cap S(D) = P$. □

As a further example, we compute all the extended Baer orderings in the real quaternions $\mathbb{H}$.

Example 3.4. The symmetric elements in $\mathbb{H}$ consist of the real numbers, which have the unique positive cone $\mathbb{R}^+$. The extensions of this unique Baer ordering of $\mathbb{H}$ are obtained by defining a cone of the form $\{ a+bi+cj+dk \mid N(bi+cj+dk)/|a| \in I \}$ where the interval I of slopes is either of the form $[0, m)$ with $m > 0$ or a closed interval $[0, m]$ with $m \geq 0$. The proof that such sets are Baer orderings is the same as for the complex numbers except for dealing with noncommutativity: i.e., $dzd^* \in P$ whenever $z \in P$. Write $z = a + z_0$ where z_0 is a pure quaternion and $a > 0$. Then $dzd^* = (a + dz_0d^{-1})dd^*$ with the same ratio of norms as z. By [O, §57], the element dz_0d^{-1} is just a rotation of z_0 and all rotations occur in this way.

4. c-orderings

For c-orderings, the situation is much like that of Baer orderings. To see this, we begin with the two examples of the previous section. For the complex numbers, the commutativity implies that the concepts of c-ordering and Baer ordering are identical. Somewhat surprisingly, for the quaternions, there are more c-orderings than Baer orderings.

Example 4.1. The rotations induced by $dPd^* \subset P$ in Example 3.4 are no longer present. For $d \in \mathbb{H}$, the element dd^* is in $\mathbb{R}^+$, so the conditions on a c-ordering P are that (1) $P \supseteq \mathbb{R}^+$, (2) $P^* = P$, (3) $P + P \subseteq P$, (4) $P\mathbb{R}^+ \subseteq P$, and (5) $0 \notin P$. These are precisely the convex cones symmetric about the central axis $\mathbb{R}^+$, where the vertex (origin) is missing and any individual boundary line may or may not be included.

For the relationship with valuation theory, we have the following analog to Propositions 2.1 and 3.1. The assumption for Baer orderings that the order valuation be a $*$-valuation (which was automatic for $*$-orderings) is obtained here by assuming the c-ordering is normal (see [Ch2] and [Ch3]).

Proposition 4.2. *Let D be a $*$-field with a normal symmetric c-ordering P and let v be the associated order valuation. We assume that v is fully compatible with P. For each convex subset $A \subset \Gamma_v^+$ (defined as in §2) containing*

$$\{ v(sdd^*s^{-1}(dd^*)^{-1} - 1) \mid s \in S(D)^\times,\ d \in D^\times \},$$

there is a c-ordering Q_A containing P defined by $Q_A = \{ s + k \mid s \in P,\ k^* = -k,\ v(k) - v(s) \in A \}$.

Proof. We note first that for $s \in S(D)^\times,\ d \in D^\times$, we have

$$v(sdd^*s^{-1}(dd^*)^{-1} - 1) > 0$$

by [Ch3, p. 3071]; also, $v(0) = \infty \in A$ as in §2. It is clear that Q_A is $*$-closed, contains 1, and $Q_A \cup -Q_A \supseteq S(D)^\times$. Closure under addition is proved as in the proof of Proposition 2.1 except that we need to know that $v(s_1+s_2)=\min(v(s_1),v(s_2))$ for any $s_1,\ s_2\in P$. This follows from the assumption of full compatibility between v and P, and is proved as in [Cr1, Lemma 2.4]. Let $s+k\in Q_A$ with $s\in P$ and $k^*=-k$, and let $d\in D^\times$. To complete the proof we must show that $(s+k)dd^*\in Q_A$. We can write $(s+k)dd^*=s_0+k_0$, where $s_0=\frac{1}{2}(sdd^*+dd^*s+kdd^*-dd^*k)$ is symmetric and $k_0=\frac{1}{2}(sdd^*-dd^*s+kdd^*+dd^*k)$ is skew. Since P is a symmetric c-ordering, we have $sdd^*+dd^*s\in P$. By [Ch3, Theorem 1.13], we have $v(sdd^*+dd^*s)=v(s)+v(dd^*)<v(kdd^*-dd^*k)$, whence compatibility implies that $s_0\in P$. Furthermore, $v(k_0)-v(s_0)\geq\min(v(sdd^*-dd^*s),v(kdd^*))-v(s)-v(dd^*)$, where $v(sdd^*-dd^*s)=v(sdd^*s^{-1}(dd^*)^{-1}-1)+v(s)+v(dd^*)$. Since $v(sdd^*s^{-1}(dd^*)^{-1}-1)\in A$, we obtain

$$v(k_0)-v(s_0)\geq\min(v(sdd^*s^{-1}(dd^*)^{-1}-1),$$

$v(k)-v(s))\in A$, whence $v(k_0)-v(s_0)\in A$. □

We next make a few comments on $*$-semiorderings. Chacron has shown that they are normal symmetric c-orderings [Ch3, Theorem 2.6], so the previous proposition holds for them. This is a generalization of [Cr2, Theorem 2.20]. On the other hand, $*$-semiorderings are also Baer orderings, so the results of §3 also hold. In particular, the extended $*$-semiorderings in $\mathbb{H}$ are the extended Baer orderings described in Example 3.4, while the normal c-orderings in $\mathbb{H}$ comprise the larger class of c-orderings described in Example 4.1.

We now turn to the remaining major unanswered question, raised in [Ch3]. This is the issue of whether an arbitrary symmetric c-ordering can be extended to a c-ordering. What is needed for the results given in [Ch3] is some sort of condition analogous to a valuation being *real* in the case of commutative fields. This is built into the definitions of $*$-semiorderings and normal symmetric c-orderings, allowing orderings to be lifted from the residue fields.

With the modified definiton of symmetric c-ordering given in §1, we shall see that symmetric orderings always have an extension.

Lemma 4.3. *The set* $\Delta=S(D)^\times\prod_{nr}=\{\,sx\mid x\in\prod_{nr},\ s\in S(D)^\times\,\}$ *is a $*$-closed subset of* $D^\times$.

Proof. Any norm dd^* can be moved around a symmetric element s using the general formula

$$(dd^*)s=s[(s^{-1}d)(s^{-1}d)^*ss^*].\qquad\square$$

Lemma 4.4. *Let P be a symmetric c-ordering with $x\in\prod_{nr}$ and $z\in\Delta$ satisfying $z+z^*\in P$. Then $zx+x^*z^*\in P$.*

Proof. Write $z=sx_0$, where $s\in S(D)^\times$ and $x_0\in\prod_{nr}$. If $s\notin P$, then $s\in-P$, so $z+z^*=sx_0+x_0^*s\in-P$ by the definition of P, a contradiction of the hypothesis $z+z^*\in P$. Therefore $s\in P$, whence $zx+x^*z^*=s(x_0x)+(x_0x)^*s\in P$. □

Theorem 4.5. *Let P be a symmetric c-ordering of a $*$-field D. Set $Q=\{\sum z_i\mid z_i\in\Delta,\ z_i+z_i^*\in P\,\}$. Then Q is a minimal c-ordering extending P.*

Proof. The properties $Q+Q\subseteq Q$, $P\subseteq Q$ and $Q\cup-Q\supseteq S(D)^\times$ are all easily seen to hold. The fact that $Q=Q^*$ follows from Lemma 4.3. To verify that Q is a c-ordering, it remains to be shown that $Qdd^*\subseteq Q$ for any $d\in D^\times$. Let $z=\sum z_i\in Q$.

Then $zdd^* = \sum z_i dd^*$, where $z_i \in \Delta$ implies that $z_i dd^* \in \Delta$. By Lemma 4.4, we obtain $z_i(dd^*) + (dd^*)z_i^* \in P$, whence $zdd^* \in Q$. The fact that Q is minimal among c-orderings extending P follows from the fact that, with respect to any c-ordering Q_0, every element of Δ must have a defined sign; if $Q_0 \cap S(D) = P$, then for $z \in \Delta$, $z + z^* \in P \iff z \in Q_0$, as $z \in -Q_0$ would imply $z + z^* \in -P$. □

Example 4.6. As a dramatic contrast to the large numbers of c-orderings seen in the work above, we now look at an example in which there are four symmetric c-orderings and each has a unique extension to a c-ordering. This $*$-field is developed in [Ch2, Theorem 3.1.4] as an example in which the natural valuation is not a c-valuation. Let F be the field $\mathbb{Q}(\sqrt{2})$ with σ the generator of $\mathrm{Gal}(F/\mathbb{Q})$. Set $D = F((t;\sigma))$, the skew field of Laurent series in t with the commutation rule $ta = \sigma(a)t$ for $a \in F$ (see [Da, §IV–4]). Then $Z_D = \mathbb{Q}((t^2))$ and D can also be viewed as the quaternion algebra $\left(\frac{2,t^2}{Z_D}\right)$. We put a nonstandard involution on D, defined by $\sqrt{2}^* = \sqrt{2}$ and $t^* = t$. Thus $(\sqrt{2}t)^* = t\sqrt{2} = -\sqrt{2}t$ is a skew element. Let v be the t-adic valuation with valuation ring $F[[t;\sigma]]$. From the quaternion algebra form of D, we see that every element can be written in the form $a + b\sqrt{2} + ct + d\sqrt{2}t$, with $a, b, c, d \in Z_D$, and where $a + b\sqrt{2} + ct \in S(D)$ and $d\sqrt{2}t \in Sk(D)$. It is easily seen that there are exactly four symmetric c-orderings, depending on the signs of $\sqrt{2}$ and t. For explicitness, we restrict our attention to the one in which both elements are positive: set $P = \{\, \sum_m q_i t^i \in S(D) \mid q_m \in P_0 \,\}$, where P_0 is the ordering of F containing $\sqrt{2}$.

By Theorem 4.5, there is a minimal c-ordering Q extending P. Note that Proposition 4.2 does not apply: the hypothesis on the valuation fails, as is shown by the fact that, taking $s = t$ and $d = 1 + \sqrt{2}$, the element $sdd^*s^{-1}(dd^*)^{-1} - 1 = 16 - 12\sqrt{2}$ has value $0 \notin \Gamma_v^+$. It is easily checked that

$$Q_1 = \left\{ \sum_{i=n} a_i t^i \mid a_n > 0 \text{ in } F,\ n \text{ even, or } a_n \text{ is totally positive in } F,\ n \text{ odd} \right\}$$

is also a c-ordering extending P. (Note that $p + q\sqrt{2}$ is a totally positive element of F iff $p > |q\sqrt{2}|$.) We shall show that Q_1 is maximal and that $Q = Q_1$, so there is a unique c-ordering extending P.

To see that Q_1 is maximal, assume there is a larger c-ordering $Q_2 \supset Q_1$. An element of Q_2 not in Q_1 would have to be of the form $\sum_{i=n} a_i t^i$ with n odd, $a_n = r + s\sqrt{2}$, $r, s \in \mathbb{Q}$, and $0 < r < |s\sqrt{2}|$. Since t^2 is a norm, we may assume that $n = 1$. Let $0 < \epsilon < |s\sqrt{2}| - r$, so $\epsilon t - \sum_{i=2} a_i t^i \in Q_1$. Then the sum $(r + \epsilon + s\sqrt{2})t \in Q_2 \setminus Q_1$. We may replace $r + \epsilon$ by r and assume that $s > 0$. Then $2s - r\sqrt{2}$ is totally positive, whence it is a sum of squares in F (i.e. norms in D). It follows that $(2s - r\sqrt{2})((r + s\sqrt{2})t) \in Q_2$. But this equals $(2s^2 - r^2)\sqrt{2}t \in Sk(D)$, a contradiction since its negative would also have to be in Q_2.

To show that $Q = Q_1$, we shall show that every element of Q_1 is a sum of elements of Δ. The general element of Q_1, $\sum_{i=n} a_i t^i$, can be considered in several cases: if n is odd, a_n is totally positive, hence a sum of norms, and is multiplied by the positive symmetric element t^n; if n is even, t^n is a norm and a_n is a positive symmetric. In either case, it will suffice to show that an element of the form $1 + a_1 t + a_2 t^2 + \cdots$ is a product of norms for any choice of $a_i \in F$. Computing a single norm $(\sum_{i=0}^{\infty} b_i t^i)(\sum_{i=0}^{\infty} b_i t^i)^*$, one sees that the low order coefficient is a square b_0^2, all the coefficients of even powers of t can be made arbitrary elements of F, and the coefficients of odd powers of t can be made arbitrary elements of $\mathbb{Q}$. However a product of two norms will give arbitrary coefficients a_i in F (nonuniquely), as can be seen by computing the necessary coefficients in consecutive pairs.

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