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FINITE FACTORIZATION DOMAINS

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ABSTRACT. An integral domain R is a *finite factorization domain* if each nonzero element of R has only finitely many divisors, up to associates. We show that a Noetherian domain R is an FFD $\Leftrightarrow$ for each overring R' of R that is a finitely generated R-module, $U(R')/U(R)$ is finite. For R local this is also equivalent to each $R/[R:R']$ being finite. We show that a one-dimensional local domain (R, M) is an FFD $\Leftrightarrow$ either R/M is finite or R is a DVR.

In their study of factorization [2], the first author, D.F. Anderson, and M. Zafrullah introduced the notion of a finite factorization domain (FFD). An integral domain R is an *FFD* if every nonzero element of R has only a finite number of nonassociate divisors. The three authors continued their investigation of FFD's in [3], and F. Halter-Koch studied FFD's and their monoid analog in [9]. Earlier, A. Grams and H. Warner [8] introduced the related concept of idf-domains. An integral domain R is an *idf-domain* (for irreducible-divisor-finite) if each nonzero element of R has only finitely many nonassociate irreducible divisors.

We adopt the following definitions and notation. For an integral domain R with quotient field K, $U(R)$ is the group of units of R and $G(R) = K^*/U(R)$, partially ordered by $aU(R) \leq bU(R) \Leftrightarrow a|b$ in R, is the group of divisibility of R. Clearly $G(R)$ is order-isomorphic to the group $\mathrm{Prin}(R)$ of nonzero principal fractional ideals of R ordered by reverse inclusion. We sometimes call an irreducible element of an integral domain an *atom* and an integral domain R is said to be *atomic* if every nonzero, nonunit element of R is a finite product of atoms. For an integral domain R, $R^* = R - \{0\}$ and $\bar{R}$ is the integral closure of R. For a survey of factorization in integral domains, the reader is referred to [2–3] and for standard definitions and results from commutative ring theory to [6] and [11].

We begin by giving several equivalent conditions for an integral domain to be an FFD.

Theorem 1. *For an integral domain R, the following conditions are equivalent:*

(1) *R is an FFD,*
(2) *every nonzero (principal) ideal of R is contained in only finitely many principal ideals,*
(3) *for each $x \in G(R)$ with $x \geq 0$, the interval $[0, x]$ is finite,*
(4) *for any infinite collection of distinct principal ideals $\{(r_\alpha)\}$ of R, $\bigcap_\alpha (r_\alpha) = 0$,*

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(5) *every nonzero element of R has only a finite number of factorizations, up to associates, and*
(6) *R is an atomic idf-domain.*

Proof. Clearly (1)–(5) are equivalent and (1) $\Rightarrow$ (6). (6) $\Rightarrow$ (1) [2, Theorem 5.1].□

We next collect a number of examples and results concerning FFD's.

Example 1. Let R be an integral domain with $R/(a)$ finite for each $0 \neq a \in R$. Then R is an FFD. In particular, if R is a subring of the integral closure of $\mathbb{Z}$ in a finite extension of $\mathbb{Q}$, R is an FFD. Rings with the property that each proper homomorphic image is finite were studied by K. Levitz and J. Mott [12]. They observed that an integral domain R has the property that R/I is finite for each proper (principal) ideal if and only if R is a field or R is a one-dimensional Noetherian domain with each residue field finite.

Example 2. R is an FFD $\Leftrightarrow R[\{X_\alpha\}]$, $\{X_\alpha\}$ a set of indeterminates over R, is an FFD [2, Proposition 5.3]. By Example 7 below we can also add the equivalence: $R[\{X_\alpha\} \cup \{X_\alpha^{-1}\}]$ is an FFD. Later, in Example 10, we will show that R an FFD $\not\Rightarrow R[[X]]$ is an FFD.

Example 3. A Krull domain is an FFD. (This is remarked in the paragraph above [2, Theorem 5.1] and also proved in [9]. This also follows from Theorem 2.)

Let us define an integral domain R to be a *strong FFD* if each nonzero element of R has only finitely many divisors. And we define R to be a *strong idf-domain* if each nonzero element of R has only finitely many divisors which are either units or atoms. Several characterizations of strong FFD's are given in Theorem 5.

Example 4. Any subring R of $k[\{X_\alpha\}]$, where $\{X_\alpha\}$ is any set of indeterminates over k with k either a finite field or $\mathbb{Z}$, is a strong FFD and hence an FFD. Also, see Theorem 5.

Example 5. Let T be an integral domain of the form $K + M$ where M is a nonzero maximal ideal of T and K is a subfield of T. Let D be a subring of K and $R = D + M$. Then R is an FFD $\Leftrightarrow T$ is an FFD, D is a field, and K^*/D^* is finite [2, Proposition 5.2]. Thus for fields $F_1 \subseteq F_2$, $R = F_1 + XF_2[X]$ (or $F_1 + XF_2[[X]]$) is an FFD $\Leftrightarrow F_2^*/F_1^*$ is finite which by Brandis' Theorem [5] is equivalent to $F_1 = F_2$ or F_2 is finite.

Example 6. An integral domain R is a *bounded factorization domain* (*BFD*) if for each nonzero nonunit x of R, there is a positive integer $N(x)$ such that whenever $x = x_1 \cdots x_n$ as a product of irreducible elements of R, then $n \leq N(x)$. Clearly an FFD is a BFD. A Noetherian domain is always a BFD [2, Proposition 2.2], but $R = \mathbb{R} + X\mathbb{C}[X]$ is not an FFD. Here $\bar{R} = \mathbb{C}[X]$ is an FFD and $U(\bar{R}) \cap R = U(R)$, but see Example 8.

Example 7 ([2, Example 5.4]). Let k be a field and $T = \{n + i/n! \mid 0 \leq i \leq n! - 1,\ n = 0, 1, 2, \dots\}$, an additive submonoid of $\mathbb{Q}^+$. Then the monoid domain $R = k[X;T]$ is a one-dimensional FFD. However, $R_S = k[X;\mathbb{Q}]$ where $S = \{X^t \mid t \in T\}$ and $\bar{R} = k[X;\mathbb{Q}^+]$ are not atomic and hence not FFD's. However, note that if R is an FFD and $R \subseteq R_S$ is an inert extension (an extension $A \subseteq B$ is an *inert extension* if whenever $xy \in A$ for nonzero $x, y \in B$, then $xu, yu^{-1} \in A$ for some $u \in U(B)$), then R_S is an FFD [3, Theorem 2.1]. An important case where $R \subseteq R_S$ is inert is when S is generated by principal primes or S is a splitting multiplicatively closed subset of R (i.e., for each $x \in R$, $x = as$ where $a \in R$, $s \in S$ and $aR \cap tR = atR$ for all $t \in S$). Conversely, if S is a splitting multiplicatively closed subset of R generated by principal primes and R_S is an FFD, then R is an FFD [3, Theorem 3.1].

Example 8. Let $R \subseteq T$ be a pair of integral domains and let K be the quotient field of R. If $U(T) \cap K = U(R)$, then the map $\varphi_+ : \operatorname{Prin}(R)_+ \to \operatorname{Prin}(T)_+$ given by $\varphi_+(xR) = xT$ is injective (the converse is also true), so T an FFD $\Rightarrow R$ is an FFD [2, page 17]. See Theorem 3 for a generalization.

Example 9 ([3, Theorem 5.2]). Let $\{R_\alpha\}$ be a directed family of FFD's such that for each $\alpha \leq \beta$, $R_\alpha \subseteq R_\beta$ is an inert extension. Then $\varinjlim R_\alpha$ is an FFD.

Theorem 2. *Let $R = \bigcap_\alpha R_\alpha$ be a locally finite intersection of FFD's $\{R_\alpha\}$. Then R is an FFD.*

Proof. Let $0 \neq d \in R$ be a nonunit. Let $\alpha_1, \dots, \alpha_n$ be the indices for which d is not a unit in R_α. If $e \in R$ with $dR \subseteq eR$, then $dR_\alpha \subseteq eR_\alpha$ for each α. Hence e is a unit in each R_α except possibly for $\alpha_1, \dots, \alpha_n$. Since for each $R_{\alpha_1}, \dots, R_{\alpha_n}$, dR_{α_i} is contained in only finitely many principal ideals (in R_{α_i}), the same holds for dR. So R is an FFD. Alternatively, note that $G(R)$ is order-isomorphic to a subgroup of $\bigoplus G(R_\alpha)$ (with the cardinal sum order) and apply Theorem 1. □

Remark 1. While a locally finite intersection of domains each satisfying ACCP also satisfies ACCP, a locally finite intersection of idf-domains need not be an idf-domain [8].

The next two theorems which generalize [9, Theorem 7] are straightforward modifications of its proof.

Theorem 3. *Let $R \subseteq S$ be a pair of integral domains where R has quotient field K. If S is an FFD with $(U(S) \cap K^*)/U(R)$ finite, then R is an FFD.*

Proof. Observe that $(U(S) \cap K^*)/U(R) = \ker \hat{\varphi}$ where $\hat{\varphi} : G(R) \to G(S)$ is given by $\hat{\varphi}(rU(R)) = rU(S)$. Now $\ker \hat{\varphi}$ is finite $\Leftrightarrow$ each $\hat{\varphi}^{-1}(sU(S))$ is finite $\Leftrightarrow$ each $\varphi^{-1}(sS)$ is finite where $\varphi : \operatorname{Prin}(R) \to \operatorname{Prin}(S)$ is given by $\varphi(rR) = rS$. Now since S is an FFD, $\{xS \mid xS \supseteq aS,\ x \in S\}$ is finite. Thus $\varphi_+^{-1}(\{xS \mid xS \supseteq aS,\ x \in S\})$ (where $\varphi_+ = \varphi|_{\operatorname{Prin}(R)_+}$) is finite. But

$$\{bR \mid bR \supseteq aR,\ b \in R\} \subseteq \varphi_+^{-1}(\{xS \mid xS \supseteq aS,\ x \in S\}).$$

Thus R is an FFD. □

Theorem 4. *Suppose that $R \subseteq S$ is a pair of integral domains with $[R :_R S] \neq 0$. Then R an FFD $\Rightarrow U(S)/U(R)$ is finite and S is an FFD.*

Proof. Let $0 \neq a \in [R : S]$ be a nonunit. For $u \in U(S)$, $a^2 = (ua)(u^{-1}a)$ and $ua, u^{-1}a \in [R : S] \subseteq R$. Hence for each $u \in U(S)$, $a^2R \subseteq uaR$. Since R is an FFD, the set $\{uaR \mid u \in U(S)\}$ is finite. So there exist $u_1, \dots, u_n \in U(S)$ so that for any $u \in U(S)$, $uaR = u_iaR$ for some i. Thus there exists $\lambda \in U(R)$ with $ua = \lambda u_i a$ and hence $u = \lambda u_i$. Thus $U(S)/U(R) = \{u_1U(R), \dots, u_nU(R)\}$ is finite.

Let $0 \neq s \in S$. Suppose that $sS \subseteq s_\alpha S$, so $s = s_\alpha s'_\alpha$ where $s, s_\alpha, s'_\alpha \in S$. Let $0 \neq d \in [R : S]$, so $(ds_\alpha)(ds'_\alpha) = d^2 s$. Hence $ds_\alpha R \supseteq d^2 sR$. Since R is an FFD, $\{ds_\alpha R\}$ is finite. Hence $\{s_\alpha R\}$ is finite and thus $\{s_\alpha S\}$ is finite. So S is also an FFD. □

Corollary 1. *Let R be an FFD and let S be the complete integral closure of R. Then $U(S)/U(R)$ is torsion. In particular, $U(\bar{R})/U(R)$ is torsion.*

Proof. Let $u \in U(S)$. Then $u \in U(R[u, u^{-1}])$ and $[R : R[u, u^{-1}]] \neq 0$. By Theorem 4, $U(R[u, u^{-1}])/U(R)$ is finite. So $uU(R)$ has finite order in $U(S)/U(R)$. □

Remark 2. Let $F_1 \subsetneq F_2$ be a pair of fields where F_1 is an infinite algebraic extension of Z_p and $[F_2 : F_1] < \infty$. Then $R = F_1 + XF_2[[X]]$ is a one-dimensional local domain with $\bar{R} = F_2[[X]]$ and $G(R) \cong \mathbb{Z} \oplus F_2^*/F_1^*$. Here $U(\bar{R})/U(R) \cong F_2^*/F_1^*$ is torsion, but R is not an FFD.

Corollary 2. *$R[[X]]$ an FFD $\Rightarrow$ R is completely integrally closed. Hence for R Noetherian, $R[[X]]$ is an FFD $\Leftrightarrow$ R is integrally closed.*

Proof. Let $\alpha \in K$, the quotient field of R, be almost integral over R, so $[R : R[\alpha]] \neq 0$. Hence $[R[[X]] : R[\alpha][[X]]] \neq 0$. Thus $R[[X]]$ an FFD $\Rightarrow U(R[\alpha][[X]])/U(R[[X]])$ is finite. Suppose that $\alpha \notin R$. Then $\{(1 + \alpha X^n)U(R[[X]])\}_{n=1}^{\infty}$ is an infinite subset of $U(R[\alpha][[X]])/U(R[[X]])$. For if $(1 + \alpha X^n)U(R[[X]]) = (1 + \alpha X^m)U(R[[X]])$ for $0 < m < n$, then $(1 + \alpha X^n)(1 + \alpha X^m)^{-1} \in U(R[[X]]) \subseteq R[[X]]$. But

$$(1 + \alpha X^n)(1 + \alpha X^m)^{-1} = 1 - \alpha X^m + \cdots ,$$

a contradiction.

Suppose that R is Noetherian. If R is integrally closed, then R is a Krull domain. Hence $R[[X]]$ is also a Krull domain and hence an FFD. □

Example 10. Let $F_1 \subsetneq F_2$ be a pair of finite fields. Then $R = F_1 + YF_2[[Y]]$ is a nonintegrally closed one-dimensional local domain with finite residue field F_1 and hence is an FFD. Since R is not integrally closed, $R[[X]]$ is not an FFD.

We next use Theorem 4 to characterize strong FFD's.

Theorem 5. *For an integral domain R the following conditions are equivalent.*

(1) *R is a strong FFD.*
(2) *R is an atomic strong idf-domain.*
(3) *R is an FFD and $U(R)$ is finite.*
(4) *For any set of indeterminates $\{X_\alpha\}$ over R, every subring of $R[\{X_\alpha\}]$ is a strong FFD.*
(5) *Every subring of $R[X]$ is an FFD.*

Proof. (1) $\Rightarrow$ (2) Clear. (2) $\Rightarrow$ (3) By Theorem 1, R is an FFD. Since 1 has only finitely many unit factors, $U(R)$ is finite. (3) $\Rightarrow$ (1) Clear. (3) $\Rightarrow$ (4) $R[\{X_\alpha\}]$ is an FFD (Example 2) and $U(R[\{X_\alpha\}]) = U(R)$ is finite. Let S be a subring of $R[\{X_\alpha\}]$. Since $0 \neq f \in S$ has only finitely many factors in $R[\{X_\alpha\}]$ by (1) $\Leftrightarrow$ (3), f certainly has only finitely many factors in S. So S is a strong FFD. (4) $\Rightarrow$ (5) Clear. (5) $\Rightarrow$ (3) R is a subring of $R[X]$ and hence is an FFD. Let R_0 be the prime subring of R and let $S = R_0 + XR[X]$. By hypothesis, S is an FFD. Now $X \in [S : R[X]]$, so by Theorem 4, $U(R[X])/U(S)$ is finite. But $U(R[X]) = U(R)$

and $U(S) = U(R_0)$, so $U(R)/U(R_0)$ is finite. Since $U(R_0)$ is finite, $U(R)$ is itself finite. □

Remark 3. (1) Let $R = \bigcup_{n=1}^{\infty} GF(p^{2^n})$, p a prime. Then R is an infinite field with every proper subring a finite field and hence a strong FFD. But R is not a strong FFD. Thus in Theorem 5 we cannot add the condition: every subring of R is an FFD. This example also shows that a direct limit of strong FFD's, while an FFD, need not be a strong FFD.

(2) Let R be a subring of the integral closure of $\mathbb{Z}$ in a finite field extension. By Example 1, every subring of R is an FFD. However, since $U(R)$ may be infinite (e.g., $R = Z[\sqrt{2}]$), R need not be a strong FFD.

We next characterize Noetherian FFD's.

Theorem 6. *For a Noetherian integral domain R, the following conditions are equivalent.*

(1) *R is an FFD.*
(2) *If S is an overring of R with S a finitely generated R-module, then $U(S)/U(R)$ is finite.*
(3) *There is an FFD overring R' of R which is integral over R such that if S is an overring of R with $R \subseteq S \subseteq R'$ where S is a finitely generated R-module, then $U(S)/U(R)$ is finite.*

Proof. (1) $\Rightarrow$ (2) Theorem 4. (2) $\Rightarrow$ (3) Take $R' = \bar{R}$, the integral closure of R. Now $\bar{R}$ is a Krull domain and hence an FFD. (3) $\Rightarrow$ (1) Suppose that R is *not* an FFD. Then there exist nonzero $d, d_1, d_2, \ldots \in R$ with $dR \subset d_nR$ such that the d_nR are distinct principal ideals of R. Now $dR' \subseteq d_nR'$ and R' is an FFD, so the set $\{d_nR'\}$ is finite. Re-indexing, if necessary, we can assume that $d_1R' = d_nR'$ for each $n \geq 1$. Now $(\{d_n\})$ is a finitely generated ideal of R, say $(\{d_n\}) = (d_1, \ldots, d_m)$. So each $\frac{d_n}{d_1} \in R\left[\frac{d_2}{d_1}, \ldots, \frac{d_m}{d_1}\right]$. Now $d_1R' = d_nR'$ gives that each $\frac{d_n}{d_1}$ is a unit in R'. Since $S = R\left[\frac{d_2}{d_1}, \ldots, \frac{d_m}{d_1}\right]$ is a finitely generated R-module, $U(S)/U(R)$ is finite. But $\frac{d_n}{d_1} \in U(S)$ (since $\frac{d_n}{d_1} \in U(R')$) and if $\frac{d_n}{d_1}U(R) = \frac{d_k}{d_1}U(R)$, then $d_nU(R) = d_kU(R)$ and hence $d_nR = d_kR$, a contradiction. □

Corollary 3 ([9, Theorem 7]). *Let R be a Noetherian domain with $\bar{R}$ a finitely generated R-module. Then R is an FFD $\Leftrightarrow U(\bar{R})/U(R)$ is finite.*

Theorem 6 is actually stronger than Corollary 3 in the sense that a Noetherian FFD R need not have $\bar{R}$ a finitely generated R-module. The existence of one-dimensional local domains (R, M) with R/M finite and $\bar{R}$ not a finitely generated R-module follows from [10, Corollary 1.27]. W. Heinzer also communicated to us that [12, Example 2.9] can be modified as follows to yield an appropriate example.

Example 11. (A one-dimensional local FFD (R, M) with $\bar{R}$ not a finitely generated R-module.) Let k be a finite field of characteristic p and let $Y \in Xk[[X]]$ with X, Y algebraically independent over k. Let $R = V[Y]$ where $V = k[[X]] \cap k(X, Y^p) \subset W = k[[X]] \cap k(X, Y)$. Here V and W are DVR's with quotient fields $k(X, Y^p)$ and $k(X, Y)$, respectively, and $k(X, Y)$ is purely inseparable over $k(X, Y^p)$ of degree p. So W is the integral closure of V in $k(X, Y)$, R has quotient field $k(X, Y)$, and $\bar{R} = W$. By the Krull-Akizuki Theorem, we see that R is a one-dimensional local domain, say with maximal ideal M, and $R/M = k$. Hence

(R, M) is an FFD. However, W is not a finitely generated R-module. For if W were a finitely generated R-module, then W would be a finitely generated V-module. But then since $W = V + XW$, we get $W = V$ by Nakayama's Lemma, a contradiction. Note that here $G(R) \cong G(W) \oplus (U(W)/U(R))$ where $G(W) \cong \mathbb{Z}$ and $U(W)/U(R)$ is a countably infinite elementary p-primary abelian group. Thus $U(W)/U(R)$ is torsion, but not finite.

We have observed that a one-dimensional local domain (R, M) with R/M finite is an FFD. This raises the question of when is a one-dimensional local domain (R, M) with R/M infinite an FFD? It follows from our next theorem that R must be a DVR.

Theorem 7. *Let (R, M) be a quasilocal domain and let R' be an overring of R that is a finitely generated R-module. Then $U(R')/U(R)$ is finite if and only if $R/[R : R']$ is finite. Hence if R/M is infinite, $U(R')/U(R)$ is finite if and only if $R = R'$.*

Proof. ($\Leftarrow$) Suppose that $R/[R : R']$ is finite. If $[R : R'] = R$, $R = R'$ and certainly $U(R')/U(R)$ is finite. So suppose that $[R : R'] \neq R$. By [1, Lemma 2], $U(R')/U(R) \cong U(R'/[R : R'])/U(R/[R : R'])$ which is finite since $R'/[R : R']$ is finite being a finitely generated $R/[R : R']$-module.

($\Rightarrow$) Suppose that $U(R')/U(R)$ is finite. First suppose that R/M is finite. The proof of [1, Theorem 1] shows that $R/[R : R']$ is finite. Next suppose that R/M is infinite. Let $Q_1, \dots, Q_n$ be the maximal ideals of R'. As in the proof of [1, Theorem 1] (with R' playing the role of $\bar{D}$), $(Q_1 \cap \dots \cap Q_n)/M$ has finite length as an R-module. The equation $\bar{q} = (1 + m)\bar{t}_i$ ([1, Theorem 1, line 13 of proof]) shows that the socle $\mathrm{Soc}((Q_1 \cap \dots \cap Q_n)/M)$ is finite. (For $(1 + m)\bar{t}_i = \bar{t}_i$ since $m\bar{t}_i = 0$ for $\bar{t}_i$ in the socle.) Since R/M is infinite, $\mathrm{Soc}((Q_1 \cap \dots \cap Q_n)/M) = M/M$, so $Q_1 \cap \dots \cap Q_n = M$. So $M \subseteq [R : R']$. Suppose that $[R : R'] \neq R$. Then by [1, Lemma 2], $U(R'/Q_1 \cap \dots \cap Q_n)/U(R/M)$ is finite. With a change of notation, put $R/M = K$ and $R'/Q_i = K_i$. So $K_1 \times \dots \times K_n$ is a vector space over K and there exist $t_i \in K_1^* \times \dots \times K_n^*$, $1 \leq i \leq l$, so that every element of $K_1^* \times \dots \times K_n^*$ has the form ut_i for some $u \in K^*$. Let $\{s_j\}$ be the set of 2^n elements of $K_1 \times \dots \times K_n$ where each coordinate is 0 or 1. Then $K_1 \times \dots \times K_n = \bigcup Kt_i s_j$—a finite union of one-dimensional subspaces. Since K is infinite, $R/M = K = K_1 \times \dots \times K_n = R'/Q_1 \cap \dots \cap Q_n = R'/M$. So $R = R'$. □

Corollary 4. *Let (R, M) be a quasilocal FFD with R/M infinite. Then R is integrally closed. Thus a local domain (R, M) with R/M infinite is an FFD $\Leftrightarrow$ R is integrally closed.*

Proof. Combine Theorems 4 and 7. □

Corollary 5. *Let (R, M) be a local domain. Then the following conditions are equivalent.*

(1) *R is an FFD.*
(2) *If R' is an overring of R which is a finitely generated R-module, then $U(R')/U(R)$ is finite.*
(3) *If R' is an overring of R which is a finitely generated R module, then $R/[R : R']$ is finite.*
(4) *Either R is integrally closed or R/M is finite and for each proper overring R' of R with $[R : R'] \neq 0$, $[R : R']$ is M-primary.*

(5) *For each simple integral overring* $R[\alpha]$ *of* R, $U(R[\alpha])/U(R)$ *is finite.*
(6) *Either* R *is integrally closed or* R/M *is finite and for each simple proper integral overring* $R[\alpha]$ *of* R, $[R : R[\alpha]]$ *is* M*-primary.*
(7) *For each simple integral overring* $R[\alpha]$ *of* R, $R/[R : R[\alpha]]$ *is finite.*

Proof. (1) $\Leftrightarrow$ (2) Theorem 6. (2) $\Leftrightarrow$ (3) and (5) $\Leftrightarrow$ (7) Theorem 7. (3) $\Leftrightarrow$ (4) Note that for R/M finite, $R/[R : R']$ is finite $\Leftrightarrow R = R'$ or $[R : R']$ is M-primary. The same proof shows that (6) $\Leftrightarrow$ (7). (4) $\Rightarrow$ (6) Clear. (6) $\Rightarrow$ (4). Suppose that R is not integrally closed, so R/M is finite. Let $R' = R[\alpha_1, \dots, \alpha_n]$ be a finitely generated R-module. By hypothesis, for each α_i, there is an M-primary ideal M_i with $M_i R[\alpha_i] \subseteq R$. Then $M_1 \cdots M_n$ is M-primary and $M_1 \cdots M_n R[\alpha_1, \dots, \alpha_n] \subseteq R$. □

Corollary 6. *A one-dimensional semilocal domain* R *is an FFD* $\Leftrightarrow$ *for each maximal ideal* M *of* R *with* R/M *infinite,* R_M *is a DVR.*

Proof. Let $M_1, \dots, M_n$ be the maximal ideals of R. Since $G(R)$ is order-isomorphic to $G(R_{M_1}) \oplus \cdots \oplus G(R_{M_n})$ (in the cardinal sum order) [4, Theorem 3.2], R is an FFD $\Leftrightarrow$ each R_{M_i} is an FFD. Now R_{M_i} has residue field R/M_i. So R_{M_i} is an FFD $\Leftrightarrow$ either R/M_i is finite (Example 1) or R_{M_i} is a DVR. □

Remark 4. Suppose in Example 7, we take k to be an infinite field. Then $R = k[X; T]$ is a one-dimensional FFD with each residue field infinite, but R is not integrally closed and $\bar{R}$ is not an FFD.

From our previous results, it seems reasonable to conjecture that a Noetherian domain is an FFD if and only if $R = \bigcap\{R_P \mid \operatorname{ht} P = 1\}$ where the intersection is locally finite and each R_P is a one-dimensional FFD. While the implication ($\Leftarrow$) does follow from Theorem 2, we show that ($\Rightarrow$) need not be true.

Example 12. Let (R, M) be a one-dimensional local domain with R/M finite that is not a DVR. Then $R[X]$ is an FFD. Since $R[X]$ is Cohen-Macaulay, $R[X] = \bigcap\{R[X]_P \mid \operatorname{ht} P = 1\}$, and the intersection is of course locally finite. Now if P is a height-one prime of $R[X]$ with $P \cap R = 0$, then $R[X]_P$ is a DVR. But for $P = M[X]$, $R[X]_{M[X]} = R(X)$ is a one-dimensional local domain with infinite residue field that is not a DVR. Hence $R(X)$ is not an FFD.

References

1. D.D. Anderson, *Integral domains with finitely generated groups of divisibility*, Proc. Amer. Math. Soc. **112** (1991), 613–618. MR **92c:**13001
2. D.D. Anderson, D.F. Anderson, and M. Zafrullah, *Factorization in integral domains*, J. Pure Appl. Algebra **69** (1990), 1–19. MR **92b:**13028
3. ———, *Factorization in integral domains* II, J. Algebra **152** (1992), 78–93. MR **94c:**13019
4. D.D. Anderson and J.L. Mott, *Cohen-Kaplansky domains: integral domains with a finite number of irreducible elements*, J. Algebra **148** (1992), 17–41. MR **93e:**13041
5. A. Brandis, *Über die multiplikative Structure von Körpererweiterungen*, Math. Z. **87** (1965), 71–73. MR **30:**1124
6. R. Gilmer, *Multiplicative ideal theory*, Marcel Dekker, New York, 1972. MR **55:**323
7. B. Glastad and J.L. Mott, *Finitely generated groups of divisibility*, Contemp. Math. **8** (1982), 231–247. MR **83h:**13001
8. A. Grams and H. Warner, *Irreducible divisors in domains of finite character*, Duke Math. J. **42** (1975), 271–284. MR **51:**12836
9. F. Halter-Koch, *Finiteness theorems for factorizations*, Semigroup Forum **44** (1992), 112–117. MR **92k:**20121

10. W. Heinzer, C. Rotthaus, and J. Sally, *Formal fibers and birational extensions*, Nagoya Math. J. **131** (1993), 1–38. MR **95a:**13008
11. I. Kaplansky, *Commutative rings*, Revised edition, University of Chicago Press, Chicago, 1974. MR **49:**10674
12. K.B. Levitz and J.L. Mott, *Rings with finite norms*, Canad. J. Math. **24** (1972), 557–565. MR **45:**6872

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