

## ON A MEASURE-THEORETIC PROBLEM OF ARVESON

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**ABSTRACT.** A probability measure  $\nu$  on a product space  $X \times Y$  is said to be bistochastic with respect to measures  $\lambda$  on  $X$  and  $\mu$  on  $Y$  if the marginals  $\pi_1(\nu)$  and  $\pi_2(\mu)$  are exactly  $\lambda$  and  $\mu$ . A solution is presented to a problem of Arveson about sets which are of measure zero for all such  $\nu$ .

If  $(X, \mathcal{F}, \lambda)$  and  $(Y, \mathcal{G}, \mu)$  are probability spaces, we shall say that a probability measure  $\sigma$  on the product  $\sigma$ -algebra  $\mathcal{F} \otimes \mathcal{G}$  is *bistochastic* (with respect to  $\lambda$  and  $\mu$ ), and write  $\sigma \in \text{Bist}(\lambda, \mu)$  if  $\sigma$  has  $\lambda$  and  $\mu$  as its marginal measures  $\pi_1(\sigma)$  and  $\pi_2(\sigma)$ , that is to say, if  $\pi_1(\sigma)(A) = \sigma(A \times Y) = \lambda(A)$  and  $\pi_2(\sigma)(B) = \sigma(X \times B) = \mu(B)$  for all  $A \in \mathcal{F}$  and  $B \in \mathcal{G}$ . In the course of his study of operator algebras associated with commutative lattices of self-adjoint projections on Hilbert space, Arveson [1] investigated the following problem: Given probability spaces  $(X, \mathcal{F}, \lambda)$  and  $(Y, \mathcal{G}, \mu)$ , and a subset  $E$  of  $X \times Y$ , when is  $E$  a null set for all  $\sigma \in \text{Bist}(\lambda, \mu)$ ? It is clear that a sufficient condition for this to be the case is that there exist  $F \in \mathcal{F}$ ,  $G \in \mathcal{G}$  with  $\lambda(F) = \mu(G) = 0$  and  $E \subseteq (F \times Y) \cup (X \times G)$ ; sets  $E$  of this type are said to be “marginally null”. Arveson showed that a converse to this statement is valid under certain circumstances: When  $X$  and  $Y$  are compact Hausdorff spaces and  $\lambda, \mu$  are regular Borel probability measures, then a closed subset of  $X \times Y$  which is a null set for all  $\sigma \in \text{Bist}(\lambda, \mu)$  is necessarily marginally null [1, Theorem 1.4.2]. The same conclusion holds if  $(X, \mathcal{F})$  and  $(Y, \mathcal{G})$  are standard Borel spaces and  $E$  is the complement in  $X \times Y$  of a countable union of Borel rectangles [1, Theorem 1.4.3]. We shall show that in both of these cases the result may be extended to a wider class of sets  $E$ . In fact, we shall give a more precise result, expressing in terms of the measures  $\lambda$  and  $\mu$  the supremum of  $\sigma(E)$  taken over all bistochastic  $\sigma$ . This may be viewed as an extension of a theorem of Sudakov [7, Theorem 9].

We use fairly standard measure-theoretic terminology. A subset of a Cartesian product  $X \times Y$  is said to be a *rectangle* if it has the form  $A \times B$  with  $A \subseteq X$  and  $B \subseteq Y$ . The *product  $\sigma$ -algebra*  $\mathcal{F} \otimes \mathcal{G}$  of  $\sigma$ -algebras  $\mathcal{F}$  on  $X$  and  $\mathcal{G}$  on  $Y$  is the  $\sigma$ -algebra generated by all rectangles  $A \times B$  ( $A \in \mathcal{F}$ ,  $B \in \mathcal{G}$ ). If  $\nu$  is a measure on a  $\sigma$ -algebra containing  $\mathcal{F} \otimes \mathcal{G}$  the *marginals* are the measures defined on  $\mathcal{F}$  and  $\mathcal{G}$  by  $\pi_1(\nu)(A) = \nu(A \times Y)$ ,  $\pi_2(\nu)(B) = \nu(X \times B)$ . In the topological theory we are concerned only with finite positive measures defined on the Borel  $\sigma$ -algebra  $\mathcal{B}(X)$

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of a Hausdorff space  $X$  (that is to say, on the  $\sigma$ -algebra generated by the open sets of  $X$ ). We say that such a measure  $\mu$  is *regular* if

$$\mu(U) = \sup\{\mu(F) : F \subseteq U \text{ and } F \text{ is closed in } X\}$$

for every open  $U \subseteq X$ . Such a measure is said to be a *Radon* measure if the closed sets  $F$  in the above expression may be replaced with compact sets  $K$ . Of course, all finite regular Borel measures on a compact space  $X$  are Radon, and these are exactly the measures that arise in the Riesz representation of positive linear functionals on  $\mathcal{C}(X)$ . A *standard Borel space* is a pair  $(X, \mathcal{F})$  such that, for a suitably chosen topology  $\mathcal{T}$ ,  $X$  is a complete separable metric space and  $\mathcal{F}$  its Borel  $\sigma$ -algebra. Any finite measure on a standard Borel space is Radon with respect to any such topology  $\mathcal{T}$ .

For arbitrary finite measure spaces  $(X, \mathcal{F}, \lambda)$  and  $(Y, \mathcal{G}, \mu)$  and an arbitrary subset  $E$  of  $X \times Y$  we define

$$\alpha(E) = \sup\{\nu^*(E) : \nu \text{ is a measure on } \mathcal{F} \otimes \mathcal{G} \text{ with } \pi_1(\nu) \leq \lambda \text{ and } \pi_2(\nu) \leq \mu\};$$

$$\beta(E) = \inf\left\{\int f d\lambda + \int g d\mu : f \in L^\infty(\lambda), g \in L^\infty(\mu), f, g \geq 0,\right.$$

$$\left. \text{and } f(x) + g(y) \geq 1 \text{ whenever } (x, y) \in E\right\};$$

$$\gamma(E) = \inf\{\lambda(A) + \mu(B) : A \in \mathcal{F}, B \in \mathcal{G} \text{ and } E \subseteq (A \times Y) \cup (X \times B)\}.$$

In the case where both  $\lambda$  and  $\mu$  are probability measures, we see easily that  $\alpha(E) = \sup\{\sigma^*(E) : \sigma \in \text{Bist}(\lambda, \mu)\}$ . Indeed, if  $\nu$  is any measure on the product  $\sigma$ -algebra  $\mathcal{F} \otimes \mathcal{G}$  with  $\pi_1(\nu) \leq \lambda$  and  $\pi_2(\nu) \leq \mu$ , and if we set  $\sigma = \nu + \theta^{-1}(\lambda' \otimes \mu')$ , where  $\lambda' = \lambda - \pi_1(\nu)$ ,  $\mu' = \mu - \pi_2(\nu)$  and  $\theta = 1 - \nu(X \times Y)$ , then  $\sigma \in \text{Bist}(\lambda, \mu)$  and  $\sigma^*(E) \geq \nu^*(E)$ .

It is obvious that  $\alpha(E) \leq \gamma(E) \leq \beta(E)$ ; we shall show that, in complete generality,  $\gamma(E) = \beta(E)$  and then that  $\alpha(E) = \gamma(E)$  in the topological case, when  $E$  is  $\mathcal{K}$ -analytic. The key to the proof of this last fact will be to show that  $\beta$  is a capacity. We recall that a function  $I$ , defined on the power set of a Hausdorff topological space  $Z$  and taking values in  $\overline{\mathbb{R}}_+$ , is said to be a *capacity* if

- (1)  $I(E) \leq I(F)$  whenever  $E \subseteq F$ ;
- (2)  $\lim_{n \rightarrow \infty} I(E_n) = I(E)$  whenever  $(E_n)$  is an increasing sequence with union  $E$ ;
- (3)  $I(K) = \inf\{I(U) : U \text{ is open and } U \supseteq K\}$  whenever  $K$  is compact.

A set  $E$  is said to be *capacitable* if, for all capacities  $I$ ,

$$I(E) = \sup\{I(K) : K \text{ is compact and } K \subseteq E\}.$$

Choquet's capacitability theorem tells us that any Borel subset of a complete separable metric space, and, more generally, any  $\mathcal{K}$ -analytic subset of a Hausdorff space is capacitable. For the definition of  $\mathcal{K}$ -analyticity and the proof of Choquet's theorem, the reader is referred to [2] or [5].

**Lemma 1.** *Let  $(X, \mathcal{F}, \lambda)$  and  $(Y, \mathcal{G}, \mu)$  be finite measure spaces and let  $(E_n)_{n \in \mathbb{N}}$  be an increasing sequence of subsets of  $X \times Y$  with union  $E$ . Then there exist  $A_0 \in \mathcal{F}$  and  $B_0 \in \mathcal{G}$  such that  $E \subseteq (A_0 \times Y) \cup (X \times B_0)$  and such that  $\lambda(A_0) + \mu(B_0) = \lim_{n \rightarrow \infty} \beta(E_n)$ .*

*Proof.* Let  $f_n \in L^\infty_+(\lambda)$  and  $g_n \in L^\infty_+(\mu)$  be chosen so that  $f_n(x) + g_n(y) \geq \mathbf{1}_{E_n}(x, y)$  and

$$\int f_n d\lambda + \int g_n d\mu - \beta(E_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Thinking of the pairs  $(f_n, g_n)$  as elements of  $L^\infty(\lambda) \times L^\infty(\mu)$ , we may extract a subsequence such that  $(f_{n_k}, g_{n_k})$  converges to a limit  $(f, g)$  for the weak-\* topology  $\sigma(L^\infty(\lambda) \times L^\infty(\mu), L^1(\lambda) \times L^1(\mu))$ . Since this topology is finer than the weak topology  $\sigma(L^1(\lambda) \times L^1(\mu), L^\infty(\lambda) \times L^\infty(\mu))$ , there exist convex combinations  $(\tilde{f}_n, \tilde{g}_n) \in \text{conv}\{(f_m, g_m) : m \geq n\}$  such that  $\tilde{f}_n \rightarrow f$  and  $\tilde{g}_n \rightarrow g$  in  $L^1$ -norm. Taking a further subsequence, we may suppose that  $\tilde{f}_n$  converges to  $f$  and  $\tilde{g}_n$  to  $g$  almost everywhere on  $X$  and  $Y$  respectively. Consequently, by modifying  $f$  on a  $\lambda$ -null set and  $g$  on a  $\mu$ -null set, we may suppose that  $f(x) + g(y) \geq 1$  for all  $(x, y) \in E$ . We also clearly have  $\int f d\lambda + \int g d\mu = \beta(E)$ . Thus we have shown that  $\beta(E) = \lim_{n \rightarrow \infty} \beta(E_n)$ , and have shown moreover that the infimum in the definition of  $\beta(E)$  is attained.

We now want to replace  $f$  and  $g$  with indicator functions. We notice that we may express  $\int f d\lambda$  and  $\int g d\mu$  as real integrals over the unit interval as

$$\begin{aligned} \int f d\lambda &= \int_0^1 \lambda\{x \in X : f(x) \geq t\} dt, \\ \int g d\mu &= \int_0^1 \mu\{y \in Y : g(y) \geq u\} du. \end{aligned}$$

Thus, making the change of variable  $u = 1 - t$ , we have

$$\beta(E) = \int_0^1 \left[ \lambda\{x \in X : f(x) \geq t\} + \mu\{y \in Y : g(y) \geq 1 - t\} \right] dt.$$

The integrand in this integral from 0 to 1 cannot be everywhere greater than  $\beta(E)$  and so, for a suitably chosen  $t_0$ , we have  $\lambda\{x \in X : f(x) \geq t_0\} + \mu\{y \in Y : g(y) \geq 1 - t_0\} \leq \beta(E)$ . We set  $A_0 = \{x \in X : f(x) \geq t_0\}$  and  $B_0 = \{y \in Y : g(y) \geq 1 - t_0\}$  and note that for every  $(x, y) \in E$  at least one of  $x \in A_0$  and  $y \in B_0$  is true, since  $f(x) + g(y) \geq 1$ . Thus  $\lambda(A_0) + \mu(B_0) \geq \beta(E)$  by the definition of  $\beta(E)$ , so that  $\lambda(A_0) + \mu(B_0) = \beta(E)$ , as claimed.  $\square$

**Corollary.** (1) For all  $E$ ,  $\gamma(E) = \beta(E)$ ;

(2) if  $(E_n)$  is an increasing sequence with union  $E$ , then  $\gamma(E_n) \rightarrow \gamma(E)$  as  $n \rightarrow \infty$ ;

(3) the infimum in the definition of  $\gamma(E)$  is attained;

(4)  $E$  is marginally null if  $\gamma(E) = 0$ .

**Lemma 2.** If  $X$  and  $Y$  are Hausdorff topological spaces and  $\lambda, \mu$  are regular Borel measures on the Borel  $\sigma$ -algebras  $\mathcal{F} = \mathcal{B}(X)$ ,  $\mathcal{G} = \mathcal{B}(Y)$ , respectively, then  $\beta$  is a capacity.

*Proof.* It is clear that  $E \subseteq F$  implies  $\beta(E) \leq \beta(F)$  and Lemma 1 shows that  $\beta(E_n) \uparrow \beta(E)$  when  $E_n \uparrow E$ . If  $E$  and  $\epsilon > 0$  are given, and the sets  $A_0$  and  $B_0$  are as in Lemma 1, then regularity of the measures  $\lambda, \mu$  allows us to find open subsets  $U$  of  $X$  and  $V$  of  $Y$  such that  $A_0 \subseteq U$ ,  $B_0 \subseteq V$  and  $\lambda(U) + \mu(V) < \beta(E) + \epsilon$ . Thus  $W = (U \times Y) \cup (X \times V)$  is an open subset of  $X \times Y$  with  $E \subseteq W$  and  $\beta(W) < \beta(E) + \epsilon$ .  $\square$

**Lemma 3.** *For a compact subset  $K$  of  $X \times Y$  there exists a regular Borel measure  $\nu$  on  $K$  with  $\pi_1(\nu) \leq \lambda$ ,  $\pi_2(\nu) \leq \mu$  and  $\nu(K) = \beta(K)$ . In particular,  $\alpha(K) = \beta(K)$ . In the case where  $\lambda$  and  $\mu$  are probability measures, there exists  $\sigma \in \text{Bist}(\lambda, \mu)$  with  $\sigma(K) = \beta(K)$ .*

*Proof.* We define a sublinear functional  $\phi$  on the space  $\mathcal{C}(K)$  of all continuous real-valued functions on  $K$  by

$$\phi(h) = \inf \left\{ \int f \, d\lambda + \int g \, d\mu : f, g \text{ are non-negative Borel functions on } X, Y, \right. \\ \left. \text{respectively, and } h(x, y) \leq f(x) + g(y) \text{ for all } (x, y) \in K \right\}.$$

By definition, we have  $\phi(\mathbf{1}_K) = \beta(K)$  so that, by the Hahn-Banach theorem, there exists a linear functional  $\nu$  on  $\mathcal{C}(K)$  such that  $\langle \nu, \mathbf{1}_K \rangle = \beta(K)$  and  $\langle \nu, h \rangle \leq \phi(h)$  for all  $h \in \mathcal{C}(K)$ . We note that  $\phi(-h) = 0$  when  $h \geq 0$ , so that  $\nu$  is positive, with  $\|\nu\| = \langle \nu, \mathbf{1}_K \rangle = \beta(K)$ . By the Riesz representation theorem,  $\nu$  may be identified with a regular Borel measure on  $K$  with  $\nu(K) = \langle \nu, \mathbf{1}_K \rangle = \beta(K)$ . The marginal measures  $\pi_1(\nu)$  and  $\pi_2(\nu)$  are regular Borel measures, supported by the compact subsets  $K_1 = \pi_1[K]$  and  $K_2 = \pi_2[K]$  of  $X$  and  $Y$ , respectively. If  $0 \leq f \in \mathcal{C}(K_1)$ , then  $\int_{K_1} f \, d(\pi_1 \nu) = \int_K (f \circ \pi_1) \, d\nu \leq \phi(f \circ \pi_1) \leq \int f \, d\lambda$ , so that  $\pi_1(\nu) \leq \lambda$ . Similarly,  $\pi_2(\nu) \leq \mu$ . The way to get from  $\nu$  to a bistochastic measure  $\sigma$  with  $\sigma(K) = \beta(K)$  by adding a suitable product measure has already been noted.  $\square$

**Theorem 1.** *Let  $X, Y$  be Hausdorff topological spaces, let  $\lambda, \mu$  be regular Borel measures on  $X$  and  $Y$ , respectively, and let  $E$  be a capacitable subset of  $X \times Y$ . Then  $\alpha(E) = \beta(E)$  and in particular  $E$  is marginally null if  $\nu(E) = 0$  for all bistochastic  $\nu$ . If  $\lambda$  and  $\mu$  are Radon measures, then  $\alpha(E) = \beta(E)$  for all sets  $E$  in the product  $\sigma$ -algebra  $\mathcal{B}(X) \otimes \mathcal{B}(Y)$ .*

*Proof.* Since  $\beta$  is a capacity, and  $E$  is capacitable, there exist compact subsets  $K$  of  $E$  with  $\beta(K)$  arbitrarily close to  $\beta(E)$ . But for any such  $K$  we have  $\alpha(E) \geq \alpha(K) = \beta(K)$  by Lemma 3. Hence  $\alpha(E) \geq \beta(E)$ , which establishes the first assertion. In the case where  $\alpha(E) = 0$  we now have  $\beta(E) = 0$  and so  $E$  is marginally null by the Corollary to Lemma 1.

It is an obvious consequence of our first assertion that  $\alpha(E) = \beta(E)$  if there is a  $\mathcal{K}$ -analytic set  $E_0 \subseteq E$  with  $E \setminus E_0$  marginally null. We shall show that this is the case for all  $E \in \mathcal{B}(X) \otimes \mathcal{B}(Y)$  provided  $\lambda$  and  $\mu$  are Radon. Using the fact that the class of  $\mathcal{K}$ -analytic sets is stable under countable unions and intersections, we see that the collection of sets defined by

$$\mathcal{H} = \{H \subseteq X \times Y : \text{there exist } \mathcal{K}\text{-analytic sets } H_0, H_1 \text{ with } H_0 \subseteq H, \\ H_1 \subseteq (X \times Y) \setminus H \text{ and } (X \times Y) \setminus (H_0 \cup H_1) \text{ marginally null}\}$$

is a  $\sigma$ -algebra. If  $\lambda$  and  $\mu$  are Radon and  $A \in \mathcal{B}(X)$ ,  $B \in \mathcal{B}(Y)$ , then it is clear that  $A \times B \in \mathcal{H}$ —we can even find  $\mathcal{K}_\sigma$  subsets  $H_0, H_1$  with the desired properties. It follows from this that  $\mathcal{B}(X) \otimes \mathcal{B}(Y) \subseteq \mathcal{H}$ , which implies our result.  $\square$

**Corollary.** *If  $(X, \mathcal{F})$  and  $(Y, \mathcal{G})$  are standard Borel spaces, equipped with finite measures  $\lambda, \mu$ , respectively, and if  $E$  is in the product  $\sigma$ -algebra  $\mathcal{F} \otimes \mathcal{G}$ , then  $\alpha(E) =$*

$\beta(E)$ . In particular, if  $\lambda$  and  $\mu$  are probability measures and  $E$  is not marginally null, then there exists a bistochastic measure  $\sigma \in \text{Bist}(\lambda, \mu)$  with  $\sigma(E) \neq 0$ .

The above theorem and its corollary give a partial answer to the problem posed by Arveson on page 436 of [1]: if  $\lambda$  and  $\mu$  are regular Borel probability measures on compact spaces  $X$  and  $Y$ , respectively, and  $E$  is a Borel subset of  $X \times Y$ , which is a null set for all regular Borel measures  $\nu$  satisfying  $\pi_1(\nu) = \lambda$ ,  $\pi_2(\nu) = \mu$ , then is  $E$  necessarily marginally null? However, the general answer to this question is negative, as is shown by the following example, which exhibits amongst other things the distinction between the product  $\sigma$ -algebra of the Borel algebras of  $X$  and  $Y$  and the Borel  $\sigma$ -algebra of  $X \times Y$ .

**Example 1.** *There is a compact space  $X$ , a regular Borel probability measure  $\mu$  on  $X$  and a Borel subset  $E$  of  $X \times X$  which is not marginally null with respect to  $\mu$  but is such that  $\nu(E) = 0$  for all regular Borel measures  $\nu$  satisfying  $\pi_1(\nu) = \pi_2(\nu) = \mu$ .*

*Proof.* We take as  $X$  a familiar counterexample, the “double-arrow” space, defined to be the set  $[0, 1] \times \{0, 1\}$  under the order topology derived from the lexicographic order. In this topology, sets of the forms

$$L_t = ([0, t) \times \{0, 1\}) \cup \{(t, 0)\} \text{ and}$$

$$R_t = ((t, 1] \times \{0, 1\}) \cup \{(t, 1)\}$$

are open. It can be shown that Borel subsets of  $X$  have a very simple form: each such set may be expressed as  $(B \times \{0, 1\}) \Delta C$  with  $B$  a Borel subset of  $[0, 1]$  (for the usual topology) and  $C$  a countable subset of  $X$ . The measure  $\mu$  which assigns to such a set the Lebesgue measure of  $B$  is a regular Borel probability measure on  $X$ .

We define the following sets:

$$G = \{((s, \delta), (t, \epsilon)) \in X \times X : s < t \text{ or } s = t \text{ and } \delta = 0, \epsilon = 1\};$$

$$F = \{((s, \delta), (t, \epsilon)) \in X \times X : s < t\};$$

$$E = G \setminus F = \{((t, 0), (t, 1)) : t \in [0, 1]\}.$$

Since  $G = \bigcup_{u \in [0, 1]} L_u \times R_u$  and  $F = \bigcup_{u < v} L_u \times R_v$ , both  $G$  and  $F$  are open, so that  $E$  is Borel. However, the topology on  $E$  induced by the product topology of  $X \times X$  is discrete, since the intersection of  $E$  with the open set  $L_t \times R_t$  is the singleton  $\{((t, 0), (t, 1))\}$ . Thus  $E$  is a null set for any atomless regular Borel measure on  $X \times X$ . To see, on the other hand, that  $E$  is not marginally null is easy using the special form of Borel sets in  $X$  already mentioned above.  $\square$

We shall next look at the question of when the supremum in the definition of  $\alpha(E)$  is attained, that is to say, when there exists a bistochastic measure  $\nu$  with  $\nu(E) = \alpha(E)$ . Even with the nicest of measures  $\lambda, \mu$ , this is not generally the case. (As remarked in [7], we may consider Lebesgue measure on  $[0, 1]$  and the subset  $E = \{(s, t) : s < t\}$ .) We shall establish a positive result for a suitable class of sets by borrowing and generalizing ideas of Sudakov.

We write  $\mathfrak{M}_t^+(X)$  for the cone of all finite Radon measures on the Hausdorff space  $X$  and recall [8, or 6] that the *narrow* topology on this cone is defined to be the coarsest topology with respect to which all the maps  $\mu \mapsto \mu(F)$ , with  $F$  closed,

are upper semicontinuous, while all the maps  $\mu \mapsto \mu(G)$ , with  $G$  open, are lower semicontinuous. In the case of a completely regular space  $X$ , this is the same as the topology induced by embedding  $\mathfrak{M}_t^+(X)$  in the weak\* dual of the space  $\mathcal{C}_b(X)$  of all bounded continuous functions on  $X$ . The most important result about the narrow topology is the Prohorov compactness criterion: a bounded subset  $\mathfrak{A}$  is relatively compact in  $\mathfrak{M}_t^+(X)$  provided, for every  $\epsilon > 0$ , there is a compact subset  $K$  of  $X$  such that  $\sup_{\mu \in \mathfrak{A}} \mu(X \setminus K) < \epsilon$ .

**Lemma 4.** *Let  $X$  and  $Y$  be Hausdorff spaces and let  $\lambda$  and  $\mu$  be finite Radon measures on  $X$  and  $Y$  respectively. Let  $\mathfrak{A}$  be the set of all  $\nu \in \mathfrak{M}_t^+(X \times Y)$  such that  $\pi_1(\nu) \leq \lambda$  and  $\pi_2(\nu) \leq \mu$ . Then  $\mathfrak{A}$  is narrowly compact.*

*Proof.* Since  $\nu(X \times Y) \leq \lambda(X)$  for all  $\nu \in \mathfrak{A}$ , the set  $\mathfrak{A}$  is bounded. We may express  $\mathfrak{A}$  as

$$\mathfrak{A} = \{\nu \in \mathfrak{M}_t^+(X \times Y) : \nu(U \times Y) \leq \lambda(U) \text{ and } \nu(X \times V) \leq \mu(V) \\ \text{whenever } U \text{ and } V \text{ are open in } X \text{ and } Y \text{ respectively}\},$$

so that  $\mathfrak{A}$  is closed by the semicontinuity of the maps  $\nu \mapsto \nu(U \times Y)$ ,  $\nu \mapsto \nu(X \times V)$ . Finally, for  $\epsilon > 0$ , there are compact subsets  $I$  of  $X$  and  $J$  of  $Y$  such that  $\lambda(X \setminus I) < \frac{1}{2}\epsilon$  and  $\mu(Y \setminus J) < \frac{1}{2}\epsilon$ . Clearly,  $\nu((X \times Y) \setminus (I \times J)) < \epsilon$  for all  $\nu \in \mathfrak{A}$ , so that the Prohorov criterion is satisfied.  $\square$

**Lemma 5.** *Let  $X, Y, \lambda, \mu$  and  $\mathfrak{A}$  be as in Lemma 4. For any Borel sets  $A \subseteq X$ ,  $B \subseteq Y$ , the map  $\nu \mapsto \nu(A \times B)$  is continuous on  $\mathfrak{A}$ .*

*Proof.* By regularity, we can choose, for given  $\epsilon > 0$ , closed sets  $I \subseteq A$  and  $J \subseteq B$ , as well as open sets  $U \supseteq A$  and  $V \supseteq B$  such that  $\lambda(U \setminus I) < \frac{1}{2}\epsilon$ ,  $\mu(V \setminus J) < \frac{1}{2}\epsilon$ . For any  $\nu \in \mathfrak{A}$  we have

$$\nu(A \times B) - \epsilon \leq \nu(I \times J) \leq \nu(A \times B) \leq \nu(U \times V) \leq \nu(A \times B) + \epsilon.$$

Thus  $\nu \mapsto \nu(A \times B)$  is uniformly approximable on  $\mathfrak{A}$  both by lower and upper semicontinuous functions, and so is continuous.  $\square$

Our next proposition extends Theorem 6 of [7], which treats the case where  $X$  and  $Y$  are compact metric spaces and  $\beta(E) = \lambda(X) = \mu(Y)$ .

**Proposition.** *Let  $\lambda$  and  $\mu$  be finite Radon measures on the Hausdorff spaces  $X$  and  $Y$  and let  $E$  be a subset of  $X \times Y$  which is the complement of the union of countably many Borel rectangles  $A \times B$ . Then there is a Radon measure  $\nu$  on  $X \times Y$  with  $\pi_1(\nu) \leq \lambda$ ,  $\pi_2(\nu) \leq \mu$  and  $\nu(E) = \beta(E)$ .*

*Proof.* Let  $E = (X \times Y) \setminus \bigcup_{n \in \mathbb{N}} A_n \times B_n$  with  $A_n \in \mathcal{B}(X)$  and  $B_n \in \mathcal{B}(Y)$ . Write  $\mathfrak{A}(E)$  for the set of Radon measures  $\nu$  on  $X \times Y$  such that  $\pi_1(\nu) \leq \lambda$ ,  $\pi_2(\nu) \leq \mu$  and  $\nu((X \times Y) \setminus E) = 0$ . By countable additivity,  $\nu((X \times Y) \setminus E) = 0$  if and only if  $\nu(A_n \times B_n) = 0$  for all  $n$ . Thus, by Lemma 5,  $\mathfrak{A}(E)$  is a closed subset of the set  $\mathfrak{A}$  and so is narrowly compact by Lemma 4. Hence the continuous function  $\nu \mapsto \nu(X \times Y)$  attains its supremum  $\alpha(E)$  on  $\mathfrak{A}(E)$ . Since  $E$  is in the product  $\sigma$ -algebra  $\mathcal{B}(X) \otimes \mathcal{B}(Y)$ , we also have  $\alpha(E) = \beta(E)$  by our earlier theorem.  $\square$

As we have just seen, sets like  $E$  in the above proposition behave in some ways like compact subsets of  $X \times Y$ . Introducing a piece of non-standard terminology, we

shall say that a subset  $F$  of the cartesian product of two measure spaces  $(X, \mathcal{F}, \lambda)$  and  $(Y, \mathcal{G}, \mu)$  is *pseudo-closed* if it can be expressed as  $(X \times Y) \setminus \bigcup_{n \in \mathbb{N}} A_n \times B_n$  with  $A_n \in \mathcal{F}$  and  $B_n \in \mathcal{G}$ . It is possible to develop the results presented in this paper in a way that makes less explicit use of topology, relying rather on the more abstract version of capacity theory presented in [4]. We shall not go into details but it turns out, for instance, that in the Corollary to Theorem 1 we may weaken the hypothesis that  $(X, \mathcal{F})$  and  $(Y, \mathcal{G})$  are standard Borel spaces, assuming only that the set-function  $\beta$  is upper semi-continuous on the pseudo-closed sets, in the sense that  $\lim_{n \rightarrow \infty} \beta(E_n) = \beta(E)$  whenever  $(E_n)$  is a decreasing sequence of pseudo-closed sets with intersection  $E$ . It is tempting to believe that such an approach might lead to results like the ones in this paper but valid for arbitrary measure spaces, with no special topological or other assumptions. Unfortunately, once we leave behind the well-behaved measure spaces that we have been looking at so far, we find that  $\beta$  need not be upper semi-continuous on pseudo-closed sets.

**Example 2.** *There exist probability spaces  $(X, \mathcal{F}, \lambda)$  and  $(Y, \mathcal{G}, \mu)$  and pseudo-closed subsets  $E_n$  ( $n \in \mathbb{N}$ ) of  $X \times Y$  such that  $\beta(E_n) = 1$  for all  $n$  but  $\bigcap_{n \in \mathbb{N}} E_n = \emptyset$ .*

*Proof.* Let  $X$  be a subset of  $[0, 1]$  with inner Lebesgue measure 0 and outer measure 1, let  $Y = [0, 1] \setminus X$  and let  $\lambda, \mu$  be the measures induced on  $X$  and  $Y$  respectively by Lebesgue measure. Let  $(q_m)_{m \in \mathbb{N}}$  be an enumeration of the rationals in  $(0, 1)$  and define

$$E_n = \{(x, y) \in X \times Y : \exists m \leq n \text{ with } x < q_m < y \text{ or } y < q_m < x\}.$$

We leave it to the reader to check that these sets have the properties claimed.  $\square$

**Open problem.** While Example 2 shows that our capacity-theoretic methods cannot be generalized very far, we do not know what generalizations of our main results are possible. In particular, we ask whether the Corollary to Theorem 1 is valid for arbitrary probability spaces  $(X, \mathcal{F}, \lambda)$  and  $(Y, \mathcal{G}, \mu)$ .

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