

**FACTORIZATION THEOREMS FOR HARDY SPACES  
OF THE BIDISC,  $0 < p \leq 1$**

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(Communicated by Palle E. T. Jorgensen)

ABSTRACT. A factorization theorem is proved in the Hardy spaces  $H^p$  of the bi-upper half plane,  $0 < p \leq 1$ . The proof is based on some fundamental work of Chang-Fefferman on atomic decompositions of  $H^p$ .

1. INTRODUCTION AND PRELIMINARIES

We are concerned with a factorization theorem which has been known (for  $p = 1$ ) for the unit disc in  $\mathbf{C}$  since the first part of the twentieth century. In 1976, Coifman, Rochberg and Weiss [4] extended it to the unit ball in  $\mathbf{C}^n$ . In 1992, Krantz and Li [6] proved that it holds on smoothly bounded strongly pseudoconvex domains for  $0 < p \leq 1$ . In this paper, we prove the analogous factorization theorem for the Hardy spaces  $H^p$  of the bi-upper half plane,  $0 < p \leq 1$ . A good reference for Hardy spaces is the book by Krantz [5, Chapter 8].

The following standard notation will be used:  $\mathbf{R}$  denotes the real numbers;  $\mathbf{C}$  denotes the complex numbers;  $x = (x_1, x_2, \dots, x_n)$  denotes an element of  $\mathbf{R}^n$ .

As a consequence of the boundedness of the Hilbert transform on  $L^q(\mathbf{R})$  for  $1 < q < \infty$  (see [8, p. 38]) it follows that  $S: L^q(\mathbf{R}) \rightarrow H^q(\mathbf{R}_+^2)$  is bounded, where  $S$  is the Szegő projection for the upper half plane:

$$Sf(z) = \frac{1}{\pi} \int_{\mathbf{R}} \frac{f(w)}{z-w} dw,$$

and  $S(z, w) = \frac{1}{\pi} \frac{1}{z-w}$  is the Szegő kernel on the upper half plane  $\mathbf{R}_+^2 := \{z = x + iy: x \in \mathbf{R}, y > 0\}$ .

It is a simple matter to extend this result to higher dimensions.

**Lemma 1.1.** *If  $q > 1$  and  $n \geq 1$ , then*

$$S: L^q(\mathbf{R}^n) \rightarrow H^q((\mathbf{R}_+^2)^n)$$

*is bounded.*

The following two lemmas will be useful.

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Received by the editors September 7, 1994.

1991 *Mathematics Subject Classification.* Primary 32A35, 42B30, 32A10, 32H10, 46E35; Secondary 30D55, 26A16.

**Lemma 1.2.** Let  $\psi \in L^2(\mathbf{R})$  with support  $\subseteq [-l, l]$  and, for  $z \in \mathbf{R}_+^2$  let

$$(1) \quad \phi(z) = \int_{\mathbf{R}} \psi(w)S(z, w)dw = \frac{1}{\pi} \int_{\mathbf{R}} \frac{\psi(w)}{z-w} dw.$$

Then  $\phi \in H^2(\mathbf{R})$  and (identifying  $\phi$  with its boundary values in  $L^2(\mathbf{R})$ )

- (1)  $\int |\phi(x)|^2 dx \leq \int |\psi(w)|^2 dw$ .  
 (2) If  $|x| > 2l$  and  $x \in \mathbf{R}$ , then

$$|\phi(x)| \leq c \frac{\|\psi\|_1}{|x|} \quad a.e.$$

- (3) If  $\int_{\mathbf{R}} \psi = 0$  and  $|x| > 2l$  and  $x \in \mathbf{R}$ , then

$$|\phi(x)| \leq \frac{cl\|\psi\|_1}{|x|^2} \quad a.e.$$

*Proof.* Part (1) of the lemma follows from the fact that  $\phi$  is the image of  $\psi$  under the Szegő projection of  $L^2(\mathbf{R})$  onto  $H^2(\mathbf{R})$ .

- (2) If  $|x| > 2l$ ,  $x \in \mathbf{R}$ , then for  $w \in [-l, l]$ ,

$$|x| \leq |x-w| + |w| \leq |x-w| + l.$$

Therefore,

$$|x-w| \geq |x| - l \geq |x| - \frac{|x|}{2} = \frac{|x|}{2},$$

and thus

$$|x-w|^{-1} \leq c|x|^{-1}.$$

So for a.e.  $x$ , we obtain from (1)

$$|\phi(x)| \leq c|x|^{-1} \int_{-l}^l |\psi(w)|dw \leq c|x|^{-1} \|\psi\|_1.$$

- (3) If  $|x| > 2l$ ,  $x \in \mathbf{R}$ ,  $\int \psi = 0$ , then for a.e.  $x$

$$\begin{aligned} \phi(x) &= c \int_{\mathbf{R}} \psi(w)S(x, w) dw \\ &= c \int_{\mathbf{R}} \psi(w)[S(x, w) - S(x, 0)] dw \\ &= c \int_{-l}^l \frac{\psi(w)w}{(x-w)x} dw. \end{aligned}$$

As in the argument of part (2), we see that

$$|\phi(x)| \leq cl|x|^{-2} \int_{-l}^l |\psi(w)|dw = cl|x|^{-2} \|\psi\|_1.$$

**Lemma 1.3.** If  $\alpha > 0$ , then

$$\int_{\mathbf{R} \setminus [-2l, 2l]} l^\alpha |x|^{-(1+\alpha)} dx = c_\alpha = \alpha^{-1} 2^{-(\alpha+1)}.$$

We shall conclude this section with a discussion of the atomic decomposition of Chang and Fefferman for  $H^p, 0 < p \leq 1$ .

In one variable (see [3], [7]) if  $f \in H^1(\mathbf{R}^1)$ , then  $f(x)$  can be written as

$$f(x) = \sum \lambda_k a_k(x)$$

where  $\sum |\lambda_k| \leq C\|f\|_{H^1}$  and  $a_k(x)$  are particularly simple functions called ‘‘atoms’’.

An analogous decomposition holds for functions  $f$  defined on  $\mathbf{R}^2$  which are boundary values of functions in  $H^1(\mathbf{R}_+^2 \times \mathbf{R}_+^2)$ , where  $\mathbf{R}_+^2$  is the upper half plane.

In what follows, we shall deal exclusively with the domain  $\mathbf{R}_+^2 \times \mathbf{R}_+^2$  and its Šilov boundary  $\mathbf{R}^2$ . A point of  $\mathbf{R}_+^2 \times \mathbf{R}_+^2$  will be denoted by  $z = (z_1, z_2)$  where  $z_j = x_j + iy_j$  and  $x_j \in \mathbf{R}, y_j > 0, j = 1, 2$ .

Now, we shall introduce some notation: Let  $\psi \in C^1(\mathbf{R})$  be supported on  $[-1, 1]$  with  $\psi$  even and  $\int_{-1}^1 \psi(x) dx = 0$ . If  $y > 0, \psi_y(x) = (1/y)\psi(x/y)$  and if  $y = (y_1, y_2)$  and  $x = (x_1, x_2) \in \mathbf{R}^2$ , then  $\psi_y(x) = \psi_{y_1}(x_1)\psi_{y_2}(x_2)$ . If  $f$  is a function defined on  $\mathbf{R}$ , then we define  $f(x, y) := f * \psi_y(x)$ ; if  $x = (x_1, x_2) \in \mathbf{R}^2$ , we denote  $\Gamma(x) := \Gamma(x_1) \times \Gamma(x_2)$ , where  $\Gamma(x_j) := \{(t_j, y_j) \in \mathbf{R}^2: |x_j - t_j| < y_j\}, j = 1, 2$ .

Given a function  $f$  on  $\mathbf{R}^2$ , we define its double Square function by

$$Q^2(f)(x) := \int \int_{\Gamma(x)} |f(t, y)|^2 \frac{dt dy}{y_1^2 y_2^2}.$$

It is a fact that for  $1 < p < \infty$

$$\|Q(f)\|_p \leq c_p \|f\|_p.$$

We may also define functions in  $H^p(\mathbf{R}_+^2 \times \mathbf{R}_+^2), 0 < p < \infty$ , as those functions  $f$  with  $Q(f) \in L^p(\mathbf{R}^2)$  and define  $\|f\|_{H^p} = \|Q(f)\|_p$ .

The following definition is due to Chang and Fefferman.

**Definition 1.4.** A  $p$ -atom ( $0 < p \leq 1$ ) is a function  $a(x), x = (x_1, x_2) \in \mathbf{R}^2$ , defined on  $\mathbf{R}^2$  whose support is contained in some open set  $\Omega$  of finite measure such that

- (1)  $\|a\|_{L^2} \leq |\Omega|^{1/2-1/p}$ .
- (2)  $a$  can be further decomposed into ‘‘elementary particles’’  $a_R$  as follows:
  - (a)  $a = \sum_{R \subset \Omega} a_R$ , where  $a_R$  is supported in a rectangle  $R \subseteq \Omega$  (say,  $R = I_1 \times I_2$ ) and the  $R$  in the sum have the property that no one  $R$  is contained in the triple of any other.
  - (b)  $\int_{I_1} a_R(x_1, x_2) x_1^k dx_1 = 0 = \int_{I_2} a_R(x_1, x_2) x_2^k dx_2$  for each  $x'_j \in I_j, j = 1, 2$ , and  $\forall k = 1, 2, \dots, k(p)$ , where  $k(p)$  is an integer depending on  $p, k(p) \leq [2/p - 3/2]$ .
  - (c)  $a_R$  is  $C^{k(p)+1}$  with

$$\begin{aligned} \|a_R\|_\infty &\leq c_R/|R|^{1/2}, \\ \|\partial^m a_R/\partial x_j^m\|_\infty &\leq c_R/|I_j|^m |R|^{1/2}, \quad j = 1, 2, \\ m &\leq k(p) + 1, \end{aligned}$$

and  $\sum_R c_R^2 \leq A|\Omega|^{1-2/p}$ , where  $A$  is an absolute constant.

With this definition of  $p$ -atom, Chang and Fefferman have shown the following result (see [1], [2]).

**Theorem 1.5.** *Let  $f \in H^p(\mathbf{R}_+^2 \times \mathbf{R}_+^2)$ ,  $0 < p \leq 1$ . Then (identifying  $f$  with its boundary values)  $f$  can be written as  $f = \sum \lambda_k a_k$  where  $a_k$  are  $p$ -atoms and  $\lambda_k \geq 0$  satisfy*

$$\sum \lambda_k^p \leq A_p \|f\|_{H^p}^p.$$

## 2. FACTORIZATION IN HARDY SPACE OF BI-UPPER HALF PLANE ( $0 < p \leq 1$ )

The main purpose of this paper is to study factorization theorems in Hardy spaces  $H^p$ ,  $0 < p \leq 1$ , on the bi-upper half plane. The case of the bidisc can then be easily obtained by use of the Cayley transform.

**Theorem 2.1** (Factorization Theorem for  $\mathbf{R}_+^2 \times \mathbf{R}_+^2$ ). *Let  $0 < p \leq 1$  and  $f \in H^p(\mathbf{R}_+^2 \times \mathbf{R}_+^2)$ . Then there exist  $g_j, h_j \in H^{2p}(\mathbf{R}_+^2 \times \mathbf{R}_+^2)$ , such that*

$$f = \sum_{j=1}^{\infty} g_j h_j$$

*in the sense of distributions, and  $\sum \|g_j\|_{H^{2p}} \|h_j\|_{H^{2p}} \approx \|f\|_{H^p}$ .*

Similarly to the idea given by Coifman, Rochberg and Weiss [4], with the aid of Theorem 1.5, the proof of Theorem 2.1 can be reduced to the following theorem, that is, it suffices to prove Theorem 2.1 for  $f = S(a)$ , where  $a$  is a  $p$ -atom.

**Theorem 2.2.** *Let  $a$  be a  $p$ -atom,  $0 < p \leq 1$ . There exist  $B_j, C_j \in H^{2p}(\mathbf{R}^2)$  such that (with  $A = S(a)$ )*

$$A = \sum_1^{\infty} B_j C_j$$

*(in the sense of distributions) and*

$$\sum_1^{\infty} \|B_j\|_{H^{2p}} \|C_j\|_{H^{2p}} \approx \|A\|_{H^p} \approx 1.$$

By Definition 1.4  $a$  can be written as follows:  $a = \sum_{R \in \mathcal{R}_a} a_R$ , where  $a_R$  is supported in a rectangle  $R \subseteq \Omega$  and the  $R$  in the sum have the property that no one  $R$  is contained in the triple of any other. Moreover,

$$\|a_R\|_{\infty} \leq c_R |R|^{-1/2}, \quad \sum_R c_R^2 \leq A |\Omega|^{1-2/p},$$

where  $A$  is an absolute constant.

Therefore, the proof of Theorem 2.2 is reduced to Theorem 2.3, as we shall show below.

**Theorem 2.3.** *Let  $a$  be a  $p$ -atom,  $0 < p \leq 1$ . Then for each  $R \in \mathcal{R}_a$ , there exist  $B_j, C_j \in H^{2p}(\mathbf{R}^2)$  such that*

$$A_R = \sum_1^4 B_j C_j, \quad \text{where } A_R = S(a_R)$$

*and*

$$\sum_1^4 \|B_j\|_{H^{2p}} \|C_j\|_{H^{2p}} \leq c c_R |R|^{1/p-1/2}.$$

*Proof.* By shift, without loss of generality, we may assume that  $R = l_1I \times l_2I$ , where  $I = [-1, 1]$ ,  $0 < l_j < \infty$  (in general,  $0 < l_j \ll 1$ ),  $j = 1, 2$ . Since

$$\begin{aligned} 1 &= \int_R \frac{1}{|R|} dw \\ &= \int_{l_1I} \frac{1}{2l_1} \int_{l_2I} \frac{1}{2l_2} dw_2 dw_1 \\ &= \prod_{j=1}^2 \int_{l_jI} \frac{1}{2l_j} \frac{1}{(z_j - w_j)} (z_j - w_j) dw_j \\ &= g_1(z) + g_2(z) - g_3(z) - g_4(z), \end{aligned}$$

where

$$\begin{aligned} g_1(z) &= z_1 z_2 \int_{l_1I} \frac{1}{4l_1 l_2} \int_{l_2I} \frac{dw_2 dw_1}{(z_1 - w_1)(z_2 - w_2)}, \\ g_2(z) &= \int_{l_1I} \int_{l_2I} \frac{w_1 w_2}{4l_1 l_2} \frac{dw_2 dw_1}{(z_1 - w_1)(z_2 - w_2)}, \\ g_3(z) &= \int_{l_1I} \int_{l_2I} \frac{w_1 z_2}{4l_1 l_2} \frac{dw_2 dw_1}{(z_1 - w_1)(z_2 - w_2)}, \\ g_4(z) &= \int_{l_1I} \int_{l_2I} \frac{z_1 w_2}{4l_1 l_2} \frac{dw_2 dw_1}{(z_1 - w_1)(z_2 - w_2)}. \end{aligned}$$

Thus

$$A_R(z) = A_R(z)g_1(z) + A_R(z)g_2(z) - A_R(z)g_3(z) - A_R(z)g_4(z).$$

We shall work on each  $A_R(z)g_j(z)$  to achieve the desired factorization as claimed in Theorem 2.3.

*Step 1.* We shall work on  $A_R(z)g_1(z)$ .

Let  $\eta$  be a number determined by

$$-1 + \frac{1}{2p} < \eta < -1 + \frac{1}{2p} + \varepsilon, \quad \text{with } 0 < \varepsilon \ll 1.$$

Let

$$\begin{aligned} B_1(z) &= z_1^{-\eta} z_2^{-\eta} \int_R \frac{dw_1 dw_2}{(z_1 - w_1)(z_2 - w_2)}; \\ C_1(z) &= z_1^{1+\eta} z_2^{1+\eta} A_R(z) |R|^{-1}. \end{aligned}$$

Then  $g_1(z)A_R(z) = B_1(z)C_1(z)$ .

We shall show first

$$(2) \quad \|B_1\|_{H^{2p}} \|C_1\|_{H^{2p}} \leq c c_R |R|^{1/p-1/2}.$$

**Lemma 2.4.**  $\int_{\mathbf{R}^2} |B_1(z)|^{2p} dz \leq c_p |R|^{1-2p\eta}$ .

*Proof.* *Case 1.*  $|z_1| > 2l_1, |z_2| > 2l_2$ .

Then (see the proof of Lemma 1.2)

$$|B_1(z)| \leq c |z_1|^{-(\eta+1)} |z_2|^{-(\eta+1)} |R|.$$

Thus, since  $2p(1 + \eta) > 1$

$$\begin{aligned} \int_{(\mathbf{R} \setminus 2l_1 I) \times (\mathbf{R} \setminus 2l_2 I)} |B_1(z)|^{2p} dz_1 dz_2 &\leq c^{2p} |R|^{2p} \int_{2l_1}^\infty \int_{2l_2}^\infty t^{-2p(1+\eta)} s^{-2p(1+\eta)} ds dt \\ &\leq c |R|^{2p} l_1^{-2p(1+\eta)+1} l_2^{-2p(1+\eta)+1} \\ &= c |R|^{1-2p\eta}. \end{aligned}$$

*Case 2.*  $|z_1| \leq 2l_1, |z_2| \leq 2l_2$ .

Let  $q = (1 + \eta_0)/\eta_0$  where  $\eta_0 > 0$  is chosen so that  $\eta < \eta_0$  and  $2p \frac{\eta}{\eta_0} (1 + \eta_0) < 1$ . Then  $q' := q/(q - 1) = 1 + \eta_0$ . Since  $\frac{\eta}{\eta_0} 2p(1 + \eta_0) < 1$  if and only if  $2p\eta(1 + \eta_0) < \eta_0$  if and only if  $2p\eta < \eta_0(1 - 2p\eta)$ , and since we know that  $0 < 1 - 2p\eta < 2p$ , we have  $\eta_0(1 - 2p\eta) < 2p\eta_0$ . So we may choose such  $\eta_0$ . By Hölder's inequality with exponent  $q$ , and by Lemma 1.1,

$$\begin{aligned} &\int_{2l_1 I \times 2l_2 I} |B_1(z)|^{2p} dz_1 dz_2 \\ &= \int_{2R} |z_1 z_2|^{-2p\eta} \left| \int_R \frac{dw_1 dw_2}{(z_1 - w_1)(z_2 - w_2)} \right|^{2p} dz_1 dz_2 \\ &\leq \left( \int_{2R} |z_1 z_2|^{-2p\eta \frac{(1+\eta_0)}{(1+\eta_0)-1}} dz \right)^{\frac{(1+\eta_0)-1}{(1+\eta_0)}} \\ &\quad \times \left( \int_{2R} \left| \int_R \frac{dw}{(z_1 - w_1)(z_2 - w_2)} \right|^{2p(1+\eta_0)} dz \right)^{\frac{1}{1+\eta_0}} \\ &\leq \left( \int_{2R} |z_1 z_2|^{-\frac{2p\eta(1+\eta_0)}{\eta_0}} dz \right)^{\frac{\eta_0}{1+\eta_0}} \left( \int_{2R} 1 dw \right)^{\frac{1}{1+\eta_0}}. \end{aligned}$$

Thus

$$\begin{aligned} \int_{2R} |B_1(z)|^{2p} dz_1 dz_2 &\leq c \left( |R|^{-\frac{2p\eta(1+\eta_0)}{\eta_0}} + 1 \right)^{\frac{\eta_0}{1+\eta_0}} |R|^{\frac{1}{1+\eta_0}} \\ &= c |R|^{\frac{-2p\eta(1+\eta_0)+\eta_0+1}{1+\eta_0}} = c |R|^{-2p\eta+1}. \end{aligned}$$

*Case 3.*  $|z_1| \leq 2l_1$  and  $|z_2| > 2l_2$ .

Using part 2 of Lemma 1.2 and Lemma 1.3, we have

$$\begin{aligned} &\int_{2l_1 I \times (\mathbf{R} \setminus 2l_2 I)} |B_1(z)|^{2p} dz_1 dz_2 \\ &= \int_{2l_1 I \times (\mathbf{R} \setminus 2l_2 I)} |z_1 z_2|^{-2p\eta} \left| \int_R \frac{dw}{(z_1 - w_1)(z_2 - w_2)} \right|^{2p} dz \\ &\leq \int_{\mathbf{R} \setminus 2l_2 I} |z_2|^{-2p\eta-2p} l_2^{2p} dz_2 \int_{2l_1 I} |z_1|^{-2p\eta} \left| \int_R \frac{dw_1}{(z_1 - w_1)} \right|^{2p} dz_1 \\ &\leq c_p l_2^{2p} l_2^{-2p(1+\eta)+1} \int_{2l_1 I} |z_1|^{-2p\eta} \left| \int_{l_1 I} \frac{dw_1}{(z_1 - w_1)} \right|^{2p} dz_1. \end{aligned}$$

By using the argument of Case 2 (but in one variable only), we have

$$\int_{2l_1 I} |z_1|^{-2p\eta} \left| \int_{l_1 I} \frac{dw_1}{(z_1 - w_1)} \right|^{2p} dz_1 \leq cl_1^{-2p\eta+1}.$$

Thus

$$\int_{2l_1 I \times (\mathbf{R} \setminus 2l_2 I)} |B_1(z)|^{2p} dz \leq c_p l_2^{-2p\eta+1} l_1^{-2p\eta+1} \leq c_p |R|^{-2p\eta+1}.$$

Case 4.  $|z_1| > 2l_1$  and  $|z_2| \leq 2l_2$ . By a similar argument to Case 3, we have

$$\int_{(\mathbf{R} \setminus 2l_1 I) \times 2l_2 I} |B_1(z)|^{2p} dz \leq c_p |R|^{-2p\eta+1}.$$

Therefore

$$\int_{\mathbf{R}^2} |B_1(z)|^{2p} dz \leq c_p |R|^{-2p\eta+1}.$$

Next we shall prove that

$$\int_{\mathbf{R}^2} |C_1(z)|^{2p} dz \leq c_p c_R^{2p} |R|^{2-p} |R|^{2p\eta-1} = c_p c_R^{2p} |R|^{1+2p\eta-p},$$

and this will complete the proof of (2). We state this as another lemma.

**Lemma 2.5.**

$$\int_{\mathbf{R}^2} |C_1(z)|^{2p} dz \leq c_p c_R^{2p} |R|^{1+2p\eta-p}.$$

*Proof. Case 1.*  $z_1, z_2 \in 2R$ .

Then

$$\begin{aligned} \int_{2R} |C_1(z)|^{2p} dz &= \int_{2R} |z_1 z_2|^{2p(1+\eta)} |A_R(z)|^{2p} dz_1 dz_2 |R|^{-2p} \\ &\leq c_p |R|^{2p(1+\eta)-2p} \int_{2R} |A_R(z)|^{2p} dz \\ &\leq c_p |R|^{2p\eta} \|a_R\|_2^{2p} |R|^{1-p} \\ &\leq c_p |R|^{2p\eta} c_R^{2p} |R|^{1-p} \\ &= c_p c_R^{2p} |R|^{2p\eta+1-p}. \end{aligned}$$

Case 2.  $|z_1| > 2l_1, |z_2| > 2l_2$ .

To prove this case, we need the following sublemmas. First let us introduce some notation. For  $z_j, w_j \in \mathbf{R}$  and  $j = 1, 2$ , let

$$S_j^* = S_j^*(z_j, w_j) := \sum_{m=0}^{k(p)} \frac{1}{m!} \frac{\partial^m S_j(z_j, 0)}{\partial w_j^m} w_j^m.$$

Note that  $S_j^*(z_j, w_j)$  is the  $k(p)$ th partial sum of the Taylor expansion of  $S_j^*(z_j, \cdot)$  at 0.

**Sublemma 2.6.** For a function  $\phi$ , let

$$\phi^*(z_j) = \int_{l_j I} [S_j(z_j, w_j) - S_j^*(z_j, w_j)] \phi(w_j) dw_j.$$

Then for  $|z_j| > 2l_j$ , we have

$$|\phi^*(z_1, w_2)| \leq cl_1^{k(p)+1} |z_1|^{-(k(p)+2)} \int_{l_1 I} |\phi(w_1)| dw_1.$$

In particular, if

$$M(z_2, w_1) = \int_{l_2 I} [S_2(z_2, w_2) - S_2^*(z_2, w_2)] a_R(w) dw_2,$$

then

$$|M(z_2, w_1)| \leq cl_2^{k(p)+1} |z_2|^{-(k(p)+2)} \int_{l_2 I} |a_R(w)| dw_2.$$

*Proof.* Since  $\forall m \in Z^+$ ,

$$\frac{\partial^m S_j(z_j, w_j)}{\partial w_j^m} = c(-1)^m m! (z_j - w_j)^{-(m+1)},$$

we have

$$S_j(z_j, w_j) - S_j^* = \frac{\partial^{k(p)+1} S_j(z_j, w'_j)}{\partial w_j^{k(p)+1}} z_j^{-(k(p)+2)} w_j^{-(k(p)+1)}$$

where  $w'_j$  lies between 0 and  $w_j$ .

Then

$$|S_j(z_j, w_j) - S_j^*| \leq c |z_j - w'_j|^{-(k(p)+2)} |w_j|^{-(k(p)+1)}.$$

*Claim.*  $|S_j(z_j, w_j) - S_j^*| \leq c |z_j|^{-(k(p)+2)} |w_j|^{-(k(p)+1)}$ .

To prove this we need:  $|z_j - w'_j|^{-(k(p)+2)} \leq c |z_j|^{-(k(p)+2)}$ , that is, we have to show  $|z_j - w'_j|^{-1} \leq c |z_j|^{-1}$ .

Since

$$\begin{aligned} |z_j| &= |z_j - w'_j + w'_j| \\ &\leq |z_j - w'_j| + |w'_j| \\ &\leq |z_j - w'_j| + l_j \quad (\text{because } |w'_j| \leq |w_j| \leq l_j), \end{aligned}$$

we obtain

$$\begin{aligned} |z_j - w'_j| &\geq |z_j| - l_j \\ &\geq |z_j| - \frac{1}{2}|z_j| \quad (\text{because } |z'_j| > 2l_j) \\ &= \frac{1}{2}|z_j|. \end{aligned}$$

Hence  $|z_j - w'_j|^{-1} \leq 2|z_j|^{-1}$ . Therefore the claim is proved.

Thus

$$\begin{aligned} |\phi^*(z_j)| &\leq c \int_{l_j I} |\phi(w_j)| |z_j|^{-(k(p)+2)} |w_j|^{k(p)+1} dw_j \\ &\leq cl_j^{k(p)+1} |z_j|^{-(k(p)+2)} \int_{l_j I} |\phi(w_j)| dw_j. \end{aligned}$$

**Sublemma 2.7.** *If  $|z_1| > 2l_1$  and  $|z_2| > 2l_2$ , then*

$$|A_R(z)| \leq cc_R |R|^{k(p)+3/2} |z_1 z_2|^{-(k(p)+2)}.$$

*Proof.* By property 2(b) of Definition 1.4,

$$\begin{aligned} A_R(z) &= \int_{l_1 I} S_1(z_1, w_1) \int_{l_2 I} S_2(z_2, w_2) a_R(w) dw_2 dw_1 \\ &= \int_{l_1 I} [S_1(z_1, w_1) - S_1^*] \int_{l_2 I} [S_2(z_2, w_2) - S_2^*] a_R(w) dw_2 dw_1 \\ &= \int_{l_1 I} [S_1(z_1, w_1) - S_1^*] M(z_2, w_1) dw_1. \end{aligned}$$

Sublemma 2.6 implies

$$\begin{aligned} |A_R(z)| &\leq c l_1^{k(p)+1} |z_1|^{-(k(p)+2)} \int_{l_1 I} |M(z_2, w_1)| dw_1 \\ &\leq c l_1^{k(p)+1} |z_1|^{-(k(p)+2)} l_2^{k(p)+1} |z_2|^{-(k(p)+2)} \int_{l_1 I} |a_R(w_1, w_2)| dw_1 dw_2 \\ &\leq c |R|^{k(p)+3} |z_1 z_2|^{-(k(p)+2)}. \end{aligned}$$

Now we return to the proof of Case 2.

$$\begin{aligned} &\int_{(\mathbf{R} \setminus 2l_1 I) \times (\mathbf{R} \setminus 2l_2 I)} |C_1(z)|^{2p} dz \\ &= \int_{(\mathbf{R} \setminus 2l_1 I) \times (\mathbf{R} \setminus 2l_2 I)} |R|^{-2p} |z_1 z_2|^{-2p} |z_1 z_2|^{2p(1+\eta)} |A_R(z)|^{2p} dz_1 dz_2 \\ &\leq c c_R^{2p} \int_{(\mathbf{R} \setminus 2l_1 I) \times (\mathbf{R} \setminus 2l_2 I)} |R|^{2p(k(p)+1/2)} |z_1 z_2|^{2p(1+\eta)-2p(k(p)+2)} dz_1 dz_2 \\ &= c c_R^{2p} |R|^{2p(k(p)+1/2)} \int_{(\mathbf{R} \setminus 2l_1 I) \times (\mathbf{R} \setminus 2l_2 I)} |z_1 z_2|^{-2p(k(p)+1-\eta)} dz_1 dz_2. \end{aligned}$$

Note that

$$\begin{aligned} 2p(k(p) + 1 - \eta) &= 2pk(p) + 2p - 2p\eta \\ &\geq 2p(2/p - 3/2 - 1) + 2p - 2p\eta \\ &= 4 - 3p + 2p - 2p\eta \\ &> 4 - p - (-2p + 1 + \varepsilon) \\ &= 4 + p - 1 - \varepsilon \\ &= 3 + p - \varepsilon > 1. \end{aligned}$$

Thus, by Lemma 1.3,

$$\int_{\mathbf{R} \setminus 2l_1 I} \int_{\mathbf{R} \setminus 2l_2 I} |z_1 z_2|^{-2p(k(p)+1-\eta)} dz_1 dz_2 \leq c(l_1 l_2)^{-2p(k(p)+1-\eta)+1}.$$

Hence

$$\begin{aligned} \int_{(\mathbf{R} \setminus 2l_1 I) \times (\mathbf{R} \setminus 2l_2 I)} |C_1(z)|^{2p} dz &\leq c_p c_R^{2p} |R|^{2p(k(p)+1/2)-2p(k(p)+1-\eta)+1} \\ &= c_p c_R^{2p} |R|^{-p+2p\eta+1}. \end{aligned}$$

*Case 3.*  $|z_1| < 2l_1, |z_2| \leq 2l_2$ .

The key to this case is the following sublemma.

**Sublemma 2.8.** *If  $z_1 \in \mathbf{R} \setminus 2l_1I$ , then*

$$\int_{2l_2I} |A_R|^{2p} dz_2 \leq c_p c_R^{2p} l_1^{2p(k(p)+3/2)} |z_1|^{-2p(k(p)+2)} l_2^{1-p}.$$

*Proof.*

$$\begin{aligned} & \int_{2l_2I} \left| \int_R S_1(z_1, w_1) S_2(z_2, w_2) a_R(w) dw \right|^{2p} dz_2 \\ &= \int_{2l_2I} \left| \int_{l_1I} [S_1(z_1, w_1) - S_1^*(z_1)] \int_{l_2I} S_2(z_2, w_2) a_R(w) dw \right|^{2p} dz_2 \\ &\leq \int_{2l_2I} \left[ c l_1^{k(p)+3/2} |z_1|^{-(k(p)+2)} \int_{l_1I} \left| \int_{l_2I} S_2(z_2, w_2) a(w) dw_2 \right| dw_1 \right]^{2p} \\ &\leq c_p l_1^{2p(k(p)+3/2)} |z_1|^{-2p(k(p)+2)} \left( \int_R |S_2(a_R)(w_1, z_2)|^2 dz_2 \right)^p \left( \int_{l_2I} dz_2 \right)^{1-p} \\ &\leq c_p l_1^{2p(k(p)+3/2)} |z_1|^{-2p(k(p)+2)} \left( \int_R |a_R(w_1, w_2)|^2 dw \right)^p |R|^{1-p} \\ &\leq c_p l_1^{2p(k(p)+3/2)} |z_1|^{-2p(k(p)+2)} c_R^{2p} l_2^{1-p} \\ &= c_p c_R^{2p} l_1^{2p(k(p)+3/2)} |z_1|^{-2p(k(p)+2)} l_2^{1-p}. \end{aligned}$$

With the aid of this sublemma we have

$$\begin{aligned} & \int_{(\mathbf{R} \setminus 2l_1) \times 2l_2I} |C_1(z)|^{2p} dz \\ &= \int_{\mathbf{R} \setminus 2l_1} \int_{2l_2I} |R|^{-2p} |z_1 z_2|^{2p(1+\eta)} |A_R(z)|^{2p} dz_2 dz_1 \\ &\leq \int_{\mathbf{R} \setminus 2l_1} |R|^{-2p} c_p c_R^{2p} l_1^{2p(k(p)+3/2)} (2l_2)^{2p(1+\eta)} l_2^{1-p} |z_1|^{-2p(k(p)+2)+2p(1+\eta)} dz_1 \\ &\leq c_p c_R^{2p} \int_{2l_1}^{\infty} l_1^{2p(k(p)+3/2)} l_2^{2p\eta+1+p} t^{-2p(k(p)+2)+2p(1+\eta)} dt |R|^{-2p} \\ &\leq c_p c_R^{2p} l_1^{2p(k(p)+3/2)-2p(k(p)+\eta+1)+1} l_2^{2p\eta+1+p} |R|^{-2p} \\ &= c_p c_R^{2p} l_1^{p+2p\eta+1} l_2^{2p\eta+1+p} |R|^{-2p} \\ &= c_p c_R^{2p} |R|^{-p+2p\eta+1}. \end{aligned}$$

*Case 4.*  $|z_1| \leq 2l_1, |z_2| > 2l_2$ .

By symmetry with Case 3, we obtain

$$\int_{\mathbf{R}^2} |C_1(z)|^{2p} dz \leq c_p c_R^{2p} |R|^{-p+2p\eta+1},$$

and hence the proof of (2) is complete.

*Step 2.* We now work on  $g_2(z)A_R(z)$ .

Put

$$g_2(z)A_R(z) = B_2(z)C_2(z),$$

where

$$B_2(z) = (z_1 z_2)^{-\eta} g_2(z) \quad \text{and} \quad C_2(z) = (z_1 z_2)^\eta A_R(z).$$

By using an argument similar to that of (2), we obtain

$$(3) \quad \|B_2\|_{H^{2p}} \|C_2\|_{H^{2p}} \leq c c_R |R|^{1/p-1/2}.$$

Similarly, if we set

$$g_3(z) A_R(z) = B_3(z) C_3(z),$$

where

$$B_3(z) = z_1^{-\eta} z_2^{-\eta-1} g_3(z) \quad \text{and} \quad C_3(z) = z_1^\eta z_2^{\eta+1} A_R(z)$$

and

$$g_4(z) A_R(z) = B_4(z) C_4(z),$$

where

$$B_4(z) = z_1^{-\eta-1} z_2^{-\eta-1} g_4(z) \quad \text{and} \quad C_4(z) = z_1^{\eta+1} z_2^{-\eta} A_R(z),$$

then we can prove

$$(4) \quad \|B_j\|_{H^{2p}} \|C_j\|_{H^{2p}} \leq c c_R |R|^{1/p-1/2}, \quad j = 3, 4.$$

Combining (2), (3), and (4) completes the proof of Theorem 2.3.

We now prove Theorem 2.2:

$$A = \sum_{R \subset \Omega} S(a_R) = \sum_{R \subset \Omega} \sum_{j=i}^4 B_j C_j$$

and because  $2/p - 1 > 1$ ,

$$\begin{aligned} \sum_{R \subset \Omega} \sum_{j=i}^4 \|B_j\|_{H^{2p}} \|C_j\|_{H^{2p}} &\leq \sum_{R \subset \Omega} c_R |R|^{1/p-1/2} \\ &\leq \left( \sum_{R \subset \Omega} c_R^2 \right)^{1/2} \left( \sum_{R \subset \Omega} |R|^{2/p-1} \right)^{1/2} \\ &\leq \left( \sum_{R \subset \Omega} c_R^2 \right)^{1/2} \left( \sum_{R \subset \Omega} |R| \right)^{(2/p-1)/2} \\ &\leq ((A|\Omega|)^{1-2/p})^{1/2} ((2|\Omega|)^{2/p-1})^{1/2} = c. \end{aligned}$$

Therefore the proof of Theorem 2.2 is complete.

ACKNOWLEDGMENT

The author wishes to thank Professors Bernard Russo and Song-Ying Li for their helpful advice and many discussions on the subject matter of this paper. The factorization theorem proved here is part of the author's Ph. D. thesis at the University of California, Irvine.

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