PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 124, Number 3, March 1996

A THEOREM OF BRIANÇON-SKODA TYPE FOR REGULAR LOCAL RINGS CONTAINING A FIELD

IAN M. ABERBACH AND CRAIG HUNEKE

(Communicated by Wolmer V. Vasconcelos)

Abstract. Let (R, m) be a regular local ring containing a field. We give a refinement of the Briançon-Skoda theorem showing that if J is a minimal reduction of I where I is m-primary, then $\overline{I^{d+w}} \subseteq J^{w+1}\mathfrak{a}$ where $d = \dim R$ and $\mathfrak{a}$ is the largest ideal such that $\mathfrak{a}J = \mathfrak{a}I$. The proof uses tight closure in characteristic p and reduction to characteristic p for rings containing the rationals.

1. Introduction

Let (R, m) be a commutative Noetherian local ring. In 1954, Northcott and Rees [NR] introduced the concept of a reduction of an ideal. Let $I \subseteq R$ be an ideal. An ideal $J \subseteq I$ is a *reduction* of I if there exists an integer $r \geq 0$ such that $I^{r+1} = JI^r$. A reduction of I which is minimal with respect to inclusion (such reductions exist) is called a *minimal reduction* of I. When the residue class field of R, R/m, is infinite then every minimal reduction J of I has the same number of minimal generators. This number is called the *analytic spread* of I, denoted $\ell(I)$. The inequalities $ht(I) \leq \ell(I) \leq \dim R$ always hold. In particular, when I is m-primary and the residue field is infinite, every minimal reduction J of I is generated by exactly $\dim(R)$ elements.

Closely related to the idea of a reduction is the integral closure of an ideal. Given an ideal I, an element x is in the *integral closure* of I, $\overline{I}$, if x satisfies an equation of the form $x^k + a_1x^{k-1} + \cdots + a_k = 0$ where $a_i \in I^i$. An ideal $J \subseteq I$ is a reduction if and only if $I \subseteq \overline{J}$. Thus $\overline{J}$ is the unique largest ideal for which J is a reduction.

It is in the context of studying the integral closure of an ideal that the theorem of Briançon and Skoda was proved. This result was proved in response to a question of Mather which we recall: let $\mathbf{O}_n = \mathbf{C}\{z_1, \ldots, z_n\}$ be the ring of convergent power series in n variables. Let $f \in \mathbf{O}_n$ be a nonunit (i.e., f vanishes at the origin). The Jacobian ideal of f is $j(f) = (\partial f/\partial z_1, \ldots, \partial f/\partial z_n)\mathbf{O}_n$. Since $f \in \overline{j(f)}$, there is an integer k such that $f^k \in j(f)$. Mather asked if there exists a bound for k which works for all nonunits f.

Briançon and Skoda answered this question affirmatively with the following stronger result:

Received by the editors June 21, 1994 and, in revised form, September 7, 1994.
1991 *Mathematics Subject Classification*. Primary 13H05; Secondary 13A35, 13B22.
Key words and phrases. Briançon-Skoda theorems, integral closure, tight closure.
Both authors were partially supported by the National Science Foundation.

Theorem [BS]. *Let $I \subseteq \mathbf{O}_n$ be an ideal which can be generated by d elements. Then for every $w \geq 0$*

$$\overline{I^{d+w}} \subseteq I^{w+1}.$$

Since $j(f)$ has at most n generators, applying the theorem with $I = j(f)$ and $w = 0$ gives $f^n \in j(f)$, answering Mather's question.

The ring $\mathbf{O}_n$ is a regular local ring so one may ask if the entirely algebraic statement of the theorem remains true in any regular ring. Lipman and Sathaye succeeded in proving the same theorem for arbitrary regular local rings [LS]. Lipman and Teissier partly extended this theorem to rings having rational singularities, cf. [LT]. Since then, there has been considerable activity in proving more precise theorems of this type, e.g., see [HH, AH, AHT, RS, Sw, L]. Our purpose in this paper is to give a stronger version of the theorem for equicharacteristic regular local rings (that is, for regular local rings which contain a field), by using ideas related to the tight closure approach of [HH]. In particular we use reduction to characteristic p. We prove that for all $w \geq -1$, $\overline{I^{d+w}} \subseteq I^{w+1}\mathfrak{a}$, where $\mathfrak{a}$ is an explicitly constructed ideal depending on I (see Definition 2.1 below).

2. Coefficient ideals and the Briançon-Skoda theorem

Definition 2.1. Let R be any commutative Noetherian ring and let $J \subseteq I$ be two ideals of R. The *coefficient ideal* of I relative to J, $\mathfrak{a}(I, J)$, is the largest ideal $\mathfrak{b}$ of R for which $I\mathfrak{b} = J\mathfrak{b}$. When the ideals I and J are understood we will write simply $\mathfrak{a}$.

Remark 2.2. The ideal $\mathfrak{a}(I, J)$ exists since if $\mathfrak{b}_1 I = \mathfrak{b}_1 J$ and $\mathfrak{b}_2 I = \mathfrak{b}_2 J$, then $(\mathfrak{b}_1 + \mathfrak{b}_2)I = (\mathfrak{b}_1 + \mathfrak{b}_2)J$, and a union of ideals $\mathfrak{b}_i$ satisfying $\mathfrak{b}_i I = \mathfrak{b}_i J$ satisfies the same equality. However, in most cases $\mathfrak{a}(I, J) = 0$. For suppose that $\mathfrak{a}(I, J)$ contains a nonzerodivisor. The fact that $I\mathfrak{a}(I, J) = J\mathfrak{a}(I, J)$ will then imply, using the 'determinant trick', that I and J have the same integral closure. Conversely, suppose that $J \subseteq I$ and these ideals have the same integral closure. In this case, J is a reduction of I and there exists an integer k such that $I^{k+1} = I^k J$. Hence $I^k \subseteq \mathfrak{a}(I, J)$.

When I is an m-primary ideal we can compute $\mathfrak{a}$ algorithmically. This computation is based on the observation that $\mathfrak{a}(I, J) \subseteq J : I$. Let $\mathfrak{a}_1 = (J : I)$. Define $\mathfrak{a}_{n+1}$ inductively as $\mathfrak{a}_{n+1} = J\mathfrak{a}_n : I$. Before stating the proposition, we recall that the *reduction number* of I with respect to J, $r_J(I)$, is the least integer $r = r_J(I)$ such that $I^{r+1} = I^r J$.

Proposition 2.3. *Let I be an ideal, let $J \subseteq I$ be a reduction and let $\mathfrak{a}_n$ be defined as above. Let $r = r_J(I)$. Then*

(1) *$I^r \subseteq \mathfrak{a}$.*
(2) *$\mathfrak{a} \subseteq \mathfrak{a}_n$ and $\mathfrak{a}_{n+1} \subseteq \mathfrak{a}_n$ for all n.*
(3) *If R is local and I is m-primary, then the sequence $\mathfrak{a}_1 \supseteq \mathfrak{a}_2 \supseteq \cdots$ stabilizes and $\mathfrak{a} = \mathfrak{a}_n$ for $n \gg 0$.*

Proof. (1) is clear since $I^r I = I^r J$. (2) follows by induction since $\mathfrak{a}_{n+1} I \subseteq J\mathfrak{a}_n \subseteq J\mathfrak{a}_{n-1}$, hence $\mathfrak{a}_{n+1} \subseteq \mathfrak{a}_n$, and $\mathfrak{a}I = \mathfrak{a}J \subseteq \mathfrak{a}_n J$ implies $\mathfrak{a} \subseteq \mathfrak{a}_{n+1}$.

We now prove (3). Since I is primary to the maximal ideal, R/I^r is Artinian, hence the descending sequence $\mathfrak{a}_1/I^r \supseteq \mathfrak{a}_2/I^r \supseteq \cdots$ stops, so $\mathfrak{a}_1 \supseteq \mathfrak{a}_2 \supseteq \cdots$ does too. Suppose that $\mathfrak{a}_n = \mathfrak{a}_{n+1}$. Then $I\mathfrak{a}_n = I\mathfrak{a}_{n+1} \subseteq J\mathfrak{a}_n \subseteq I\mathfrak{a}_n$. Hence $I\mathfrak{a}_n = J\mathfrak{a}_n$, so $\mathfrak{a}_n \subseteq \mathfrak{a}$. But $\mathfrak{a} \subseteq \mathfrak{a}_n$ by (2), so $\mathfrak{a} = \mathfrak{a}_n$.

Remark 2.4. Proposition 2.3 shows that when I is m-primary then calculating $\mathfrak{a}(I,J)$ can be done easily using a computer program such as MACAULAY.

Before continuing we make several remarks concerning reductions, integral closures and powers of ideals.

Remark 2.5. Let R be a Noetherian ring, let $I \subseteq R$ be an ideal and let $J = (a_1, \dots, a_n)$ be a reduction of I. Then

(1) J^t is a reduction of I^t for all t.
(2) If $x \in \overline{I}$, then there exists $k \geq 0$ such that for all $m \geq 0$, $x^{k+m} \in I^{m+1}$.
(3) For all $h \geq 0$, $J^{nh} \subseteq (a_1^h, \dots, a_n^h)J^{h(n-1)}$.

Proof. If $JI^r = I^{r+1}$, then raising to the t^{th} power shows (1). For (2) see [HH, section 5.1]. For (3) see [AH, Lemma 3.4].

The ideal $\mathfrak{a}(I,J)$ has another nice property relative to I, which will be crucial in the proof of Theorem 2.7. Before giving this theorem we recall some basic facts about the Frobenius map and the tight closure of an ideal. Let R be a ring of characteristic p. Let $I \subseteq R$ be an ideal and let $x \in R$. Then we say that x is in the *tight closure* I^* of I if there exists an element $c \in R$, not in any minimal prime, such that for all sufficiently large integers q of the form p^e, where e is a nonnegative integer, $cx^q \in I^{[q]}$, the ideal of R generated by the qth powers of the elements of I. An important point we need is that when R is regular, every ideal is tightly closed. This can be seen easily from the fact that if R is regular and I and J are ideals, then $I^{[q]} : J^{[q]} = (I : J)^{[q]}$; the last equality is due to the fact that the Frobenius morphism is flat when R is regular of positive characteristic. (Here and in the rest of this paper, when we write q we will always mean $q = p^e$, a power of the characteristic of R.) Now if $x \in I^*$, then choose $c \in R$ such that $cx^q \in I^{[q]}$ for all sufficiently large q. Then $c \in I^{[q]} : x^q = (I : x)^{[q]}$ for all large q. Since c is not zero, $x \in I$.

Theorem 2.6. *Let (R,m) be a regular ring of characteristic p and dimension d having an infinite residue field, and let I be an m-primary ideal with minimal reduction J. Then for all $q = p^e$*

$$\overline{I^{(d-1)q}} \subseteq \mathfrak{a}(I,J)^{[q]}.$$

Proof. We will show that $\overline{I^{(d-1)q}} \subseteq \mathfrak{a}_n^{[q]}$ for all n. Then by Proposition 2.3(3), $\overline{I^{(d-1)q}} \subseteq \mathfrak{a}^{[q]}$. Let $c \neq 0$ be an element such that $cI^t \subseteq J^t$ for all t ($c \in I^{r_J(I)}$ will work).

When $n = 1$ we wish to prove that $\overline{I^{(d-1)q}} \subseteq (J : I)^{[q]} = (J^{[q]} : I^{[q]})$. The equality holds because R is regular. Let $x \in \overline{I^{(d-1)q}}$ and let k be as in Remark 2.5(2). We have $cx^k x^{q'} I^{[qq']} \subseteq cI^{dqq'} \subseteq J^{dqq'} \subseteq J^{[qq']}$ for all $q' \geq 1$.
Thus $\overline{I^{(d-1)q}} \subseteq ((J : I)^{[q]})^* = (J : I)^{[q]}$.

Assume now that the claim has been shown for $n-1$. Then

$$cx^k x^{q'} I^{[qq']} \subseteq J^{dqq'} \subseteq J^{[qq']} J^{(d-1)qq'} \subseteq J^{[qq']} I^{(d-1)qq'} \subseteq (J\mathfrak{a}_{n-1})^{[qq']}$$

for all $q' \geq 1$. Hence $\overline{I^{(d-1)q}} \subseteq ((J\mathfrak{a}_{n-1})^{[q]} : I^{[q]})^* = ((J\mathfrak{a}_{n-1} : I)^{[q]})^* = (\mathfrak{a}_n^{[q]})^* = \mathfrak{a}_n^{[q]}$.

We are now ready to prove the main theorem.

Theorem 2.7. *Let (R, m) be a regular local ring of dimension d containing a field and having infinite residue field. Let I be an m-primary ideal and let J be a minimal reduction of I. Then for all $w \geq -1$,*

$$\overline{I^{d+w}} \subseteq J^{w+1}\mathfrak{a}(I, J).$$

We will first give the proof in characteristic $p > 0$ and then discuss how to reduce the case of characteristic 0 to characteristic p.

Proof in characteristic p. The case $w = -1$ follows immediately from Theorem 2.6 with $q = p^0 = 1$. (Here we interpret $J^0 = R$.) Therefore we assume for the remainder of the proof in characteristic p that $w \geq 0$.

Let $x \in \overline{I^{d+w}} = \overline{J^{d+w}}$. By applying Remark 2.5(2) we obtain k for which $x^{k+m} \in J^{(d+w)(m+1)}$ for all $m \geq 0$. Then for $q = p^e$

$$x^k x^q \in J^{(d+w)q} \subseteq (J^{[q]})^{w+1} J^{(d-1)q} \subseteq (J^{[q]})^{w+1}\mathfrak{a}(I, J)^{[q]},$$

using Remark 2.5(3) and Theorem 2.6. Since $(J^{[q]})^{w+1} = (J^{w+1})^{[q]}$, we have $x \in (J^{w+1}\mathfrak{a})^* = J^{w+1}\mathfrak{a}$.

Proof in characteristic 0. Let (R, m) be a regular local ring containing a field of characteristic 0. If I is an ideal of R with reduction J for which the theorem does not hold, then $\hat{R}$ will also have a counter-example. This is because $(\overline{IR})\hat{R} \subseteq \overline{I\hat{R}}$, and $\mathfrak{a}_n(I, J)\hat{R} = \mathfrak{a}_n(I\hat{R}, J\hat{R})$ ($\hat{R}$ is flat over over R). Thus we may assume that R is complete. By the Cohen structure theorem, $R = k[[x_1, \ldots, x_d]]$. Let $S = k[x_1, \ldots, x_d]_{(x_1,\ldots,x_d)}$. Let $x \in \overline{I^{d+w}} - J^{w+1}\mathfrak{a}$. Form the ring B by adjoining to S the element x, generators for I and J, elements to give the containments $J \subseteq I$ and $I^{r+1} \subseteq JI^r$, and elements giving an integral dependence of x on I^{d+w}.

Now let $C \supseteq B$ be a regular ring which is smooth and finitely generated over S (given by [Sp, Theorem 9.1] which shows that $\hat{R}$ is the direct limit of smooth S-subalgebras of $\hat{R}$ – answering a question raised by Artin after the proof of [Ar, Theorem 1]). Note that C is now a ring with a counterexample since $\mathfrak{a}(IC, JC)R \subseteq \mathfrak{a}(I, J)$ and $R = \hat{C}$ is faithfully flat over C. We now need to write down equations over **Z** expressing that:

(1) $J \subseteq I \subseteq m_C = (x_1, \ldots, x_d)C$ and J is a reduction of I.
(2) $\mathfrak{a}_1 = (J : I), \ldots, \mathfrak{a}_{n+1} = (J\mathfrak{a}_n : I)$, and $\mathfrak{a}_n = \mathfrak{a}_{n+1}$ (this insures that $\mathfrak{a}(I, J) = \mathfrak{a}_n$).
(3) $x \in \overline{I^{d+w}}$.
(4) x is not in $J^{w+1}\mathfrak{a}(I, J)$.

See [AH] for a more detailed explanation of how to achieve these objectives. One may then apply a theorem of Hochster [AH, Theorem 3.7] to obtain a counterexample in a regular ring of characteristic p. This is a contradiction. $\square$

It is possible to use a more standard reduction to characteristic p (see [AH] for example), but Spivakovsky's result makes the argument simpler.

When the ideal I is integrally closed we can cut the ideal $\mathfrak{a}(I, J)$ to $A = \mathfrak{a} \cap I$ and obtain a sharper version of Theorem 2.7. We first need to state a theorem due independently to Itoh and Huneke (in the equicharacteristic case).

Proposition 2.8 [It, Theorem 1], [Hu, Appendix]. *Let J be an ideal of a Noetherian ring R which is generated by a regular sequence. Then for all $n \geq 1$, $J^n \cap \overline{J^{n+1}} = J^n \overline{J}$.*

Theorem 2.9. *Let (R, m) be a regular local ring of dimension d containing a field and having an infinite residue field. Let I be an m-primary ideal. Let $A = \mathfrak{a}(I, J) \cap \overline{I}$. Then for any minimal reduction J of I,*
(1) *If I is integrally closed, then $JA = IA$.*
(2) *If I is integrally closed, then $r_J(J + A) \leq 1$.*
(3) *If R contains a field, then for all $w \geq 0$*

$$\overline{I^{d+w}} \subseteq J^{w+1}A.$$

Proof. Assume that I is integrally closed. We let $\mathfrak{a} = \mathfrak{a}(I, J)$. We have $IA = I(\mathfrak{a} \cap I) \subseteq I\mathfrak{a} \cap I^2 = J\mathfrak{a} \cap I^2 \subseteq J\mathfrak{a} \cap (J \cap I^2) = J\mathfrak{a} \cap JI$. The last equality follows from Proposition 2.8. Let $J = (x_1, \ldots, x_d)$ where $x_1, \ldots, x_d$ is a regular sequence. If $r \in J\mathfrak{a} \cap JI$, then $r = \sum x_i a_i = \sum x_i u_i$ where $a_i \in \mathfrak{a}$ and $u_i \in I$. Then $a_i - u_i \in J$ for all i, so $a_i \in \mathfrak{a} \cap I$. Hence $J\mathfrak{a} \cap JI = J(\mathfrak{a} \cap I)$. Thus $IA = JA$. This proves (1).

Since $J \subseteq J + A \subseteq I$, J is a minimal reduction of $J + A$. Then $(J + A)^2 = J^2 + JA + A^2 \subseteq J^2 + JA + AI \subseteq J^2 + JA = J(J + A)$, using (1). Therefore $r_J(J + A) \leq 1$.

Assume now that R contains a field (but I may not be integrally closed). We now prove (3). If $d \leq 1$, then $I = 0$ or I is principal and the assertion is clear. Hence we may assume that $d \geq 2$. By Theorem 2.7 we have $\overline{I^{d+w}} \subseteq J^{w+1}\mathfrak{a}$. Thus $\overline{I^{d+w}} \subseteq J^{w+1}\mathfrak{a} \cap \overline{J^{w+2}} \subseteq J^{w+1}\mathfrak{a} \cap J^{w+1}\overline{I}$ by Proposition 2.8. Let $r \in J^{w+1}\mathfrak{a} \cap J^{w+1}\overline{I}$ and write $r = \sum a_i \mathbf{x}_i = \sum u_i \mathbf{x}_i$ where $a_i \in \mathfrak{a}$, $u_i \in \overline{I}$, and $\mathbf{x}_i$ is a monomial of degree $w + 1$ in the generators of the regular sequence generating J (the index i is really a $(w + 1)$-tuple). Then $a_i - u_i \in J$ since J is generated by a regular sequence. Thus $a_i \in \mathfrak{a} \cap \overline{I} = A$. The desired conclusion follows. □

An obvious question to consider is whether or not the results in Theorems 2.7 and 2.9 may be extended. We do not know if the theorems hold for either regular rings which do not contain a field or for weakly F-regular rings (rings in which every ideal is tightly closed). Lipman's work mentioned below in Remark 2.11 lends some plausability to the mixed characteristic case. The difficulty in proving Theorem 2.7 when R is weakly F-regular but not regular is that $(J : I)^{[q]}$ is generally strictly smaller than $(J^{[q]} : I^{[q]})$.

Example 2.10. In this example we restrict our discussion to the case where R is a 2-dimensional regular local ring. Using work in either [HS] or [L], one obtains that if I is an integrally closed m-primary ideal with minimal reduction J, then $I(J : I) = J(J : I)$. In particular, in the notation of Proposition 2.3, $J : I = \mathfrak{a}_1 = \mathfrak{a}_2 = \ldots = \mathfrak{a}(I, J)$. Even more precisely, I can be written as the n by n minors of a n by $n + 1$ matrix A by the theorem of Hilbert and Burch. For any minimal reduction J of I, the coefficient ideal is the ideal generated by the $n - 1$ by $n - 1$ minors of A. This follows as the $n - 1$ size minors of A are shown in [HS] to generate $J : I$. In particular the coefficient ideal $\mathfrak{a}(I, J)$ is independent of J. A simple example is provided by the nth power m^n of the maximal ideal of R. In this case, the coefficient ideal with respect to any minimal reduction is exactly m^{n-1}.

These results in dimension two suggest that $\mathfrak{a}(I, J)$ could be independent of the minimal reduction J, at least in the case that I is integrally closed.

Remark 2.11. Recently Lipman [L] has proved (using completely different methods) closely related results for regular local rings by introducing the adjoint of an ideal. His methods show the existence of an ideal, called the adjoint of I, denoted $\mathrm{adj}(I)$, such that $\mathrm{adj}(I^{d-1}) \subseteq \mathfrak{a}(I, J)$ for any integrally closed ideal I with minimal reduction J having d generators, and such that $\overline{I^{d+w}} \subseteq \mathrm{adj}(I^{d+w}) = J^{w+1}\mathrm{adj}(I^{d-1})$. We refer the reader to his paper for details. Lipman needs to be in a situation where he can apply certain vanishing theorems which are now known only for rings essentially of finite type over the complex numbers, for analytic local rings, or in dimension two. However, Lipman has pointed out to us that the adjoint behaves well with respect to completion of excellent regular local rings provided enough desingularizations exist: one needs that for every ideal $I \subseteq R$ there exists a desingularization of $\mathrm{Spec}(R)$ on which I becomes invertible. Hence Lipman's results probably work for any excellent local ring containing a field of characteristic zero.

If R is a regular local ring of dimension two, then Lipman proves [L, 3.3] that $\mathrm{adj}(I) = J : I$ for any minimal reduction J of an integrally closed ideal I. By Example 2.10, it follows that in this case, $J : I = \mathrm{adj}(I) = \mathfrak{a}(I, J)$, so that our results are the same as Lipman's in this case. (Although in dimension two, Lipman's results do not need the assumption that R contain a field.) Comparing his proofs to those in this paper again points out the connection between analytic methods and characteristic p methods in commutative algebra.

References

[AH] I. M. Aberbach and C. Huneke, *An improved Briançon-Skoda theorem with applications to the Cohen-Macaulayness of Rees rings*, Math. Ann. **297** (1993), 343–369. MR **95b:**13005

[AHT] I. M. Aberbach, C. Huneke, and N. V. Trung, *Reduction numbers, Briançon-Skoda theorems and the depth of Rees rings*, Compositio Math. **97** (1995), 403–434.

[Ar] M. Artin, *Algebraic structure of power series rings*, Contemp. Math. **13** (1982), 223–227. MR **84b:**13014

[BS] J. Briançon and H. Skoda, *Sur la clôture intégrale d'un idéal de germes de fonctions holomorphes en un point de* C^n, C. R. Acad. Sci. Paris Sér. A **278** (1974), 949–951. MR **49:**5394

[HH] M. Hochster and C. Huneke, *Tight closure, invariant theory, and the Briançon-Skoda theorem*, J. Amer. Math. Soc. **3** (1990), 31–116. MR **91g:**13010

[Hu] C. Huneke, *Hilbert functions and symbolic powers*, Michigan Math. J. **34** (1987), 293–318. MR **89b:**13037

[HS] C. Huneke and I. Swanson, *Cores of ideals in two-dimensional regular local rings*, Michigan Math. J. **42** (1995), 193–208.

[It] S. Itoh, *Integral closures of ideals generated by regular sequences*, J. Algebra **117** (1988), 390–401. MR **90g:**13013

[L] J. Lipman, *Adjoints of ideals in regular local rings*, Math. Res. Letters **1** (1994), 1–17. CMP 95:05

[LS] J. Lipman and A. Sathaye, *Jacobian ideals and a theorem of Briançon-Skoda*, Michigan Math. J. **28** (1981). MR **83m:**13001

[LT] J. Lipman and B. Teissier, *Pseudo-rational local rings and a theorem of Briançon-Skoda about integral closures of ideals*, Michigan Math. J. **28** (1981), 97–116. MR **82f:**14004

[NR] S. Northcott and D. Rees, *Reductions of ideals in local rings*, Math. Proc. Cambridge Philos. Soc. **50** (1954), 145–158. MR **15:**596a

[RS] D. Rees and J. Sally, *General elements and joint reductions*, Michigan Math. J. **35** (1988), 241–254. MR **89h:**13034

[Sp] M. Spivakovsky, *Smoothing of ring homomorphisms, approximation theorems and the Bass-Quillen conjecture*, preprint.
[Sw] I. Swanson, *Joint reductions, tight closure, and the Briançon-Skoda theorem*, J. Algebra **147** (1992), 128–136. MR **93g:**13001

Department of Mathematics, University of Missouri, Columbia, Missouri 65211
E-mail address: aberbach@msindy8.cs.missouri.edu

Department of Mathematics, Purdue University, W. Lafayette, Indiana 47907
E-mail address: huneke@math.purdue.edu