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INVERSION FORMULA AND SINGULARITIES OF THE SOLUTION FOR THE BACKPROJECTION OPERATOR IN TOMOGRAPHY

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ABSTRACT. Let $R^*\mu := \int_{S^2} \mu(\alpha, \alpha \cdot x) d\alpha$, $x \in \mathbb{R}^n$, be the backprojection operator. The range of this operator as an operator on non-smooth functions $R^* : X := L_0^\infty(S^{n-1} \times \mathbb{R}) \to L^2_{\text{loc}}(\mathbb{R}^n)$ is described and formulas for $(R^*)^{-1}$ are derived. It is proved that the operator R^* is not injective on X but is injective on the subspace X_e of X which consists of even functions $\mu(\alpha, p) = \mu(-\alpha, -p)$. Singularities of the function $(R^*)^{-1}h$ are studied. Here h is a piecewise-smooth compactly supported function. Conditions for μ to have compact support are given. Some applications are considered.

I. INTRODUCTION

In applications one may be interested in calculating the integral $I := \int_D f(x)dx$ in terms of tomographic data, that is, the Radon transform $Rf := \hat{f}$ of f. For example, in medicine the integral I describes a property of a tumor and this property can be used in medical diagnostics. It is shown in this paper that calculation of I in terms of tomographic data is closely related to the properties of the solution to the equation $R^*\mu = h$ where R^* is the operator adjoint to R. The purpose of this paper is to study this equation: to prove existence of $(R^*)^{-1}$ on a subspace of even, not necessarily smooth, functions, to give formulas for the inverse operator $(R^*)^{-1}$, to study singularities of the solution μ in terms of the singularities of h, to give conditions for the solution μ to be compactly supported when h is compactly supported, and to give an example of analytical solution of the basic equation for the case when D is a ball and h is the characteristic function of this ball. Thus our results are of practical interest.

Let us introduce the basic notation. Let $X := L_0^\infty(S^{n-1} \times \mathbb{R})$, where S^{n-1} is the unit sphere in $\mathbb{R}^n$, L_0^∞ is the space of bounded and compactly supported functions, and X_e is the subspace of X which consists of even functions $\mu(\alpha, p) = \mu(-\alpha, -p)$. Our scheme is quite flexible and allows one to consider X as a space of distributions with compact support. An[4] example of such consideration is given in section III

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(formula (13)). This example is of practical interest. Define the backprojection operator

$$R^*\mu = \int_{S^{n-1}} \mu(\alpha, \alpha \cdot x) d\alpha, \quad R^* : X \to L^2_{\text{loc}}(\mathbb{R}^n),$$

and consider the equation

$$R^*\mu = h. \tag{1}$$

The operator R^* is the adjoint to the Radon transform operator R:

$$Rf := \hat{f}(\alpha, p) := \int_{\mathbb{R}^n} f(x)\delta(p - \alpha \cdot x) dx$$

where δ is the delta function. It is known that the operator R^*R is an integral operator in $L^2(\mathbb{R}^n)$ with the kernel $\text{const}|x-y|^{-1}$. A characterization of the range of R defined on the Schwartz space $\mathcal{S}$ of infinitely smooth and rapidly decaying (with all their derivatives) functions is given by I. Gelfand and M. Graev in 1960 [GGV] (see also [N]). No characterization of the range of R^* was known. Equation (1) is of interest in applications. Recently some new applications of (1) were suggested in [FWZ]. In [S] an inversion formula for R^* is given on the Schwartz's type space of infinitely smooth functions.

The results of this paper are:

1) a necessary and sufficient condition for h to be of the form (1) with $\mu \in X$ is given and analytic formulas for solving equation (1) are obtained for non-smooth h; usually (see [GGV], [He], [N] and [S], for example) the inversion formulas are discussed on classes of infinitely smooth functions;

2) the null-space of R^* is described and injectivity of the map $R^* : X_e \to L^2(\mathbb{R}^n)$ is proved;

3) singularities of the solution μ are described, these singularities depend on the dimension of the space;

4) conditions for the solution μ to have compact support are given, and some applications are demonstrated.

In particular, it is proved that if $x \in \mathbb{R}^n$, n is odd, $h(x)$ is compactly supported, $h = 0$ for $|x| > a$, and $\mu \in X_e$ solves (1), then $\mu(\alpha, p) = 0$ for $|p| > a$. It is also proved that if n is even, $h(x)$ is not identically zero and is compactly supported, and μ solves (1), then μ cannot be compactly supported. These results should be compared with the known results for the Radon transform: for both odd and even n the Radon transform of a sufficiently nice function f is compactly supported if and only if $\hat{f}$ is compactly supported. The "if" part is trivial, and the "only if" part is the known "hole theorem" (see e.g. [N]). Also, we prove that if $h(x)$ is a characteristic function of a compact domain with a smooth boundary and $n = 3$, then $\mu(\alpha, p)$ is not locally integrable: it is a distribution. In [R1-3], [RK], [RZ1-4] various aspects of the theory of singularities of the Radon transform are studied. The basic results of this paper are given in sections II and III.

II. The inversion formulas

Let us represent $\mu(\alpha, p)$ as

$$\mu(\alpha, p) = (2\pi)^{-1} \int_{-\infty}^{\infty} m(\alpha, \lambda) \exp(-i\lambda p) d\lambda := F^{-1}m. \tag{2}$$

Let $\mathcal{F}h := \tilde{h}(\xi) := \int_{\mathbb{R}^n} h(x)\exp(ix\cdot\xi)dx$ be the Fourier transform of h, $x\cdot\xi$ is the dot product. Take the Fourier transform of (1), using (2), and get

$$\tilde{h}(\xi) = (2\pi)^{-1}\int_{S^{n-1}} d\alpha \int_{-\infty}^{\infty} d\lambda m(\alpha,\lambda)\int_{\mathbb{R}^n} dx \exp[ix\cdot(\xi-\lambda\alpha)]. \tag{3}$$

This implies

$$\tilde{h}(\xi) = (2\pi)^{n-1}\int_{S^{n-1}} d\alpha \int_{-\infty}^{\infty} d\lambda m(\alpha,\lambda)\delta(\lambda\alpha-\xi) \tag{4}$$

where $\delta(\lambda\alpha-\xi)$ is the delta-function in $\mathbb{R}^n$, $\delta(\lambda\alpha-\xi) = |\xi|^{-(n-1)}\delta(\lambda-|\xi|)\delta(\alpha-\xi^0)$, where $\xi := |\xi|\xi^0$. Thus (4) yields

$$(2\pi)^{n-1}[m(\xi^0,|\xi|)+m(-\xi^0,-|\xi|)] = |\xi|^{n-1}\tilde{h}(\xi^0|\xi|). \tag{5}$$

It follows from (5) that the odd part of $m(\alpha,\lambda)$ cannot be recovered and one should look for the solution to (5) in the space of even functions $m(\alpha,\lambda)$. For even m equation (5) implies $m(\alpha,\lambda) = \gamma|\lambda|^{n-1}\tilde{h}(\lambda\alpha)$, where $\gamma := \frac{1}{2(2\pi)^{n-1}}$. Let $\alpha = \xi^0$. Then (5) and (2) imply

$$\mu(\alpha,p) = \frac{\gamma}{2\pi}\int_{-\infty}^{\infty} |\lambda|^{n-1}\tilde{h}(\lambda\alpha)\exp(-i\lambda p)d\lambda = \gamma F^{-1}|\lambda|^{n-1}\mathcal{F}h. \tag{6}$$

Denote

$$H(\alpha,p) := (2\pi)^{-1}\int_0^{\infty} \lambda^{n-1}\tilde{h}(\lambda\alpha)\exp(-i\lambda p)d\lambda. \tag{7}$$

Then (6) can be written as

$$\mu(\alpha,p) = \gamma[H(\alpha,p)+H(-\alpha,-p)]. \tag{8}$$

Here we took into account the equation

$$\begin{aligned}\int_{-\infty}^{0} |\lambda|^{n-1}\tilde{h}(-|\lambda|\alpha)\exp(-i\lambda p)d\lambda &= \int_0^{\infty} \lambda^{n-1}\tilde{h}(\lambda(-\alpha))\exp[-i\lambda(-p)]d\lambda \\ &= 2\pi H(-\alpha,-p).\end{aligned} \tag{9}$$

Equation (8) implies that (1) has infinitely many solutions in X: If $\mu\in X$ is a solution to (1) and $\psi(\alpha,p) = -\psi(-\alpha,-p)\in X$, then $\mu+\psi$ solves (1).

If $\mu\in X_e$ solves (1), then (8) implies

$$\mu(\alpha,p) = \gamma[H(\alpha,p)+H(-\alpha,-p)] = \frac{\gamma}{2\pi}\int_{-\infty}^{\infty} |\lambda|^{n-1}\tilde{h}(\lambda\alpha)\exp(-i\lambda p)d\lambda. \tag{10}$$

Thus there exists a solution to (1) $\mu\in X_e$ iff the integral in (10) belongs to X. Note that this integral is an even function of (α,p) and thus it belongs to X_e if it belongs to X. It follows from (10) that the solution to (1) is unique in X_e. The integrals in (2)-(10) are understood in the sense of distribution theory. One could rewrite

the given proof using the space of tempered distributions and the Schwartz space of test functions ϕ. Let [,] denote the inner product in $L^2(S^{n-1}\times\mathbb{R})$. Equation (1) implies:

$$(2\pi)^{n-1}[m,\tilde{\phi}] = (2\pi)^{n-1}[m,FR\phi] = (2\pi)^n[\mu,R\phi] = (2\pi)^n(R^*\mu,\phi) = (2\pi)^n(h,\phi) = (\tilde{h},\tilde{\phi}),$$

where the Fourier slice theorem and the definition of the Fourier transform of distributions via Parseval's equality were used. If one writes the inner product on the right of this chain of equalities in the spherical coordinates and transforms the integral on the left over the whole axis with respect to the variable λ to the integral over the positive semiaxis, then one obtains equation (5). The above calculations are valid for tempered distributions h and μ. We have proved the following result:

Theorem 1. *Equation* (1) *has a solution $\mu\in X$ iff the integral in* (10) *belongs to X. The solution to* (1) *is unique in X_e and can be calculated by formula* (8). *The general solution to* (1) *in X is given by the formula*

$$M(\alpha,p) = Q(\alpha,p) + \Psi(\alpha,p), \tag{11}$$

where $\Psi(\alpha,p) = -\Psi(-\alpha,-p)\in X$ is an arbitrary function and Q is the integral in (10). *The null-space of the operator R^* on a functional space X consists of the subspace X_0 of all odd functions belonging to X and the range of R^* consists of all $h(x)$ for which*

$$\int_{-\infty}^{\infty}\tilde{h}(\lambda\alpha)\exp(-i\lambda p)|\lambda|^{n-1}d\lambda\in X. \tag{12}$$

Corollary. *The singular support S_μ of the solution to* (1) *can be calculated from the singular support S_h of h as follows: assume S_h to be a hypersurface and let $x_n = g(x')$ be its equation in the local coordinates, $x'\in\mathbb{R}^{n-1}$. Let $q(\beta) = Lg(x')$, where $\beta\in\mathbb{R}^{n-1}$ and L is the Legendre transform. Then $q = q(\beta)$ is the equation of S_μ in the local coordinates (q,β).*

Proof. The function $\psi := \mathcal{F}^{-1}(|\lambda|^{n-1}\tilde{h})$ has the same singular support as h since the map $h\to\psi$ is an elliptic PDO [H]. Formula (10) shows that

$$\mu = \gamma F^{-1}\mathcal{F}\psi = \gamma\hat{\psi}.$$

Thus, $\text{singsupp}\mu = \text{singsupp}\hat{\psi}$. This and Theorem 2 in [RZ1] yield the Corollary.

Remark 1. We have chosen X as one of the possible spaces on which R^* is defined. However, our argument is applicable to many other spaces and yields basically the conclusion that μ belongs to X if the given function h generates the integral in (10) which belongs to the space X. Note that the function (12) is in one-to-one correspondence with the data $h(x)$.

Remark 2. One can give an operator-theoretical derivation of a formula for the inverse operator $(R^*)^{-1}$. This formula has the following advantages: it is short,

general and can be used for theoretical analyses of the properties of the solution μ, in particular, the nature of the singularities of the solution. It requires some knowledge of the properties of the Radon transform and, in particular, it uses a formula for R^{-1}. Namely, assuming n odd and denoting $\nu := (n-1)/2$, one has:

$$(R^*)^{-1} = (R^{-1})^* = [c_n^o(\Delta)^\nu R^*]^* = c_n^o R(\Delta)^\nu.$$

Here R is the Radon transform, and we used the known formula $R^{-1} = c_n^o(\Delta)^\nu R^*$, where Δ is the Laplacian in $\mathbb{R}^n$ and $c_n^o := (-1)^\nu \gamma$. In particular, $c_3^o = -(8\pi^2)^{-1}$.

Thus, the solution of equation (1) in X_e which is given by formula (10) can be written, for odd n, also in the form:

$$\mu = c_n^o R(\Delta)^\nu h = c_n^o \hat{h}_p^{(n-1)}, \quad n = 2\nu + 1, \quad \nu > 0, \tag{10'}$$

while for even n the formula is:

$$\mu = c_n^e \mathcal{H} \hat{h}_p^{(n-1)}, \quad n = 2\nu, \quad \nu > 0, \tag{10''}$$

where $\mathcal{H}$ is the Hilbert transform, $\mathcal{H}g := (\pi)^{-1} \int_{-\infty}^{\infty} (p-q)^{-1} g(p) dp$, and $\hat{h}_p^{(n-1)}$ denotes the $(n-1)$th derivative of $\hat{h}$ with respect to p, $c_n^o = \gamma i^{n-1}$ for $n > 1$ odd and $c_n^e = \gamma i^n$ for $n > 0$ even. In particular, $c_2^e = -(4\pi)^{-1}$. The reason for the difference in formulas (10′) and (10″) can be seen if one recalls that the inversion formula for the Radon transform contains the non-local Hilbert transform in place of a derivative with respect to p, which is a local operation. Note that the Hilbert transform preserves the singular support of $\hat{h}$, as the differentiation with respect to p does. Therefore formulas (10′) and (10″) can be used for a description of the singular support and the nature of singularities of the solution to equation (1). Namely, the singular support of μ is the same as that of $\hat{h}$, and the relation between the singular support of $\hat{h}$ and that of h is described in detail in the papers [RZ] and [R1]. In particular, in these papers it is proved that if S is the singular support of a piecewise-smooth function h, then the singular support of its Radon transform $\hat{h}$ is a variety $\hat{S}$ dual to S in the sense defined in [RZ]. One can also analyze the nature of singularities of μ using formulas (10′) or (10″). An example of such an analysis is given in section III . Note that formal derivations, of the type given in Remark 2, may lead to errors. For example, the Radon transform on the Schwartz class of test functions is injective and satisfies the identity

$$R = F^{-1}\mathcal{F}, \tag{13}$$

which is equivalent to the Fourier slice theorem. Although both F^{-1} and $\mathcal{F}$ can be continuously extended to the space $\mathcal{D}'$ of distributions, and both are injective, the classically understood operator R is not injective on $\mathcal{D}'$ (see [Z]). Let us give the assumptions sufficient for a justification of the formulas (10′) and (10″). The assumption $h \in L_0^2(\mathbb{R}^n)$, where the subindex 0 stands for functions with compact support, is sufficient for (10′) and (10″) to hold. Then $\hat{h} \in H_0^{(n-1)/2}(Z)$ is the Sobolev space, $Z := \mathbb{S}^{n-1} \times \mathbb{R}$, and $\hat{h}_p^{(n-1)} \in H^{-(n-1)/2}(Z)$, where $H^{-\ell}$ is the space of distributions dual to H_0^ℓ. The Hilbert transform $\mathcal{H}$ is well defined on $H^{-\ell}$,

so that formulas (10′) and (10″) are well defined on $L_0^2(\mathbb{R}^n)$. The proof of Theorem 1 shows that these formulas give the even solution to (1).

Remark 3. Let us derive formulas (10′) and (10″) from formula (10). Let $n = 2\nu+1$. Note that $F^{-1}\mathcal{F} = R$. Formula (10) yields:

$$\begin{aligned}\mu = \gamma F^{-1}(|\lambda|^{n-1}\tilde{h}) = \\ \gamma F^{-1}\mathcal{F}[(-\Delta)^{\nu}h] = \gamma R[(\Delta)^{\nu}h] = \\ (-1)^{(n-1)/2}(1/2)(2\pi)^{-(n-1)}\hat{h}_p^{n-1} := \\ c_n^o\hat{h}_p^{(n-1)}, \quad n = 2\nu+1.\end{aligned}$$

This is formula (10′).

Let $n = 2\nu$. Note that $F(\frac{1}{p}) = i\pi\operatorname{sgn}\lambda$. By the convolution theorem for the Fourier transform one obtains:

$$\begin{aligned}F^{-1}[|\lambda|w(\lambda)] = i\frac{\partial}{\partial p}F^{-1}[(\operatorname{sgn}\lambda)w(\lambda)] = \\ i\frac{\partial}{\partial p}[\frac{1}{i\pi p} * W(p)] = \\ -\frac{\partial}{\partial p}\mathcal{H}W = -\mathcal{H}W_p\end{aligned}$$

where $W := W(p) := F^{-1}w(\lambda), W_p := \frac{\partial}{\partial p}W$, * denotes the convolution, the commutativity of $\mathcal{H}$ and $\frac{\partial}{\partial p}$ was used, and $\mathcal{H}$ was defined in Remark 2. Therefore

$$\begin{aligned}\mu = \gamma F^{-1}(|\lambda|^{2\nu-1}\tilde{h}) = \gamma F^{-1}|\lambda|\mathcal{F}[(-\Delta)^{\nu-1}h] = \\ -\gamma\frac{\partial}{\partial p}\mathcal{H}F^{-1}\mathcal{F}[(-\Delta)^{\nu-1}h] = (-1)^{\nu}\gamma\mathcal{H}\hat{h}_p^{n-1} = \\ c_n^e\mathcal{H}\hat{h}_p^{n-1}, \quad n = 2\nu.\end{aligned}$$

This is formula (10″).

III. Applications

In this section we prove the following results.

1) First, we prove that: a) if n is even and $h(x)$ is compactly supported, then equation (1) does not have a compactly supported solution μ, and b) if n is odd, μ is even and solves equation (1) with a compactly supported $h(x)$, $h(x) = 0$ for $|x| > a$, then $\mu(\alpha, p) = 0$ for $|p| > a$.

2) Secondly, we prove that equation (1) has no locally integrable solutions if $h(x)$ is the characteristic function of a bounded domain D with a smooth boundary. Moreover, we describe precisely the singularities of the solution μ to equation (1). As an illustative example of some independent interest we consider the case when D is a ball, in which case

$$h = \chi_D(x) := \begin{cases} 1, & |x| \le a, \\ 0, & |x| > a. \end{cases}$$

For this $h(x)$, $x \in \mathbb{R}^3$, we calculate an explicit analytical solution which is a distribution.

Let us state these results:

Theorem 2. *If n is even, then equation* (1) *with a compactly supported $h(x) \not\equiv 0$ cannot have a solution $\mu \in X_e$ which is compactly supported. If n is odd, $\mu(\alpha, p)$ is even and solves equation* (1) *with $h(x) = 0$ for $|x| > a$, then $\mu(\alpha, p) = 0$ for $|p| > a$.*

Theorem 3. *Let D be a bounded domain with a smooth convex boundary S whose Gaussian curvature is strictly positive, $h(x) = \chi_D(x)$, $\hat{h}(\alpha, p) := Rh$, $\operatorname{singsupp} h := \hat{S}$, and $(\alpha, q) \in \hat{S}$. Then the leading term in the expansion in smoothness of the solution to* (1) *as $p \to q$,α being fixed, is:*

$$\mu \sim c_n^o A_{2\nu+1} \nu! \delta^{(\nu)}(q-p) = \frac{-K^{1/2}}{2(2\pi)^\nu} \delta^{(\nu-1)}(p-q), \qquad n = 2\nu + 1, \tag{14}$$

(15)
$$\mu \sim c_n^e A_{2\nu} \mathcal{H} \frac{\partial^{2\nu-1}}{\partial p^{2\nu-1}} (q-p)_+^{\nu-0.5} = \frac{(-1)^\nu K^{1/2} (2\nu-3)!!}{(2\pi)^\nu 2^{\nu-1/2}} (p-q)_+^{1/2-\nu}, \qquad n = 2\nu,$$

where

$$x_+^\lambda := \begin{cases} x^\lambda, & x > 0 \\ 0, & x < 0, \end{cases}$$

$$A_{2\nu+1} := \frac{(2\pi)^\nu}{\nu!} K^{1/2}, \quad K := \rho_1 \cdots \rho_n, \tag{16}$$

ρ_j are the principal radii of curvature of S at the point $x(\alpha) \in S$ at which the plane $\alpha \cdot x(\alpha) = q$ is tangent to S, α is the exterior normal to S at $x(\alpha)$, and

$$A_{2\nu} := \frac{(2\pi)^{\nu-1/2}}{\Gamma(\nu + 1/2)} K^{1/2}. \tag{17}$$

Proof of Theorem 2. Let $\mu \in X_e$ solve equation (1). Then equation (10) yields:

$$\mu(\alpha, p) = \frac{\gamma}{2\pi} \int_{-\infty}^{\infty} |\lambda|^{n-1} \tilde{h}(\lambda\alpha) \exp(-i\lambda p) d\lambda.$$

Thus,

$$\eta(\alpha, \lambda) := \int_{-\infty}^{\infty} \mu(\alpha, p) \exp(ip\lambda) dp = \gamma |\lambda|^{n-1} \tilde{h}(\lambda\alpha). \tag{18}$$

If $\mu(\alpha, p)$ is compactly supported, then $\eta(\alpha, \lambda)$ is an entire function of λ for almost all $\alpha \in S^{n-1}$. Since $h(x)$ is compactly supported, the function $\tilde{h}(\lambda\alpha)$ is an entire function of λ for almost all α. Equation (18) shows that, for almost all α, the meromorphic function $\frac{\eta(\alpha,\lambda)}{\gamma \tilde{h}(\lambda\alpha)}$ of the variable λ equals to the function $|\lambda|^{n-1}$ which is not meromorphic if n is even. This contradiction proves that $\mu(\alpha, p)$ cannot be compactly supported.

Note that this result can also be derived from formula (10″) if one uses the following claim: if $g(\alpha, p)$ is a compactly supported function, then $\mathcal{H}g$ cannot be compactly supported unless $g = 0$. Here $\mathcal{H}$ is the Hilbert transform. A proof of this claim uses an argument similar to the one given below formula (18). Namely, if g

and $\mathcal{H}g$ are both compactly supported, then the ratio of their Fourier transforms is a meromorphic function which is proportional to the non-meromorphic function sgn λ.

If n is odd and $h(x) = 0$ for $|x| > a$, then $|\lambda|^{n-1}\tilde{h}(\lambda\alpha)$ is an entire function of λ of exponential type $\leq a$. By formula (10) and the well-known Paley-Wiener theorem [Ru, p. 407] it follows that $\mu(\alpha, p) = 0$ for $|p| > a$. Note that this result can also be obtained from formula (10′) . Indeed, if h is compactly supported, then $\hat{h}$ is also compactly supported, and formula (10′) shows that in this case μ is compactly supported. □

Remark 4. Suppose that $n = 2$, $\mu = Rf$ solves equation (1). Taking the Fourier transform of (1) yields:

$$\begin{aligned}\tilde{h}(\xi) &= \int_{\mathbb{R}^2} dx \exp(i\xi\cdot x)\int_{S^1} d\alpha \int_{\mathbb{R}^2} f(y)\delta(\alpha\cdot x - \alpha\cdot y)dy \\ &= \int_{\mathbb{R}^2} dy f(y)\int_{\mathbb{R}^2} dx \exp(i\xi\cdot x)\int_{S^1}\delta(\alpha\cdot\beta r)d\alpha\big|_{r=|x-y|=|z|,\beta:=\frac{x-y}{|z|}} \\ &= 2\int_{\mathbb{R}^2} dy f(y)\exp(i\xi\cdot y)\int_{\mathbb{R}^2} dz \exp(i\xi\cdot z)|z|^{-1} \\ &= 4\pi\tilde{f}(\xi)|\xi|^{-1}.\end{aligned} \tag{19}$$

Here we used the known formulas:

$$\int_{\mathbb{R}^2}\exp(i\xi\cdot z)|z|^{-1}dz = 2\pi|\xi|^{-1}, \quad \int_{S^1}\delta(\alpha\cdot\beta r)d\alpha = 2r^{-1}. \tag{20}$$

It follows from (19) that

$$|\xi| = 4\pi\tilde{f}(\xi)/\tilde{h}(\xi), \tag{21}$$

assuming that $\tilde{h}(\xi) \not\equiv 0$. If f and h are compactly supported, then the right-hand side of (21) is a meromorphic function of ξ, while the left-hand side is not.

This contradiction proves that if $h(x) \not\equiv 0$ is compactly supported and $\mu = Rf$ solves equation (1), then $f(x)$ cannot be compactly supported.

Note that if $x \in \mathbb{R}^3$, the above argument leads to the identity $\tilde{h}(\xi) = c|\xi|^{-2}\tilde{f}(\xi)$, $c = \text{const}$, and this identity can hold for $f(x)$ and $h(x)$ compactly supported. A similar argument shows that if n is even, equation (1) cannot have a solution $\mu = Rf$ with a compactly supported $f(x)$ if $h(x)$ is compactly supported, but in $\mathbb{R}^n$ with n odd, equation (1), with a compactly supported $h(x)$, may have a solution $\mu = Rf$ with a compactly supported f. A similar argument was used first in [R, p. 154].

Proof of Theorem 3. Let $n = 2\nu + 1$. Then, by (10′), $\mu = c^o_{2\nu+1}\hat{h}^{(2\nu)}_p$, and by [RZ1,p.113], $\hat{h}(\alpha, p) \sim A_{2\nu+1}(q-p)^{\nu}_+$ as $p \to q$, where $A_{2\nu+1}$ is given in (16). This implies (14).

Let $n = 2\nu$. Then, by (10″), $\mu = c^e_{2\nu}\mathcal{H}\hat{h}^{(2\nu-1)}_p$, and by [RZ1, p.113], $\hat{h}(\alpha, p) \sim A_{2\nu}(q-p)^{\nu-1/2}_+$. This implies (15) if one uses the commutativity of $\mathcal{H}$ and ∂_p and the equation $\mathcal{H}(q-p)^{-1/2}_+ \sim -(p-q)^{-1/2}_+$ as $p \to q$. □

Examples. Let $n = 3$ and D be a ball of radius a centered at the origin. If $h(x) = \chi_D(x)$, then one can check that $\tilde{h}(\xi) = \frac{4\pi}{\lambda^3}[\sin(\lambda a) - (\lambda a)\cos(\lambda a)]$, $\lambda := |\xi|$, and

$$H(\alpha, p) = H(p) := \int_0^\infty \frac{2}{\lambda}[\sin(\lambda a) - \lambda a \cos(\lambda a)] \exp(-i\lambda p) d\lambda.$$

Thus

$$\begin{aligned} \mu(\alpha, p) &= (4\pi^2)^{-1} \int_0^\infty [\sin(\lambda a) - \lambda a \cos(\lambda a)] \cos(\lambda p) \lambda^{-1} d\lambda \\ &= (4\pi)^{-1}[-a\delta(|p| - a) + \phi(p)], \quad \phi(p) := \begin{cases} 1, & |p| < a, \\ \frac{1}{2}, & p = a, \\ 0, & |p| > a, \end{cases} \end{aligned} \tag{22}$$

where δ is the delta-function. Here we used a formula [BE, 1.6.1]. It is clear from (22) that $\mu(\alpha, p)$ is a distribution, μ is not locally integrable, and $\mu = 0$ for $|p| > a$. Equation (22) can be derived from (10′). Indeed, the Radon transform of the characteristic function of the ball of radius a centered at the origin is $\hat{h} = \pi(a^2 - p^2)_+$. Using this formula, one obtains directly from (10′), with $n = 3$ and $c_3^o = -(8\pi^2)^{-1}$, formula (22). This gives an alternative way of derivation of formula (22). The nature of the singularities of the solution to equation (1) is similar for all strictly convex bounded domains D with a smooth boundary S, as follows from Theorem 3. Namely, the solution to (1), with h being the characteristic function of the domain D, has the following structure: if $n = 3$ and S is strictly convex, then, for all α, one has $\mu = c\delta(a - |p|) + \phi$ where $c = c(\alpha)$ and ϕ is a bounded function as p approaches a. More precisely, $c = -(4\pi K)^{-1}$, $a = a(\alpha)$ is the distance from the origin to the plane tangent to S and normal to α, and K is the Gaussian curvature of S at the point of tangency. If D is a ball of radius a centered at the origin, then the coefficient c is independent of α: it is equal to $-a/(4\pi)$ in agreement with formula (22). The nature of the singularity of the solution to (1) remains the same for h piecewise smooth with a jump across S. The jump will enter as a factor in the formula for c. For the characteristic function this factor is 1. Note that if $n = 2$, then the leading singularity of μ is different from that in the case $n = 3$.

Let $n = 2$. Using notation similar to the case when $n = 3$, one can describe the leading term of the behavior of the Radon transform of the characteristic function of a plane domain D with a smooth strictly convex boundary S. Let r denote the radius of curvature of S at the tangency point, corresponding to the chosen α. Then the leading term of the Radon transform of the characteristic function of the domain D with the boundary S, for this α and for p in a neighborhood of $a = a(\alpha)$, is $[8r(a - |p|)]^{1/2}$ for $|p| < a$ and is 0 for $|p| > a$. The corresponding μ is given by formula (10″) with $n = 2$. Since $\hat{h}' \sim -(2r)^{1/2}(a - p)^{-1/2}$ as $p \to a$ from the left, the leading singularity of μ as $p \to a$ is given by the term proportional to $\int_{-a}^{a} dq(q - p)^{-1}(a - q)^{-1/2}$, $\quad a > 0$, and this integral is bounded as $p \to a$, $\quad p < a$, if p is a real number [G, section 8.3]. If $p \to a$ from the right, that is $p > a$, then the above integral is no longer bounded: it is of order $(p - a)^{-1/2}$.

Remark 5. The following calculation is used in [FWZ]: if $Rf = \hat{f}$ and $R^*\mu = \chi_D$, then $\int_D f dx = (f, \chi_D) = (f, R^*\mu) = [\hat{f}, \mu]$, where $(f, h) := \int_{\mathbb{R}^3} f(x)h(x)dx$, $[\hat{f}, \mu] := \int_{-\infty}^{\infty} dp \int_{S^2} d\alpha \hat{f}(\alpha, p)\mu(\alpha, p)$ are inner products in the Hilbert spaces of

real-valued functions. This observation allows one to calculate $\int_D f dx$ knowing the tomographic data $\hat{f}(\alpha, p)$ and the solution to equation (1) with $h = \chi_D(x)$. This solution, as we have shown above, is a distribution. In [GGV, formula (I.2.1)] a formula for calculation of the integral of a function f over a bounded region D is given in terms of the Radon transforms of f and of the characteristic function of the domain D. This formula allows one to avoid solving equation (1) for the purpose of calculating the integral of f over the domain D in terms of tomographic data. Let us give the above formula in the case $n = 3$:

$$\int_D f(x)dx = -(8\pi^2)^{-1} \int_{S^2} d\alpha \int_{-\infty}^{\infty} dp S_{pp}(\alpha, p)\hat{f}(\alpha, p)$$

where $S(\alpha, p)$ is the area of the section of the body D by the plane $\alpha \cdot x = p$. In other words, $S(\alpha, p)$ is the Radon transform of the characteristic function of the domain D. Let us give a similar formula for even n, using $(10'')$:

$$\int_D f(x)dx = \int_{-\infty}^{\infty} dp \int_{S^{n-1}} d\alpha c_n^e \mathcal{H}\hat{\chi}_p^{(n-1)}(\alpha, p)\hat{f}(\alpha, p),$$

where $\chi := \chi_D$. In particular, if $n = 2$, then

$$\int_D f(x)dx = (4\pi^2)^{-1} \int_{S^1} d\alpha \int_{-\infty}^{\infty} dp \hat{f}(\alpha, p) \int_{-\infty}^{\infty} dq (p-q)^{-1} \hat{\chi}_q'(\alpha, q).$$

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