PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 124, Number 2, February 1996

A NOTE ON THE WEIGHTED NORM INEQUALITY FOR THE ONE-SIDED MAXIMAL OPERATOR

LAI QINSHENG

(Communicated by J. Marshall Ash)

ABSTRACT. Let M_g^+ be the one-sided maximal function. In this note we obtain some necessary and sufficient conditions in order that the weighted weak type inequality holds for M_g^+. Meanwhile, some necessary or sufficient conditions for the weighted inequality for M_g^+ are given.

1. INTRODUCTION

Given a positive and locally integrable function g on the real line R, the one-sided maximal function M_g^+ is defined by

$$M_g^+ f(x) = \sup_{h>0} \frac{1}{g(x, x+h)} \int_x^{x+h} |f(y)| g(y)\, dy \tag{1.1}$$

for $f \in L^1_{\text{loc}}(g(x)\, dx)$, where $g(x, x+h) = \int_x^{x+h} g(y)\, dy$.

Symmetrically, the left maximal function M_g^- can be defined in the obvious way.

Recently, weighted inequalities for the operators M_g^+ and M_g^- have been extensively studied (see [9], [6], [5], [4]). In this note we will characterize the pairs of weights (w, v) such that M_g^+ (or M_g^-) is of weak type $(1, q)$ with $1 \le q$. Secondly, we will give a necessary condition on the pairs (w, v) in order that the weak type (p, q) inequality with $0 < p \le q < \infty$ holds for M_g^+ (or M_g^-) with respect to the measures wdx and vdx. This result shows that most weight functions do not verify the weighted weak type (p, q) inequality when $1 < p < q < \infty$. In the case of $1 \le p$ and $0 < q < p$ we will introduce a $B_g^+(p, q)$ (or $B_g^-(p, q)$) condition and prove that this condition is sufficient for the weak type (p, q) inequality and necessary for the strong type (p, q) inequality for the operator M_g^+ (or M_g^-). Finally, we will prove analogies of Sawyer's theorem in [10] and Verbitsky's theorem in [11]. They characterize the weak type (p, q) inequality when $1 \le p < \infty$ and $0 < q < p$.

Throughout this paper, w and v always are weight functions by which we mean nonnegative measurable functions taking values in $[0, \infty]$. For a given weight w and measurable set $E, w(E) = \int_E w(x)\, dx$ and $\chi_E(x)$ denotes the characteristic function of E. Particularly, $|E|$ is the Lebesgue measure of E and $w((a, b))$ is written by $w(a, b)$. For $p \ge 1$, let $p' = \frac{p}{p-1}$ be its conjugate index, and we adopt the usual

Received by the editors January 28, 1993 and, in revised form, September 7, 1994.
1991 *Mathematics Subject Classification.* Primary 42B25.
Key words and phrases. One-sided maximal function, weighted inequality, weight.

conventions for multiplication in $[0,\infty]$, namely, $\infty\cdot t = t\cdot\infty = \infty$ for $0 < t \le \infty$, $0\cdot\infty = \infty\cdot 0 = 0, \infty^{-1} = 0$ and $0^{-1} = \infty$.

Now we state our results. We just present our theorems concerning M_g^+, since their counterparts for M_g^- can be obtained similarly.

Theorem 1. *Suppose $1 \le q < \infty$. Then the weak type $(1,q)$ inequality*

$$(\lambda^q w(\{x\in R\colon M_g^+ f(x) > \lambda\}))^{1/q} \le C\int_R |f(x)|v(x)\,dx \tag{1.2}$$

holds for all $\lambda > 0$ and measurable f, if and only if $(w,v)\in A_g^+(1,q)$, that is,

$$\sup_{a<b<c}\left(\frac{w(a,b)^{1/q}}{g(a,c)}\operatorname*{ess\,sup}_{x\in(b,c)}\left(\frac{g}{v}\right)\right) = A < \infty. \tag{1.3}$$

Moreover, for the best constants A and C, we have $A \le C \le 4A$.

Remark 1. The good weights for the weak type (p,p) inequalities ($p \ge 1$) were established in [6], [9] and [5]. However, in those articles the condition $A_g^+(1,1)$ was represented by

$$M_g^-\left(\frac{w}{g}\right)(x) \le C\frac{v(x)}{g(x)} \quad \text{for a.e. } x\in R. \tag{1.4}$$

Later on, in [4], we established the weak type inequalities for M_g^+ in Orlicz classes. In the particular case of $1 < p \le q < \infty$ we obtained the $A_g^+(p,q)$ condition, i.e.

$$\sup_{a<b<c}\frac{w(a,b)^{1/q}\sigma(b,c)^{1/p'}}{g(a,c)} = A < \infty,$$

where $\sigma = g^{p'1}v^{1-p'}$. Our $A_g^+(1,q)$ condition can be considered as the limiting case of $A_g^+(p,q)$ as $p\to 1$. Furthermore, it has an extended version in the case $0 < q < p = 1$ (cf. Theorem 4). This sort of $A_g^+(1,q)$ conditions for M_g^+ are new in the literature, although the parallel conditions for usual Hardy-Littlewood maximal operator are well known.

Theorem 2. *Let $0 < p \le q < \infty$ and (w,v) is a pair of weight functions. Suppose the weak type (p,q) inequality*

$$(\lambda^q w(\{x\in R\colon M_g^+ f(x) > \lambda\}))^{1/q} \le C\left(\int_R |f(x)|^p v(x)\,dx\right)^{1/p} \tag{1.5}$$

holds for all $\lambda > 0$ and measurable f. Then we have:

(i) *For a.e. $x\in R$, either $w(x) = 0$ or $v(x) = \infty$ if $p < q$.*

(ii) *There exists a constant B such that*

$$w(x) \le Bv(x) \quad \text{a.e. } x\in R \tag{1.6}$$

if $p = q$. Moreover, $B \le (2C)^p$ for the best constants B and C in (1.6) and (1.5) respectively.

Remark 2. Theorem 2 indicates that if v is finite a.e. (e.g. locally integrable) and $p < q$, then the weight w must be zero a.e. But, the $A_g^+(p,q)$ ($1 \le p \le q < \infty$) always contain nontrivial pairs of weight functions, i.e. neither $w(x)\equiv 0$ nor $v(x)\equiv\infty$. Here we shall give an example.

Example 1. Since g is locally integrable, there exists $M > 0$ such that

$$|E| = |\{x \in R\colon g(x) \le M\}| > 0.$$

Choose an interval (a_1, a_2) satisfying $|(a_1, a_2) \cap E| > 0$, and fix $a_3 < a_4 < a_1$ arbitrary. Let $w(x) = \chi_{[a_3,a_4]}(x), v(x) = 1$ for $x \in (a_1, a_2) \cap E$ and $v(x) = \infty$ elsewhere. Then $(w, v) \in A_g^+(p, q)$ for all $1 \le p \le q < \infty$.

Indeed, we only need to verify the $A_g^+(p, q)$ conditions for $a < b < c$ with $(a, b) \cap (a_3, a_4) \ne \varnothing$ and $(b, c) \cap (a_1, a_2) \ne \varnothing$. In these cases, it is obvious that $g(b, c) \ge g(a_4, a_1), w(a, b) \le a_4 - a_3, \sigma(b, c) \le M^{p'}(a_2 - a_1)(p > 1)$ and $\operatorname{ess\,sup}_{x\in(b,c)} \frac{g(x)}{v(x)} \le M(p = 1)$. Then the conclusion follows easily.

Definition 1. Suppose $1 \le p < \infty$ and $0 < q < p$. We say that a pair of weight functions $(w, v) \in B_g^+(p, q)$ if

$$\left[\sum_j \left(\frac{w(a_j, b_j)^{1/q}\sigma(b_j, c_j)^{1/p'}}{g(a_j, c_j)}\right)^r\right]^{1/r} \le B < \infty \quad (\text{when } p > 1), \tag{1.7}$$

$$\left[\sum_j \left(\frac{w(a_j, b_j)^{1/q}}{g(a_j, c_j)} \operatorname*{ess\,sup}_{x\in(b_j,c_j)} \left(\frac{g(x)}{v(x)}\right)\right)^r\right]^{1/r} \le B < \infty \quad (\text{when } p = 1) \tag{1.8}$$

for all sequences $\{(a_j, c_j)\}$ of pairwise disjoint intervals and $b_j \in (a_j, c_j)$. Where $\sigma = \sigma_{g,v,p} = g^{p'} v^{1-p'}$ and $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$. We shall keep these notations throughout this paper.

Theorem 3. *Suppose $1 \le p < \infty$, $0 < q < p$ and (w, v) is a pair of weight functions. If $(w, v) \in B_g^+(p, q)$, then the weak type (p, q) inequality (1.5) holds for all $\lambda > 0$ and measurable f. Moreover, for the best constants B in Definition 1 and C in (1.5) we have $C \le 2^{2+1/p'} B$.*

On the other hand, if the strong type (p, q) inequality

$$\left(\int_R [M_g^+ f(x)]^q w(x)\, dx\right)^{1/q} \le C \left(\int_R |f(x)|^p v(x)\, dx\right)^{1/p} \tag{1.9}$$

holds for all measurable f, then$(w, v) \in B_g^+(p, q)$ with $B \le C$.

Remark 3. We will give some examples of $B_g^+(p, q)$ functions, which shows that the $B_g^+(p, q)$ condition is not sufficient for the strong type inequality.

Example 2. Suppose $(u, v) \in A_g^+(p, p)(p \ge 1)$. Choose $\Omega \in \mathbf{R}$ with $u(\Omega) < \infty$. Let $w = u\chi_\Omega$. Then $(w, v) \in B_g^+(p, q)$ for all $0 < q < p$.

Example 3. Choose $\Omega \subset \mathbf{R}$ bounded. Let $w = g$, and $v = g$ on Ω and ∞ elsewhere. Then $(w, v) \in B_g^+(p, q)$ for all $1 \le q < p$. Particularly, setting $g = 1$ and $\Omega = [0, 1]$, we obtain a simple example which satisfies $B_g^+(p, 1)$ (therefore the weak type $(p, 1)$ inequality), but the function $f = \chi_\Omega$ causes the strong type $(p, 1)$ inequality to fail.

For $q > 1$, an example follows from the corresponding example given by Muckenhoupt in the case $p = q = 2$ (see [7], p. 218).

Example 4. Set $g = 1$. Let $w(x) = x \log(-x)$ on $[-\frac{1}{2}, 0)$ and 0 elsewhere, and $v(x) = x^2(\log(-x))^4$ on $[-\frac{1}{2}, 0)$ and ∞ elsewhere. Then $(w, v) \in B_g^+(3, 2)$, but the function $f(x) = -\chi_{[-\frac{1}{2},0)}(x)/(x \log^2(-x))$ violates the strong type $(3, 2)$ inequality.

Definition 2. Suppose $1 \le p < \infty$ and $0 < q < p$. For every $(a,b) \subset \mathbf{R}$, let

$$\rho_{(a,b)} = \sup_{c>b} \frac{w(a,b)^{1/p}\sigma(b,c)^{1/p'}}{g(a,c)} \qquad (p>1),$$

$$\rho_{(a,b)} = \sup_{c>b} \frac{w(a,b)}{g(a,c)} \operatorname*{ess\,sup}_{x\in(b,c)} \left(\frac{g(x)}{v(x)}\right) \qquad (p=1).$$

Define $\Phi(x) = \Phi_{p,q,w,v}(x) = \sup\{\rho_{(a,b)}\chi_{(a,b)}(x)\}$.

Theorem 4. *Suppose $1 \le p < \infty$ and $0 < q < p$. The following statements are equivalent.*

(i) *The weak type inequality* (1.5) *holds.*

(ii) *There exists a constant $B > 0$ such that for every sequence $\{(a_j,b_j)\}$ of pairwise disjoint (a_j,b_j) and $c_j > b_j$ arbitrary, the inequality*

$$w\left(\bigcup_j (a_j,b_j)\right)^{1/q} \le B\left\{\sum_j \left(\frac{g(a_j,c_j)}{\sigma(b_j,c_j)^{1/p'}}\right)^p\right\}^{1/p} \qquad (p>1)$$

or

$$w\left(\bigcup_j (a_j,b_j)\right)^{1/q} \le B\sum_j \frac{g(a_j,c_j)}{\operatorname{ess\,sup}_{x\in(b_j,c_j)}\left(\frac{g(x)}{v(x)}\right)} \qquad (p=1)$$

holds.

(iii) *The function $\Phi(x)$ defined in Definition 2 is in $L^{r,\infty}(w)$, i.e.*

$$\|\Phi\|_{r,\infty,w} = \sup_{\lambda>0} \lambda w(\{|\Phi(x)| > \lambda\})^{1/r} < \infty.$$

Moreover, for the best constants B and C we have

$$2^{-1/r}\|\Phi\|_{r,\infty,W} \le B \le C \le (\|\Phi\|^r_{r,\infty,W} + 4^p)^{1/r}.$$

In spite of Theorem 4, Theorem 3 is still worthwhile, because the $B_g^+(p,q)$ condition is more convenient in applications. For instance, it is not easy to test Examples 3 and 4 by condition (ii) or (iii).

Remark 4. The equivalence between (i) and (ii) holds even for arbitrary $0 < q < \infty$, and this is a Sawyer's type characterization for the one-sided maximal function (cf. [10]). The condition (iii) is a Verbitsky's type condition (cf. [11]). However, the case of $p = 1$ is new and the proof is new.

2. Proofs of the theorems

Proof of Theorem 1. *Necessity.* The argument is based on the following elementary fact about the weak type (p,q) inequality (1.5).

For every $a < b < c$ and measurable set $S \subset (b,c)$, it follows from the weak type inequality (1.5) easily that

$$\frac{g(S)}{g(a,c)} w(a,b)^{1/q} \le Cv(S)^{1/p}, \tag{2.1}$$

on setting $f(x) = \chi_S(x)$.

Given $-\infty < a < b < c < \infty$, without loss of generality, we may assume $0 < \operatorname{ess\,sup}_{x\in(b,c)}(\frac{g}{v}) < \infty$. Then $0 < \alpha_{g,v,(b,c)} = \alpha = \operatorname{ess\,inf}_{x\in(b,c)}(\frac{v}{g}) < \infty$. Let $\lambda > \alpha$. It is obvious that

$$|S_\lambda| = \left|\left\{x \in (b,c)\colon \frac{v(x)}{g(x)} < \lambda\right\}\right| > 0.$$

It follows from the inequality (2.1) that

$$\frac{w(a,b)^{1/q}}{g(a,c)} \le C\frac{v(S_\lambda)}{g(S_\lambda)} < C\lambda.$$

Letting $\lambda \to \alpha$, we get

$$\frac{w(a,b)^{1/q}}{g(a,c)} \le C\alpha. \tag{2.2}$$

Sufficiency. We may assume that $f \ge 0$ is bounded and has compact support. Let $\lambda > 0$ and $\Omega_\lambda = \{x\colon M_g^+ f(x) > \lambda\}$. It is well known that (see [6]) $\Omega_\lambda = \bigcup_j (a_j, b_j)$ where the intervals (a_j, b_j) are bounded pairwise disjoint and

$$\lambda \le \frac{1}{g(x,b_j)}\int_x^{b_j} f(x)g(x)\,dx \quad \text{for every } x \in [a_j, b_j). \tag{2.3}$$

Following the idea used in [5], we set $x_j^0 = a_j$ and $x_j^k \in (a_j, b_j)$ such that

$$\int_{x_j^k}^{b_j} f(x)g(x)\,dx = \frac{1}{2^k}\int_{a_j}^{b_j} f(x)g(x)\,dx \qquad (k = 1, 2, \dots).$$

Then $\{x_j^k\}_k$ is increasing, $\lim_{k\to\infty} x_j^k = b_j$ and

$$\lambda \le \frac{4}{g(x_j^k, x_j^{k+2})}\int_{x_j^{k+1}}^{x_j^{k+2}} f(x)g(x)\,dx. \tag{2.4}$$

Then we have

$$\begin{aligned}
\lambda^q w(\Omega_\lambda) &= \sum_j \sum_{k=0}^{\infty} \lambda^q w(x_j^k, x_j^{k+1}) \\
&\le 4^q \sum_j \sum_{k=0}^{\infty} \frac{w(x_j^k, x_j^{k+1})}{g(x_j^k, x_j^{k+2})^q}\left(\int_{x_j^{k+1}}^{x_j^{k+2}} fg\right)^q \qquad \text{(by (2.4))} \\
&\le 4^q \sum_j \sum_{k=0}^{\infty} \frac{w(x_j^k, x_j^{k+1})}{g(x_j^k, x_j^{k+2})^q}\left(\operatorname*{ess\,sup}_{x\in(x_j^{k+1}, x_j^{k+2})}\left(\frac{g}{v}\right)\right)^q \left(\int_{x_j^{k+1}}^{x_j^{k+2}} fv\right)^q \\
&\le (4A)^q \sum_j \sum_{k=0}^{\infty}\left(\int_{x_j^{k+1}}^{x_j^{k+2}} fv\right)^q \qquad \text{(by } A_g^+(1,q) \text{ condition (1.3))} \\
&\le \left(4A\int f(x)v(x)\,dx\right)^q.
\end{aligned}$$

This completes the proof of Theorem 1. □

Remark 5. The necessity of Theorem 1 is still true when $0 < q < 1$. An immediate consequence is this: if the weak type $(1,q)$ inequality holds for M_g^+ with respect to (w,v) and $\operatorname{ess\,sup}_{(a,b)}(\frac{g}{v}) = \infty$, then $w(x) = 0$ a.e. $x \in (-\infty, a)$.

Proof of Theorem 2. The result is derived from the following propositions.

Proposition 1. *Suppose weak type* (p,q) *inequality* (1.5) *holds for given* $0<p$, $q<\infty$. *If* w *is not locally integrable, then there exists* $a\in[-\infty,\infty]$ *such that* w *is locally integrable on* $(-\infty,a)$ *and* $v(x)=\infty$ *a.e.* $x\in(a,\infty)$. *On the other hand, if* $v(x)=0$ *on* S *with* $|S|>0$, *then there exists* $b\in[-\infty,\infty]$ *such that* $w(x)=0$ *a.e.* $x\in(-\infty,b)$ *and* $v(x)>0$ *a.e.* $x\in(b,\infty)$.

Proposition 2. *Under the same conditions as those in Theorem* 2, *suppose* $v(x)>0$ *a.e. and* $w(x)$ *is locally integrable on an interval* (a,b). *Then we have*

(i) *For a.e.* $x\in(a,b)$, *either* $w(x)=0$ *or* $v(x)=\infty$ *if* $p<q$.
(ii) $w(x)\le(2C)^p v(x)$ *a.e.* $x\in(a,b)$ *if* $p=q$. (*Where* C *is the best constant in* (1.6).)

Proof of Proposition 1. Let

$$E=\{x\in R\colon \text{there exists } \delta>0 \text{ such that } w(x-\delta,x+\delta)<\infty\}$$

and put $a=\inf\{R\backslash E\}$. It is obvious that w is locally integrable on $(-\infty,a)$.

Meanwhile, for $a<c<d$ and measurable $S\subset(c,d)$ arbitrary, there exists $e\in R\backslash E$ with $a\le e<c$. It follows from the definition of $R\backslash E$ that $w(e-\delta,c)=\infty$ for any $\delta>0$. Hence the inequality (2.1) shows that $v(S)=\infty$, which implies $v(x)=\infty$ a.e. $x\in(a,\infty)$, since $S\subset(c,d)$ and $(c,d)\subset(a,\infty)$ are arbitrary.

In order to prove the rest, we set $\widetilde{E}=\{x\in R\colon v(x)=0\}$. Let

$$b=\operatorname{ess\,sup}(I(x)\chi_{\widetilde{E}}(x)),$$

where $I(x)=x$. For every $x<b$, we have $|\widetilde{E}\cap(x,b)|>0$. Then the inequality (2.1) implies $w(x_1,x)=0$ for all $x_1<x$. The required conclusion follows. □

Proof of Proposition 2. In order to produce a contradiction to (i) we assume that there exists a constant $M>0$ such that

$$|E|=\left|\left\{x\in(a,b)\colon w(x)>\frac{1}{M},v(x)<M\right\}\right|>0.$$

Write $v_E(x)=v(x)\chi_E(x)$, then v_E is locally integrable. Choose $x_0\in E$ being a Lebesgue point of $g(x),w(x),v_E(x)$ and a point of density of E.

Since x_0 is a point of density of E, for every positive integer n, there exists a $h_n>0$ such that

$$1-\frac{1}{n}<\frac{|E\cap(x_0-h_n,x_0+h_n)|}{2h_n}\le 1$$

and $h_n\to 0$ as $n\to\infty$. For these h_n we have

$$\begin{aligned}1-\frac{1}{n}<&\frac{|E\cap(x_0-h_n,x_0)|}{2h_n}+\frac{|E\cap(x_0,x_0+h_n)|}{2h_n}\\ \le&\frac{1}{2}+\frac{|E\cap(x_0,x_0+h_n)|}{2h_n},\end{aligned}$$

therefore

$$\frac{1}{2}-\frac{1}{n}<\frac{|E\cap(x_0,x_0+h_n)|}{2h_n}. \tag{2.5}$$

Let $S = E \cap (x_0, x_0 + h)$. It follows from (2.1) that

$$\frac{g(E \cap (x_0, x_0 + h_n))}{g(x_0 - h_n, x_0 + h_n)} w(x_0 - h_n, x_0)^{1/q} \le C v_E(x_0, x_0 + h_n)^{1/p}.$$

By use of (2.5) we have

$$\begin{aligned}\left(\frac{1}{2} - \frac{1}{n}\right) \frac{2h_n}{g(x_0 - h_n, x_0 + h_n)} \frac{g(E \cap (x_0, x_0 + h_n))}{|E \cap (x_0, x_0 + h_n)|} \left(\frac{w(x_0 - h_n, x_0)}{h_n}\right)^{1/q} \\ \le C h_n^{1/p - 1/q} \left(\frac{v_E(x_0, x_0 + h_n)}{h_n}\right)^{1/p}.\end{aligned}$$

Inequality (2.5) shows that the sequence $\{E \cap (x_0, x_0 + h_n)\}$ shrinks to x_0 nicely (see [8], p. 140). On taking $n \to \infty$, it follows from the Lebesgue differential theorem, more precisely its slightly generalized version (see [8], p. 141), that

$$\frac{1}{2} w(x_0) \le 0.$$

This contradicts the fact that $w(x_0) > \frac{1}{M}$.

Now we prove (ii). For contradiction we suppose

$$|S| = |\{x \in (a, b) \colon w(x) > (2C)^p v(x)\}| > 0.$$

Write $v_s(x) = v(x)\chi_s(x)$. Choose $x_0 \in S$ in the same way as that in the proof of (i). Then the same argument as above shows

$$\frac{1}{2} w(x_0)^{1/p} \le C v(x_0)^{1/p}.$$

This contradicts the fact that $(2C)^p v(x_0) < w(x_0)$. □

Remark 6. It follows from Proposition 1 that we may assume that $w(x)$ is locally integrable on R and $v(x) > 0$ a.e. $x \in R$ in advance, when we study the weak type or strong type inequality for the one-sided maximal operator.

Proof of Theorem 3. In order to prove the first part, without loss of generality we may assume that $f \ge 0$ is bounded and has compact support. For $\lambda > 0$, let $\Omega_\lambda, (a_j, b_j)$ and (x_j^k, x_j^{k+1}) be the same as those in the proof of the sufficiency of Theorem 1. Then we can write

$$\begin{aligned}\lambda^q w(\Omega_\lambda) &= \sum_j \sum_{k=0}^{\infty} \lambda^q w(x_j^k, x_j^{k+1}) \\ &= \sum_j \sum_{k=0}^{\infty} \lambda^q w(x_j^{2k}, x_j^{2k+1}) + \sum_j \sum_{k=0}^{\infty} \lambda^q w(x_j^{2k+1}, x_j^{2k+2}) \\ &= \mathrm{I} + \mathrm{II}, \quad \text{say.}\end{aligned} \tag{2.6}$$

Suppose $p > 1$. We shall only estimate term I in (2.6), and the rest is similar.

(2.7)

$$\begin{aligned}
\mathrm{I} \le & 4^q \sum_j \sum_{k=0}^{\infty} \frac{w(x_j^{2k}, x_j^{2k+1})}{g(x_j^{2k}, x_j^{2k+2})^q} \left(\int_{x_j^{2k+1}}^{x_j^{2k+2}} fg \right)^q \quad \text{(by (2.4))} \\
\le & 4^q \sum_j \sum_{k=0}^{\infty} \frac{w(x_j^{2k}, x_j^{2k+1})}{q(x_j^{2k}, x_j^{2k+2})^q} \left(\int_{x_j^{2k+1}}^{x_j^{2k+2}} f^p v \right)^{q/p} \sigma(x_j^{2k+1}, x_j^{2k+2})^{q/p'} \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Hölder's inequality)} \\
\le & 4^q \left[\sum_j \sum_{k=0}^{\infty} \left(\frac{w(x_j^{2k}, x_j^{2k+1})^{1/q} \sigma(x_j^{2k+1}, x_j^{2k+2})^{1/p'}}{g(x_j^{2k}, x_j^{2k+2})} \right)^r \right]^{q/r} \\
& \times \left(\sum_j \sum_{k=0}^{\infty} \int_{x_j^{2k+1}}^{x_j^{2k+2}} f^p v \right)^{p/q} \quad \text{(by Hölder's inequality again)} \\
\le & (4B)^q \left(\sum_j \sum_{k=0}^{\infty} \int_{x_j^{2k+1}}^{x_j^{2k+2}} f^p v \right)^{q/p}.
\end{aligned}$$

Thus we get

$$(\lambda^q w(\Omega_\lambda))^{1/q} \le 2^{1/p'} 4B \left(\int_R r^p v \right)^{1/p}.$$

With a few obvious changes (cf. the proof of the sufficiency of Theorem 1) the foregoing procedure is still available for the case of $p = 1$. The first part of Theorem 3 is proved.

Now we prove the rest of Theorem 3. Let $p > 1$. Given $\{(a_j, c_j)\}$ pairwise disjoint and $b_j \in (a_j, c_j)$, we may assume $0 < \sigma(b_j, c_j) < \infty$ for all j. Indeed, if $\sigma(b_j, c_j) = \infty$ for some j, which means $\int_R (\frac{g}{v} \chi_{(b_j, c_j)})^{p'} v \, dx = \infty$, then there exists f such that $\int_R |f(x)|^p v(x) \, dx$ and $\int_{b_j}^{c_j} fg = \infty$, which implies $M_g^+ f(x) = \infty$ on $(-\infty, b_j)$. Therefore the strong type (p, q) inequality holds only if $w(x) = 0$ a.e. $x \in (-\infty, b_j)$, and we have $w(a_j, b_j) = 0$. Let

$$f_j(x) = \left(\frac{w(a_j, b_j)^{1/q} \sigma(b_j, c_j)^{1/q'}}{g(a_j, c_j)} \right)^{r/p} \left(\frac{g(x)}{v(x)} \right)^{p'/p} \chi_{(b_j, c_j)}(x)$$

and $f_N(x) = \sum_{j=1}^N f_j(x)$ for given natural number N. Observe that

$$M_g^+ f(x) \ge \frac{1}{g(a_j, c_j)} \int_{b_j}^{c_j} fg \quad \text{for every } f \ge 0 \text{ and } x \in (a_j, b_j). \tag{2.8}$$

We have

$$\begin{aligned}
C^p \sum_{j=1}^{N} \left(\frac{w(a_j,b_j)^{1/q}\sigma(b_j,c_j)^{1/p'}}{g(a_j,c_j)} \right)^r &= C^p \int_R f_N(x)^p v(x)\,dx \\
&\geq \left(\int_R (M_g^+ f_N(x))^q w(x)\,dx \right)^{p/q} \\
&\geq \left(\sum_{j=1}^{N} \left[\frac{\int_{b_j}^{c_j} f_N g}{g(a_j,c_j)} \right]^q w(a_j,b_j) \right)^{p/q} \qquad \text{(by (2.8))}. \\
&= \left\{ \sum \frac{w(a_j,b_j)}{g(a_j,c_j)^q} \left[\left(\frac{w(a_j,b_j)^{1/q}\sigma(b_j,c_j)^{1/q'}}{g(a_j,c_j)} \right)^{r/p} \sigma(b_j,c_j) \right]^q \right\}^{p/q} \\
&= \left\{ \sum \left(\frac{w(a_j,b_j)^{1/q}\sigma(b_j,c_j)^{1/p'}}{g(a_j,c_j)} \right)^r \right\}^{p/q}.
\end{aligned}$$

This is

$$\left[\sum_{j=1}^{N} \left(\frac{w(a_j,b_j)^{1/q}\sigma(b_j,c_j)^{1/p'}}{g(a_j,c_j)} \right)^r \right]^{1/r} \leq C.$$

Letting $N \to \infty$, we get $(w,v) \in B_g^+(p,q)$ with $B \leq C$.

Suppose $p = 1$. Keeping Remark 5 in mind, we assume

$$0 < \beta_j = \operatorname{ess\,sup}_{(b_j,c_j)}\left(\frac{g}{v}\right) < \infty \quad \text{for every } j.$$

Given $\lambda > 1$, set $E_j = \{x \in (b_j,c_j)\colon \frac{g(x)}{v(x)} > \frac{\beta_j}{\lambda}\}$. Then $v(E_j) < \infty$ and $|E_j| > 0$ which implies $v(E_j) > 0$ (cf. Remark 6 we may assume $v(x) > 0$ a.e.). Put

$$f_j(x) = \left(\frac{w(a_j,b_j)^{1/q}}{g(a_j,c_j)} \beta_j \right)^r \frac{\chi_{E_j}(x)}{v(E_j)}$$

and $f_N(x) = \sum_{j=1}^{N} f_j(x)$. Then

$$\int_R f_N(x) v(x)\,dx = \sum_{j=1}^{N} \left(\frac{w(a_j,b_j)^{1/q}}{g(a_j,c_j)} \beta_j \right)^r$$

and

$$\begin{aligned}
\left[\frac{\int_{b_j}^{c_j} f_N g}{g(a_j,c_j)} \right]^q w(a_j,b_j) &= \frac{w(a_j,b_j)}{g(a_j,c_j)^q} \left[\left(\frac{w(a_j,b_j)^{1/q}}{g(a_j,c_j)} \beta_j \right)^r \frac{g(E_j)}{v(E_j)} \right]^q \\
&\geq \left(\frac{1}{\lambda} \right)^q \left[\left(\frac{w(a_j,b_j)^{1/q}}{g(a_j,c_j)} \beta_j \right)^r \frac{\beta_j}{g(a_j,c_j)} \right]^q w(a_j,b_j) \\
&\qquad \left(\text{since } g(E_j) > \frac{\beta_j}{\lambda} v(E_j) \right) \\
&= \left(\frac{1}{\lambda} \right)^q \left(\frac{w(a_j,b_j)^{1/q}}{g(a_j,c_j)} \beta_j \right)^r \quad \left(\text{since } \frac{1}{r} = \frac{1}{q} - 1 \right).
\end{aligned}$$

The argument similar to that used above shows

$$\left[\sum_{j=1}^{N}\left(\frac{w(a_j,b_j)^{1/q}}{g(a_j,c_j)}\beta_j\right)^r\right]^{1/r} \le \lambda C.$$

Then we obtain $(w,v) \in B_g^+(1,q)$ with $B \le C$. This completes the proof of Theorem 3. □

Proof of Theorem 4. We shall prove the theorem only in the case $p > 1$, since the necessary modifications for the case $p = 1$ are indicated in the proof of Theorems 1 and 3.

(i) ⇒ (ii) Similarly to the proof of Theorem 3, the condition $\sigma(a,b) = \infty$ together with the weak type inequality (1.5) implies that $w(x) = 0$ for a.e. $x \in (-\infty, b)$. Therefore we may assume that $0 < \sigma(b_j, c_j) < \infty$ for all j. Let

$$f = \left\{\sum_j \left(\frac{g(a_j,c_j)}{\sigma(b_j,c_j)}\right)^p \frac{\sigma(x)}{v(x)}\chi_{(b_j,c_j)}\right\}^{1/p}.$$

It follows from (2.8) that $M_g^+ f(x) \ge 1$ on $\bigcup(a_j,b_j)$ and the weak type inequality (1.5) yields

$$w\left(\bigcup(a_j,b_j)\right)^{p/q} \le C\int_R f^p v = C\sum_j\left(\frac{g(a_j,c_j)}{\sigma(b_j,c_j)^{1/p'}}\right)^p.$$

This is (ii) with $B \le C$.

(ii) ⇒ (iii) For $\lambda > 0$, write $E_\lambda = \{\Phi(x) > \lambda\}$. For each $x \in E_\lambda$ there exist $a_x < b_x < c_x$ such that $x \in (a_x, b_x)$ and $\frac{w(a_x,b_x)^{1/p}\sigma(b_x,c_x)^{1/p'}}{g(a_x,c_x)} > \lambda$. Then for arbitrary compact set $K \subset E_\lambda$, from $\{(a_x,b_x)\}_{x\in K}$ we can choose a finite subfamily of open intervals $\{I_i\}$, which still covers K. By use of the covering lemma in [3] (Lemma 4.4, p. 25), one can select a finite subsequence $\{(a_j,b_j)\}$ from $\{I_i\}$ so that $\{(a_j,b_j)\}$ are pairwise disjoint and $w(\bigcup_i I_i) \le 2\sum_j w(a_j,b_j)$. For the set $\{(a_j,b_j)\}$, we have

$$\begin{aligned} w\left(\bigcup_j(a_j,b_j)\right)^{p/q} &\le B^p\sum_j\left(\frac{g(a_j,c_j)}{\sigma(b_j,c_j)^{1/p'}}\right)^p \quad \text{(by (ii))}\\ &\le B^p\lambda^{-p}\sum_j w(a_j,b_j). \end{aligned}$$

Thus we get

$$w(K) \le w\left(\bigcup_i I_i\right) \le 2\sum_j w(a_j,b_j) \le 2B^r\lambda^{-r},$$

which implies $\|\Phi\|_{r,\infty,w} \le 2^{1/r}B$.

(iii) ⇒ (i) Let f, Ω_λ and $\{(x_j^k, x_j^{k+1})\}$ be the same as those in the proof of Theorem 1. In addition, we may assume $\int_R f^p v = 1$. Write

$$\begin{aligned} w(\Omega_\lambda) =& w(\Omega_\lambda \cap \{\Phi(x) > \lambda^{q/r}\}) + w(\Omega_\lambda \cap \{\Phi(x) \le \lambda^{q/r}\})\\ =& \mathrm{I} + \mathrm{II}, \quad \text{say.} \end{aligned} \tag{2.9}$$

For item I in (2.9) we have

$$\mathrm{I} \le w(\{\Phi(x) > \lambda^{q/r}\}) \le \|\Phi\|_{r,\infty,w}^{r}\lambda^{-q}. \tag{2.10}$$

To estimate II in (2.9), we consider (x_j^k, x_j^{k+1}) having non-empty intersection with $\{\Phi(x) \le \lambda^{q/r}\}$. It follows from (2.4) and Hölder's inequality that

$$\begin{aligned} w(x_j^k, x_j^{k+1})^{1/p}\lambda \le & 4\frac{w(x_j^k, x_j^{k+1})^{1/p}\sigma(x_j^{k+1}, x_j^{k+2})^{1/p'}}{g(x_j^k, x_j^{k+2})}\left(\int_{x_j^{k+1}}^{x_j^{k+2}} f^p v\right)^{1/p} \\ \le & 4\lambda^{q/r}\left(\int_{x_j^{k+1}}^{x_j^{k+2}} f^p v\right)^{1/p}, \end{aligned} \tag{2.11}$$

since $(x_j^k x_j^{k+1}) \cap \{\Phi(x) \le \lambda^{q/r}\} \ne \varnothing$. Raising (2.11) to the power of p and then summing it over j and k, we have

$$\mathrm{II} \le 4^p\lambda^{-q}. \tag{2.12}$$

Combining (2.10) and (2.12), we obtain (1.5) with $C \le (\|\Phi\|_{r,\infty,w}^{r} + 4^p)^{1/q}$. The proof of Theorem 4 is complete. □

Acknowledgment

The author wishes to express many thanks to Dr. J. R. Partington for a lot of helpful conversations and suggestions. Thanks are also due to the referee for informing me of Verbitsky's paper [11] and suggesting to me to consider Verbitsky's type condition.

References

1. R. R. Coifman and C. Fefferman, *Weighted norm inequalities for maximal functions and singular integrals*, Studia Math. **5** (1974), 241–250. MR **50:**10670
2. J. Garcia-Cuerva and J. L. Rubio de Francia, *Weighted norm inequalities and related topics*, North-Holland, Amsterdam, 1985. MR **87d:**42023
3. J. B. Garnett, *Bounded analytic functions*, Academic Press, New York, 1981. MR **83g:**30037
4. Lai Qinsheng, *Two weight mixed Φ-inequalities for the one-sided maximal function*, *preprint*.
5. F. J. Martin-Reyes, *New proofs of weighted inequalities for the one-sided Hardy-Littlewood maximal functions*, preprint. MR **93d:**42016
6. F. J. Martin-Reyes, P. Ortega Salvador, and A. De La Torre, *Weighted inequalities for one-sided maximal functions*, Trans. Amer. Math. Soc. **319** (1990), 517–534. MR **90j:**42047
7. B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. **165** (1972), 207–227. MR **45:**2461
8. W. Rudin, *Real and complex analysis*, 3rd ed., McGraw-Hill, New York, 1978.
9. E. Sawyer, *Weighted inequalities for the one-sided Hardy-Littlewood maximal functions*, Trans. Amer. Math. Soc. **297** (1986), 53–61. MR **87k:**42018
10. ———, *Weighted norm inequalities for fractional maximal operators*, CMS Conf. Proc., vol. 1 Amer. Math. Soc., Providence, RI, 1981 pp. 283–309. MR **83k:**42020a
11. I. E. Verbitsky, *Weighted norm inequalities for maximal operators and Pisier's theorem on factorization through $L^{p,\infty}$*, Integral Equations Operator Theory **15** (1992), 124–153. MR **93d:**42017

Department of Pure Mathematics, The University of Leeds, Leeds, LS2 9JT, England

Current address: Department of Mathematics & Statistics, University of Edinburgh, King's Buildings, Mayfield Road, Edinburgh EH9 3JZ, United Kingdom

E-mail address: pmt5lq@sun.leeds.ac.uk

E-mail address: qlai@maths.ed.ac.uk