RADIAL SYMMETRY OF LARGE SOLUTIONS
OF NONLINEAR ELLIPTIC EQUATIONS

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ABSTRACT. We give conditions under which all C^2 solutions of the problem

$$\Delta u = f(|x|, u), \qquad x \in \mathbb{R}^n,$$
$$\lim_{|x|\to\infty} u(x) = \infty$$

are radial. We assume $f(|x|, u)$ is positive when $|x|$ and u are both large and positive. Since this problem with $f(|x|, u) = u$ has non-radial solutions, we rule out this possibility by requiring that $f(|x|, u)$ grow superlinearly in u when $|x|$ and u are both large and positive. However we make no assumptions on the rate of growth of solutions.

1. INTRODUCTION

In this paper we give conditions under which all C^2 solutions of the problem

(1a) $$\Delta u = f(|x|, u), \qquad x \in \mathbb{R}^n \ (n \geq 3),$$

(1b) $$\lim_{|x|\to\infty} u(x) = \infty$$

are radial. We assume $f(|x|, u)$ is positive when $|x|$ and u are both large and positive. Since problem (1a,b) with $f(|x|, u) = u$ has non-radial solutions (see Section 2), we rule out this possibility by requiring that $f(|x|, u)$ grow superlinearly in u when $|x|$ and u are both large and positive (see condition (f2) in Section 2). However we make no assumptions on the rate of growth of solutions.

In contrast to the above-mentioned necessity of a growth condition on f in problem (1a,b), Li and Ni [LN2] proved the following theorem concerning the well-studied problem

(2a) $$\Delta u = f(|x|, u), \qquad x \in \mathbb{R}^n,$$

(2b) $$\lim_{|x|\to\infty} u(x) = 0$$

which makes assumptions on neither the rate of growth of $f(|x|, u)$ in u nor the rate of decay of solutions.

Theorem [LN2]. *Suppose for $0 \leq r < \infty$ and $0 \leq s < \infty$ that $f(r,s)$ is continuous, Lipschitz in s, and non-decreasing (strictly increasing) in r. If $f_s(r,s) \geq 0$ for r large and for s small and positive, and $u(x)$ is a positive C^2 solution of (2a,b), then $u(x)$ must be radial about some point (about the origin).*

Our procedure for studying the radial symmetry of solutions of problem (1a,b) has two parts. The first part, which contains most of the technical difficulties, consists of showing that the difference of any two solutions of problem (1a,b) tends to zero as $|x| \to \infty$. (Note that this is trivially true for problem (2a,b).) Here we use some results in [T] on the asymptotic behavior as $|x| \to \infty$ of solutions of (1a,b). For the convenience of the reader, these results are stated in the appendix.

The second part consists of standard applications of the maximum principle and an adaptation to problem (1a,b) of the Alexandrov-Serrin moving plane procedure as applied by Li and Ni [LN2] to problem (2a,b).

When $n = 2$ the problem analogous to problem (1a,b) is

$$\Delta u = f(|x|, u), \qquad x \in \mathbb{R}^2,$$

$$\lim_{|x| \to \infty} \frac{u(x)}{\log |x|} = \infty.$$

Theorem 1 below holds for this two-dimensional problem provided the expression $r^{2n-2}g(r,s)$ in condition (f3) below is replaced with $r^2(\log r)^3 g(r, s \log r)$.

Radial properties of solutions on $\mathbb{R}^n$ of non-linear elliptic equations have also been studied in [CL, GNN, LN1, Z1, Z2].

2. STATEMENT OF RESULTS

To state our results for problem (1a, b), we introduce the following assumptions on $f(r,s)$:

(f1) f and f_s are continuous on $[0, \infty) \times \mathbb{R}$;
(f2) there exist $\lambda > 1, r_0 > 0$, and $s_0 > 0$ such that

$$f(r, vs) \geq v^\lambda f(r,s) > 0$$

for $v \geq 1$, $r \geq r_0$, and $s \geq s_0$;
(f3) $\frac{f(r,s)}{g(r,s)} \to 1$ as $(r,s) \to (\infty, \infty)$ for some function $g(r,s)$ with the property that it and its partial derivative $g_s(r,s)$ are defined, continuous, and positive for $r \geq r_0$ and $s \geq s_0$ and with the property that $r^{2n-2}g(r,s)$ is monotone[1] in r on $[r_0, \infty) \times [s_0, \infty)$.

Our main result is

Theorem 1. *Suppose that $f(r,s)$ satisfies conditions (f1–3) and that $u(x)$ is a C^2 solution of problem (1a,b). Let $s_m = \min\limits_{x \in \mathbb{R}^n} u(x)$.*

(i) *If f_s is non-negative on $[0, \infty) \times [s_m, \infty)$, then $u(x)$ is radial about the origin.*
(ii) *If $f(r,s)$ is monotone (strictly monotone) in r on $[0, \infty) \times [s_m, \infty)$, then $u(x)$ is radial about some point (about the origin).*

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[1]We say a function $g(r,s)$ is monotone in r on $[r_0, \infty) \times [s_0, \infty)$ if either (i) for each $s \in [s_0, \infty)$, $g(r,s)$ is non-decreasing in r on $[r_0, \infty)$, or (ii) for each $s \in [s_0, \infty)$, $g(r,s)$ is non-increasing in r on $[r_0, \infty)$.

If $\alpha_1, \ldots, \alpha_n$ are positive constants such that $\alpha_1^2 + \cdots + \alpha_n^2 = 1$, then a non-radial solution of

$$\Delta u = u, \qquad \lim_{|x| \to \infty} u(x) = \infty$$

is $u(x_1, \ldots, x_n) = (\cosh \alpha_1 x_1) \cdots (\cosh \alpha_n x_n)$. Thus condition (f2) in Theorem 1 cannot be weakened by allowing $\lambda = 1$.

By dividing the displayed inequalities in condition (f2) by $(vs)^\lambda$ we find that condition (f2) is equivalent to the following condition:

(f2)′ there exist $\lambda > 1, r_0 > 0$, and $s_0 > 0$ such that the expression $s^{-\lambda} f(r, s)$ is positive and non-decreasing in s on $[r_0, \infty) \times [s_0, \infty)$.

Theorem 2. *Suppose that f satisfies conditions* (f1–3). *If the problem*

(3a) $\qquad\qquad \Delta u = f(|x|, u), \qquad x \in \mathbb{R}^n \ (n \geq 3) \ and \ |x| \ large,$

(3b) $\qquad\qquad \lim_{|x| \to \infty} u(x) = \infty$

has a C^2 solution, then

(4) $$\int_{r_0}^{\infty} r f(r, s) \, dr < \infty, \qquad for \ all \quad s \geq s_0$$

and problem (3a,b) *has a C^2 radial solution $u_0(|x|)$, which is strictly increasing in $|x|$, such that all C^2 solutions $u(x)$ of* (3a,b) *satisfy*

$$\lim_{|x| \to \infty} [u(x) - u_0(|x|)] = 0.$$

3. Proofs

In this section we prove Theorems 1 and 2. First we show how Theorem 1 follows from Theorem 2 and then we prove Theorem 2.

Proof of part (i) *of Theorem* 1. Let T be a unitary operator on $\mathbb{R}^n$ and let $u_T(x) = u(Tx)$. Since $\Delta(u_T) = (\Delta u)_T$, we find that u_T is also a solution of (1a,b). Thus

(5a) $\qquad\qquad \Delta(u_T - u) = f(|x|, u_T) - f(|x|, u)$

and by Theorem 2,

(5b) $\qquad\qquad \lim_{|x| \to \infty} (u_T - u)(x) = 0.$

Since $f_s(r, s) \geq 0$ on $[0, \infty) \times [s_m, \infty)$, it follows from (5a,b) and the maximum principle applied to $u_T - u$ that $u_T - u \equiv 0$ on $\mathbb{R}^n$. Since T is an arbitrary unitary operator, u is radial about the origin.

Proof of part (ii) *of Theorem* 1. With the aid of Theorem 2, part (ii) of Theorem 1 can be proved by using the Alexandrov-Serrin moving plane procedure similar to the way it was used by Li and Ni [LN2]. We can show the essential differences between the procedure for our problem and the procedure for the problem in [LN2] by starting the procedure near ∞. The remainder of the procedure will not be given

because it only involves making similar changes to the remainder of the procedure in [LN2].

For each $\lambda \in \mathbb{R}$, denote the reflection of a point $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ about the hyperplane $T_\lambda = \{x \in \mathbb{R}^n \colon x_1 = \lambda\}$ by x^λ. Then $x^\lambda = (2\lambda - x_1, x_2, \ldots, x_n)$. Let $\Sigma_\lambda = \{x \in \mathbb{R}^n \colon x_1 < \lambda\}$ and

$$\Lambda = \left\{ \lambda \in \mathbb{R} \colon u(x) < u(x^\lambda) \text{ for all } x \in \Sigma_\lambda \text{ and } \frac{\partial u}{\partial x_1} > 0 \text{ on } T_\lambda \right\}.$$

We now start the moving plane procedure near ∞; that is, we show $\lambda \in \Lambda$ for λ large and positive. By (f1) and (f2) there exists $R_0 > 0$ such that

$$(6) \qquad\qquad f_s(r, s) \geq 0 \qquad \text{for} \quad r \geq R_0 \quad \text{and} \quad s \geq R_0.$$

Choose $R_2 > R_1 > R_0$ such that

$$(7) \qquad\qquad \min_{|x| \geq R_1} u(x) > R_0 \qquad \text{and} \qquad \min_{|x| \geq R_2} u(x) > \max_{|x| \leq R_1} u(x).$$

Choose $\lambda \geq R_2$ and let $w_\lambda(x) = u(x) - u(x^\lambda)$ for $x \in \Sigma_\lambda$. Let $u_0(|x|)$ be as in Theorem 2. Since

$$(8) \qquad\qquad |x^\lambda| > |x| \qquad \text{for} \quad x \in \Sigma_\lambda,$$

we find by Theorem 2 that

$$\limsup_{\substack{|x| \to \infty \\ x \in \Sigma_\lambda}} w_\lambda(x)$$

$$= \limsup_{\substack{|x| \to \infty \\ x \in \Sigma_\lambda}} \left[\left(u(x) - u_0(|x|) \right) - \left(u(x^\lambda) - u_0(|x^\lambda|) \right) + \left(u_0(|x|) - u_0(|x^\lambda|) \right) \right]$$

$$\leq 0.$$

Since $f(r, s)$ is monotone in r on $[0, \infty) \times [s_m, \infty)$ and (4) holds, we see that $f(r, s)$ is non-increasing in r on $[0, \infty) \times [s_m, \infty)$. Thus by (8),

$$\Delta w_\lambda(x) \geq f(|x|, u(x)) - f(|x|, u(x^\lambda))$$
$$= c(x) w_\lambda(x), \quad x \in \Sigma_\lambda,$$

where $c(x) = \int_0^1 f_s(|x|, u(x^\lambda) + t(u(x) - u(x^\lambda))) dt$. It follows from (6), (7), and (8) that

$$(9) \qquad\qquad c(x) \geq 0 \qquad \text{for} \quad x \in \Sigma_\lambda - B_{R_1}(0)$$

and

$$(10) \qquad\qquad w_\lambda(x) < 0 \qquad \text{for} \quad |x| \leq R_1.$$

Therefore w_λ satisfies

$$\Delta w_\lambda - c(x) w_\lambda \geq 0 \qquad\qquad \text{on} \qquad \Sigma_\lambda - \overline{B_{R_1}(0)},$$

$$w_\lambda \leq 0 \qquad\qquad \text{on} \quad \partial\left(\Sigma_\lambda - \overline{B_{R_1}(0)} \right),$$

$$\limsup_{\substack{|x| \to \infty \\ x \in \Sigma_\lambda - \overline{B_{R_1}(0)}}} w_\lambda(x) \leq 0.$$

Thus it follows from (9), (10), the strong maximum principle, and the Hopf boundary lemma that $w_\lambda(x) < 0$ in $\Sigma_\lambda - \overline{B_{R_1}(0)}$ and $\frac{\partial w_\lambda}{\partial x_1} > 0$ on T_λ. Hence by (10), $\lambda \in \Lambda$; that is $[R_2, \infty) \subset \Lambda$. The moving plane procedure is now completed similar to the way it was completed in [LN2].

Proof of Theorem 2. Let g be as in condition (f3). Then g satisfies conditions (g1) and (g3) in the appendix. Let λ be as in (f2) and $d > 1$. Then by (f2) and (f3)

$$\liminf_{(r,s)\to(\infty,\infty)} \inf_{v>d} \frac{g(r,vs)}{v^{(1+\lambda)/2}g(r,s)} \geq d^{(\lambda-1)/2} > 1.$$

Hence g also satisfies (g2). So by (f3), f and g satisfy the hypotheses of Lemma A.1 in the appendix. Since, by assumption, problem (3a,b) has a C^2 solution, it follows from part (ii) of Lemma A.1 that problem (20a,b) has a C^2 radial solution $y(|x|)$. Thus by part (iii) of Lemma A.1 and (f3), we have that (4) holds, and by part (i) of Lemma A.1, the ratio of any two solutions of (3a,b) tends to one as $|x| \to \infty$. Hence, since under the change of variables

$$(11) \qquad\qquad z(t) = u(|x|), \qquad t = -|x|^{2-n},$$

the problem of finding radial solutions $u(|x|)$ of problem (3a,b) transforms into the problem of finding solutions $z(t)$ of

$$(12a) \qquad\qquad z'' = F(t,z), \qquad t \text{ negative and small,}$$
$$(12b) \qquad\qquad \lim_{t\to 0^-} z(t) = \infty$$

where $f(r,s) = (n-2)^2 r^{2-2n} F(-r^{2-n}, s)$, it follows that the ratio of any two solutions of problem (12a,b) tends to one as $t \to 0^-$.

From this last conclusion we show through each point $(t_1, s_1) \in [t_0, 0) \times [s_0, \infty)$, where $t_0 = -r_0^{2-n}$, that there can be at most one solution of (12a,b) whose graph is contained in $[t_1, 0) \times [s_0, \infty)$. Suppose to the contrary that $z_1(t) \not\equiv z_2(t)$ are two such solutions. Then, since f is C^1 in s, $z_1'(t_1) \neq z_2'(t_1)$ and we can assume $z_1'(t_1) > z_2'(t_1)$. Let

$$\zeta = \zeta(t) = -\int_t^0 \frac{d\tau}{z_2(\tau)} \qquad \text{and} \qquad v(\zeta) = \frac{z_1(t)}{z_2(t)}.$$

Let $\zeta_1 = \zeta(t_1)$. Then $v(\zeta_1) = 1$,

$$v'(\zeta_1) = s_1[z_1'(t_1) - z_2'(t_1)] > 0,$$

and by (f2),

$$v''(\zeta) = v(\zeta)z_2(t)^3 F(t, z_2(t)) \left[\frac{F(t, v(\zeta)z_2(t))}{v(\zeta)F(t, z_2(t))} - 1 \right]$$
$$\geq 0 \qquad \text{for} \quad v(\zeta) \geq 1.$$

Hence $v''(\zeta) \geq 0$ for $\zeta_1 \leq \zeta < 0$ and $\liminf_{\zeta\to 0^-} v(\zeta) > 1$, which contradicts the fact that the ratio of any two solutions of problem (12a,b) tends to one as $t \to 0^-$.

Next we show that the difference of any two solutions of problem (12a,b) tends to zero as $t \to 0^-$. Let $z_1(t) \not\equiv z_2(t)$ be two solutions of (12a,b). Clearly there exists $t_1 \in [t_0, 0)$ such that for $t_1 \leq t < 0$ we have $z_1(t) > s_0$, $z_2(t) > s_0$, $z_1'(t) > 0$, and $z_2'(t) > 0$. By the previous paragraph, $z_1(t) \neq z_2(t)$ for $t_1 \leq t < 0$. Hence we can assume that $z_1(t) > z_2(t)$ for $t_1 \leq t < 0$. Suppose $z_1'(t_2) > z_2'(t_2)$ for some $t_2 \in [t_1, 0)$. Let $z_3(t)$ be the solution of (12a,b) satisfying $z_3(t_2) = z_2(t_2)$ and $z_3'(t_2) = z_1'(t_2)$. Since by (f2), $f(r,s)$ is non-decreasing in s, it follows from a simple comparison argument that $z_2(t) < z_3(t) < z_1(t)$ for $t \in (t_2, 0)$. Thus $z_3(t)$ is a solution of (12a,b) which contradicts the previous paragraph. Therefore $z_1(t) - z_2(t)$ is positive and non-increasing on the interval $[t_1, 0)$.

Let $a = \lim_{t \to 0^-} z_1(t) - z_2(t)$. Suppose $a > 0$. By (f2) we find that

$$z_1''(t) - z_2''(t) \geq F(t, z_2(t) + a) - F(t, z_2(t))$$

$$\geq \left[\left(\frac{z_2(t) + a}{z_2(t)} \right)^\lambda - 1 \right] F(t, z_2(t))$$

(13)
$$\geq \frac{\lambda a}{z_2(t)} F(t, z_2(t)) \qquad \text{for} \quad t_1 \leq t < 0.$$

Integrating (13) from t to 0 we get

(14) $$\qquad -[z_1'(t) - z_2'(t)] \geq \lambda a \int_t^0 \frac{F(\tau, z_2(\tau))}{z_2(\tau)} \, d\tau \qquad \text{for} \quad t_1 \leq t < 0.$$

Integrating (14) from t_1 to 0 and interchanging the order of integration gives

$$[z_1(t_1) - z_2(t_1)] - a \geq \lambda a \int_{t_1}^0 (\tau - t_1) \frac{F(\tau, z_2(\tau))}{z_2(\tau)} \, d\tau$$

$$\geq \frac{\lambda a |t_1|}{2} \int_{t_1/2}^0 \frac{F(\tau, z_2(\tau))}{z_2(\tau)} \, d\tau.$$

Hence

(15) $$\qquad \int_{t_1}^0 \frac{F(\tau, z_2(\tau))}{z_2(\tau)} \, d\tau < \infty.$$

But integrating $z_2''(t) = F(t, z_2(t))$ twice from t_1 to t gives

$$|z_2(t)| \leq c + |t_1| \int_{t_1}^t \frac{F(\tau, z_2(\tau))}{z_2(\tau)} z_2(\tau) \, d\tau$$

where $c = z_2(t_1) + |t_1| z_2'(t_1)$. Therefore by Gronwall's inequality and (15) we have

$$|z_2(t)| \leq c \exp \left(|t_1| \int_{t_1}^0 \frac{F(\tau, z_2(\tau))}{z_2(\tau)} \, d\tau \right) < \infty$$

for $t_1 \leq t < 0$ — a contradiction. Thus $a = 0$ and we have therefore shown that the difference of any two solutions of problem (12a,b) tends to zero as $t \to 0^-$. Hence,

since under the change of variables (11) the problem of finding solutions $z(t)$ of problem (12a,b) transforms into the problem of finding radial solutions $u(|x|)$ of problem (3a,b), it follows that the difference of any two solutions of problem (3a,b) which are radial about the origin tends to zero as $|x| \to \infty$. Also, since any solution $z(t)$ of (12a,b) is clearly strictly increasing for t negative and near zero, it follows that any radial solution $u(|x|)$ of (3a,b) is strictly increasing in $|x|$ for $|x|$ large.

Let $u(x)$ be a solution of (3a,b). Then, as shown above, (4) holds. Since f satisfies conditions (f1) and (f2), it follows that f satisfies the conditions imposed on g in conditions (g1) and (g2) in the appendix. So f satisfies the hypotheses of Lemma A.2 in the appendix. Let $s_1 > s_0$. Let $r_1 > r_0$ be as in Lemma A.2. Since $u(x)$ satisfies (3b), there exists $r_2 > r_1$ such that $u(x) > s_1 + 1$ for $|x| \geq r_2$. Choose $s_2 \in (s_1+1, \min_{|x|=r_2} u(x))$. Let $u_0(r), u_1(r), \dots$ be as in Lemma A.2. Then, for $k \geq 1$, we have $u(x) - u_k(|x|) > 0$ for $|x| = r_2$, $\lim_{|x| \to \infty} u(x) - u_k(|x|) = \infty$ and

$$\Delta(u(x) - u_k(|x|)) = f(|x|, u(x)) - f(|x|, u_k(|x|))$$
$$< 0 \qquad \text{for} \quad u(x) < u_k(|x|)$$

because, by (f2), $f(r,s)$ is strictly increasing in s on $[r_0, \infty) \times [s_0, \infty)$. It therefore follows from the maximum principle that $u(x) \geq u_k(|x|)$ for $|x| \geq r_2$ and letting $k \to \infty$ we find that

$$(16) \qquad\qquad u(x) \geq u_0(|x|) \qquad \text{for} \quad |x| \geq r_2.$$

Now choose $s_2 \in (\max_{|x|=r_2} u(x), \infty)$ and let $U_0(r), U_1(r), \dots$ be as in Lemma A.2. Then, for $k \geq 1$, we have $u(x) - U_k(|x|) < 0$ for $|x| = r_2$, $\lim_{|x| \to \rho_k^-} u(x) - U_k(|x|) = -\infty$ and $\Delta(u(x) - U_k(|x|)) > 0$ for $u(x) > U_k(|x|)$. Therefore, by the maximum principle, $u(x) \leq U_k(|x|)$ for $r_2 \leq |x| < \rho_k$ and letting $k \to \infty$ we see that

$$(17) \qquad\qquad u(x) \leq U_0(|x|) \qquad \text{for} \quad |x| \geq r_2.$$

Since, as shown above, the difference of any two solutions of problem (3a,b) which are radial about the origin tends to zero as $|x| \to \infty$, Theorem 2 follows from (16) and (17).

APPENDIX

In [T] we studied the radial properties and asymptotic behavior of solutions of the problem

$$(18a) \qquad\qquad \Delta u = f(|x|, u), \qquad x \in \mathbb{R}^n \ (n \geq 3) \text{ and } |x| \text{ large,}$$
$$(18b) \qquad\qquad \lim_{|x| \to \infty} u(x) = \infty.$$

Some of our results required that

$$(19) \qquad\qquad \frac{f(r,s)}{g(r,s)} \to 1 \qquad \text{as} \quad (r,s) \to (\infty, \infty)$$

for some function $g(r, s)$ such that

(g1) for some $r_0 > 0$ and $s_0 > 0$, $g(r, s)$ and $g_s(r, s)$ are defined, continuous, and positive for $r \geq r_0$ and $s \geq s_0$;

(g2) there exists $\gamma > 1$ such that for all $d > 1$

$$\liminf_{(r,s) \to (\infty, \infty)} \ \inf_{v > d} \ \frac{g(r, vs)}{v^\gamma g(r, s)} > 1;$$

(g3) $r^{2n-2} g(r, s)$ is monotone in r on $[r_0, \infty) \times [s_0, \infty)$.

In [T] we proved

Lemma A.1. *Suppose g satisfies* (g1–3), *$f(r, s)$ is continuous on $[r_0, \infty) \times [s_0, \infty)$, and* (19) *holds.*

(i) *If $u(x)$ is a C^2 solution of* (18a,b) *then for each C^2 radial solution $y(|x|)$ of*

(20a) $$\Delta y = g(|x|, y), \qquad x \in \mathbb{R}^n \text{ and } |x| \text{ large,}$$

(20b) $$\lim_{|x| \to \infty} y(x) = \infty$$

$u(x)$ satisfies

$$\lim_{|x| \to \infty} \frac{u(x)}{y(|x|)} = 1.$$

(ii) *If problem* (20a,b) *has no C^2 radial solution $y(|x|)$, then problem* (18a,b) *has no C^2 solution.*

(iii) *Problem* (20a,b) *has a C^2 radial solution $y(|x|)$ if and only if*

$$\int_{r_0}^{\infty} r g(r, s)\, dr < \infty$$

for all $s \geq s_0$.

Lemma A.2. *Suppose that $f(r, s)$ satisfies the hypotheses imposed on $g(r, s)$ in* (g1–2) *and that*

$$\int_{r_0}^{\infty} r f(r, s)\, dr < \infty$$

for all $s \geq s_0$. Then for each $s_1 > s_0$ there exists $r_1 > r_0$ such that if $s_2 \geq s_1 + 1$ and $r_2 \geq r_1$, then there exist a sequence $\{\rho_k\}_{k=1}^{\infty}$ in (r_2, ∞) satisfying $\lim_{k \to \infty} \rho_k = \infty$ and C^2 functions $u_0(r), u_1(r), \ldots$ and $U_0(r), U_1(r), \ldots$ whose ranges are contained in (s_1, ∞) such that

(i) *$U_0(|x|)$ and $u_0(|x|), u_1(|x|), \ldots$ are solutions of*

$$\Delta u = f(|x|, u), \qquad |x| \geq r_2,$$
$$u(x) = s_2, \qquad |x| = r_2;$$

(ii) *for each positive integer k, $U_k(|x|)$ is a solution of*

$$\Delta u = f(|x|, u), \qquad r_2 \leq |x| < \rho_k,$$
$$u(x) = s_2, \qquad |x| = r_2,$$
$$\lim_{|x| \to \rho_k^-} u(x) = \infty;$$

(iii) $\lim_{r\to\infty} u_0(r) = \lim_{r\to\infty} U_0(r) = \infty$;

(iv) *for each positive integer k, $u_k(r)$ has a finite limit as $r \to \infty$;*

(v) *for each $r \geq r_2$, we have*

$$\lim_{k\to\infty} u_k(r) = u_0(r) \qquad and \qquad \lim_{k\to\infty} U_k(r) = U_0(r).$$

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