

## A NOTE ON QUASI-FROBENIUS RINGS

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**ABSTRACT.** It is shown that a semiperfect ring  $R$  is quasi-Frobenius if and only if every closed submodule of  $R(\omega)$  is non-small, where  $R(\omega)$  denotes the direct sum of  $\omega$  copies of the right  $R$ -module  $R$  and  $\omega$  is the first infinite ordinal.

### 1. INTRODUCTION

A ring  $R$  is called a *quasi-Frobenius* ring (briefly, a QF-ring), if  $R$  is right (or left) artinian and right (or left) self-injective. The class of QF-rings is an interesting generalization of semisimple rings, and the number of papers devoted to them is so large that we are unable to give all the references here. Instead we refer to Faith [4] and Kasch [9] for the basic properties of QF-rings.

Now let  $R$  be a QF-ring. Then every projective right  $R$ -module  $P$  is injective (see Faith [4, Theorem 24.20]). Hence any *closed* submodule  $U$  of  $P$  is an injective direct summand of  $P$ , in particular,  $U$  is a *non-small* module. From this it is natural to ask the question: Which rings  $R$  have the property that all closed submodules of any projective right  $R$ -module are non-small?

In this note we give an answer for a part of this question by proving the following theorem.

**Theorem 1.** *Let  $R$  be a semiperfect ring and let  $R(\omega)$  denote the direct sum of  $\omega$  copies of the right  $R$ -module  $R$ , where  $\omega$  is the first infinite ordinal. Then the following statements are equivalent:*

- (a)  $R$  is a QF-ring.
- (b) Every closed submodule of  $R(\omega)$  is non-small.
- (c)  $R$  has finite right uniform dimension and every closed uniform submodule of  $R(\omega)$  is non-small.
- (d)  $R$  has finite right uniform dimension, no non-zero projective right ideal of  $R$  is contained in the Jacobson radical  $J(R)$  of  $R$  and every closed uniform submodule of  $R(\omega)$  is a direct summand.

Since a right *continuous* semiperfect ring  $R$  is the direct sum of finitely many uniform right ideals and  $J(R)$  is a *singular* right  $R$ -module (see e.g. Mohamed-Müller [10]) which cannot contain non-zero projective submodules (cf. Goodearl [6]), the following result is an immediate consequence of Theorem 1:

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**Corollary 2.** *A right continuous semiperfect ring  $R$  is QF if and only if every closed uniform submodule of  $R(\omega)$  is a direct summand.*

Corollary 2 improves a part of [3, Theorem 1]. Concerning Corollary 2 we should mention that there is a commutative self-injective semiperfect non-QF-ring and any right and left self-injective right perfect ring is QF (see Osofsky [12]). The question on one-sided self-injectivity for perfect rings remains open, even assuming that the ring is semiprimary. This is now known as Faith's Conjecture:

*A right self-injective semiprimary ring is QF.*

This conjecture motivates several investigations in the area. For more about this and related questions we refer to Faith [5].

## 2. PRELIMINARIES

Throughout this note all rings are assumed to be associative rings with identity and all modules are unitary.

A submodule  $N$  of a module  $M$  is called small in  $M$ , or a small submodule of  $M$ , if for each submodule  $H$  of  $M$ , the relation  $N + H = M$  implies  $H = M$  (or equivalently, for each proper submodule  $L$  of  $M$ ,  $M \neq N + L$ ). A submodule  $S$  is said to be a small module, if  $S$  is small in its injective hull. If  $S$  is not a small module, we say that  $S$  is non-small. By this definition we may consider the *zero module as a non-small module* although it is small in each non-zero module.

Small modules and non-small modules have been considered by many authors, in particular, Harada [7] and Oshiro [11] used these and related concepts of modules to characterize several interesting classes of rings including artinian serial rings and QF-rings.

Dually, a submodule  $E$  of a module  $M$  is called essential in  $M$ , or an essential submodule of  $M$ , if for any non-zero submodule  $T$  of  $M$ ,  $E \cap T \neq 0$ . A non-zero module  $U$  is uniform if any non-zero submodule of  $U$  is essential in  $U$ .

Now let  $A \subseteq B$  be a submodule of a module  $M$  such that  $A$  is essential in  $B$ . Then we say that  $B$  is an essential extension of  $A$  in  $M$ . A submodule  $C$  of  $M$  is called a closed submodule of  $M$  if  $C$  has no proper essential extension in  $M$ . By Zorn's Lemma, each submodule of  $M$  is contained essentially in a closed submodule of  $M$ .

If a module  $M$  has only one maximal submodule which contains all proper submodules of  $M$ , then  $M$  is called a local module.

**Lemma 3.** (i) *Let  $M$  be a local module such that any closed submodule of  $M$  is non-small. Then  $M$  is uniform.*

(ii) *Let  $A$  and  $B$  be modules with  $A \simeq B$ . Then  $A$  is small if and only if  $B$  is small.*

*Proof.* (i) is obvious.

(ii) Since  $A \simeq B$ , there is an isomorphism  $\varphi$  from the injective hull of  $A$  onto the injective hull of  $B$  with  $\varphi(A) = B$ . Hence the statement follows from [10, Lemma 4.2 (3)].  $\square$

It is easy to see that if  $N$  is a *non-zero* small submodule of a module  $M$ , then  $N$  is a small module. This simple fact is useful in our consideration in the next section.

A module  $M$  is called a CS-module (or an extending module) if every submodule of  $M$  is essential in a direct summand of  $M$ , or equivalently if any closed submodule of  $M$  is a direct summand.

### 3. THE PROOFS

From now on we assume that  $R$  is a semiperfect ring. By [1, Chapter 27] or [4, Theorem 22.23],  $R$  contains a complete set of primitive orthogonal idempotents  $\{e_1, \dots, e_n\}$  such that

$$(1) \quad R = e_1R \oplus \cdots \oplus e_nR$$

and each  $e_iR$  is a local module with local endomorphism ring. Moreover, the maximal submodule of each  $e_iR$  is a small submodule of  $e_iR$ . We keep this decomposition of  $R$  throughout the consideration below.

**Lemma 4.** *If  $R$  satisfies (b) or (c) of Theorem 1, then:*

- (i) *Every  $e_iR$  is uniform.*
- (ii) *Each  $e_iR$  is not embedded properly in  $e_jR$ ,  $j = 1, 2, \dots, n$ .*
- (iii) *Every closed uniform submodule of  $R(\omega)$  is a direct summand.*

*Proof.* We assume that  $R$  satisfies (b) (of Theorem 1). The case when  $R$  satisfies (c) can be proved similarly.

(i) By (1) and [2, Proposition 2.2], each closed submodule  $U$  of  $e_iR$  is closed in  $R(\omega)$ , hence  $U$  is non-small by (b). By Lemma 3, if  $U$  is non-zero, then  $e_iR = U$ , proving that  $e_iR$  is uniform.

(ii) By (i) and (1) each  $e_iR$  is a closed uniform submodule of  $R(\omega)$ . Hence  $e_iR$  is non-small by (b). Then by Lemma 3, each  $e_iR$  cannot be embedded properly in  $e_jR$ ,  $j = 1, 2, \dots, n$ .

(iii) For convenience we write  $R(\omega)$  in the form

$$(2) \quad R(\omega) = \bigoplus_{\alpha \in I} P_\alpha$$

where each  $P_\alpha$  is isomorphic to some  $e_iR \in \{e_1R, \dots, e_nR\}$  and  $I$  is an infinite countable set. By (i), each  $P_\alpha$  is uniform. Let  $U$  be a closed uniform submodule of  $R(\omega)$ . For each  $\alpha$  we denote by  $\pi_\alpha$  the projection of  $R(\omega)$  onto  $P_\alpha$ . Then there is an  $\beta \in I$  such that  $U \cap \text{Ker}\pi_\beta = 0$ . Hence

$$U \simeq \pi_\beta(U) \subseteq P_\beta.$$

By hypothesis,  $U$  is non-small. Hence  $\pi_\beta(U)$  is also non-small by Lemma 3. It follows that  $P_\beta = \pi_\beta(U)$ , since  $P_\beta$  is a local module. From this it is easy to see that  $R(\omega) = U \oplus \text{Ker}\pi_\beta$ , as desired.  $\square$

*Proof of Theorem 1.* (a) $\Rightarrow$ (b) is clear by [4, Theorem 24.20].

(b) $\Rightarrow$ (c) $\Rightarrow$ (d) by using Lemma 4.

(d) $\Rightarrow$ (a). It is easy to see that  $R$  has the form (1) where each  $e_iR$  is (local and) uniform. Hence in (2) each  $P_\alpha$  ( $\alpha \in I$ ) is uniform and the endomorphism ring of each  $P_\alpha$  is local.

Let  $J$  be any subset of  $I$ . For convenience we put

$$R(J) = \bigoplus_{\alpha \in J} P_\alpha.$$

We first show that the decomposition (2) of  $R(\omega)$  complements direct summands, i.e. for each direct summand  $A$  of  $R(\omega)$ , there is a subset  $I'$  of  $I$  such that  $R(\omega) = A \oplus R(I')$  (see [1]).

Thus we assume now that  $A$  is a direct summand of  $R(\omega)$ . By Zorn's Lemma, there is a subset  $H$  of  $I$  which is maximal with respect to  $A \cap R(H) = 0$ . Since each  $P_\alpha$  is uniform, it follows that  $B = A \oplus R(H)$  is essential in  $R(\omega)$ . We aim to show that  $B = R(\omega)$ , completing the proof of the claim.

Suppose on the contrary that  $B \neq R(\omega)$ . Then there exists an element  $k \in I$  such that  $P_k \not\subseteq B$ . It follows that  $T = P_k \cap B$  is a uniform submodule of  $B$  with  $T \neq P_k$ . Let  $T^*$  be a maximal essential extension of  $T$  in  $B$ . Since  $B$  is isomorphic to a direct summand of  $R(\omega) \oplus R(\omega) \simeq R(\omega)$ , we may use our assumption (d) to see that  $T^*$  is a direct summand of  $B$  and so  $T^*$  is a uniform projective right  $R$ -module. By [1, Theorem 27.11],  $T^*$  is isomorphic to some  $e_t R \in \{e_1 R, \dots, e_n R\}$ . Since  $J(R)$  does not contain non-zero projective right ideals of  $R$  by (d),  $e_t R$  is not embedded properly in any  $e_i R$ .

Since  $R(\omega) = P_k \oplus R(I \setminus \{k\})$  we have by modularity

$$P_k + T^* = P_k \oplus T_1$$

where  $T_1 = (P_k + T^*) \cap R(I \setminus \{k\})$ . From this we see that  $T_1 \neq 0$ , since if  $T_1 = 0$ ,  $T^*$  is contained in  $P_k$  and then by the previous remark about  $e_t R$  we must have  $P_k = T^* \subseteq B$ , a contradiction. Moreover, from the definition of  $T_1$  it follows that  $T_1 \cap T^* = 0$ .

Let  $M$  be the maximal submodule of  $P_k$ . Then by (d),  $T^*$  is not embedded in  $M$ . Hence  $T^* \oplus T_1 \not\subseteq M \oplus T_1$ , in particular, in the factor module  $(P_k \oplus T_1)/T_1$  we have

$$(T^* \oplus T_1)/T_1 \not\subseteq (M \oplus T_1)/T_1$$

and since  $(P_k \oplus T_1)/T_1$  is a local module with the maximal submodule  $(M \oplus T_1)/T_1$  we must have

$$(T^* \oplus T_1)/T_1 = (P_k \oplus T_1)/T_1,$$

implying  $T^* \oplus T_1 = P_k \oplus T_1$ . Hence  $P_k + T^* = T^* \oplus T_1$ .

Now by modularity we have:

$$B \cap (P_k + T^*) = (B \cap P_k) + T^* = T + T^* = T^* = B \cap (T^* \oplus T_1) = T^* \oplus (B \cap T_1).$$

Therefore  $B \cap T_1 = 0$ , a contradiction to the fact that  $T_1 \neq 0$  and  $B$  is essential in  $R(\omega)$ . Thus  $B = R(\omega)$ , as desired.

We have shown that the decomposition (2) of  $R(\omega)$  complements direct summands. Then by Harada's Theorem (see [10, Theorem 2.25]) every local direct summand of  $R(\omega)$  is a direct summand (for the definition of local direct summands we refer to [10]). We use this to show below that  $R(\omega)$  is a CS-module.

Let  $Q$  be a non-zero closed submodule of  $R(\omega)$ . Then  $Q$  contains a closed uniform submodule  $U$  which is also closed in  $R(\omega)$  by [2, Proposition 2.2]. Hence  $U$  is a direct summand of  $R(\omega)$  by (d). From this we may use Zorn's Lemma to verify that  $Q$  contains a maximal local direct summand  $L$  of  $R(\omega)$ . It follows that  $L$  is a direct summand of  $R(\omega)$ , say  $R(\omega) = L \oplus P$  for some submodule  $P$  of  $R(\omega)$ . Hence  $Q = L \oplus P'$  where  $P' = P \cap Q$ . Clearly,  $P'$  is closed in  $R(\omega)$ . If  $P' \neq 0$ ,  $P'$  contains a uniform direct summand  $V$  of  $R(\omega)$ . Then it is clear that  $L \oplus V$  is a direct summand of  $R(\omega)$  with  $L \oplus V \subseteq Q$ , a contradiction to the maximality of  $L$ . Thus  $P' = 0$ , and so  $Q = L$  is a direct summand of  $R(\omega)$ . This shows that  $R(\omega)$  is CS.

Now  $R$  is QF by [8, Corollary 2], proving (a).  $\square$

In [3, Lemma 6] it was shown that a *right quasi-continuous semiperfect ring with nil Jacobson radical is right continuous*. From this and Corollary 2 it follows that a *right quasi-continuous semiperfect ring  $R$  is QF if and only if  $J(R)$  is nil and any closed uniform submodule of  $R(\omega)$  is a direct summand*.

We would like to notice further that in light of [1, Theorem 28.14] the following equivalences have been essentially established in [3] for a semiperfect ring  $R$ : (i)  $R$  is QF; (ii)  $R$  is right self-injective and each uniform submodule of  $R(\omega)$  is contained in a finitely generated submodule (of  $R(\omega)$ ); (iii)  $R$  is right continuous,  $R_R \oplus R_R$  is CS and each uniform submodule of  $R(\omega)$  is contained in a finitely generated submodule.

It is easy to see that (i) $\Leftrightarrow$ (iii) is a consequence of Corollary 2.

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