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WELL-BOUNDED OPERATORS ON NONREFLEXIVE BANACH SPACES

CHENG QINGPING AND IAN DOUST

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Abstract. Every well-bounded operator on a reflexive Banach space is of type (B), and hence has a nice integral representation with respect to a spectral family of projections. A longstanding open question in the theory of well-bounded operators is whether there are any nonreflexive Banach spaces with this property. In this paper we extend the known results to show that on a very large class of nonreflexive spaces, one can always find a well-bounded operator which is not of type (B). We also prove that on any Banach space, compact well-bounded operators have a simple representation as a combination of disjoint projections.

1. Introduction

Well-bounded operators were introduced by Smart [Sm] in order to provide a theory for Banach space operators which was similar to the successful theory of self-adjoint operators on Hilbert space, but which included operators whose spectral expansions may only converge conditionally. Well-bounded operators are defined as those which possess a functional calculus for the absolutely continuous functions on some compact interval $[a, b]$ of the real line (more formal definitions will be given in section 2). Smart and Ringrose [R1] proved that on reflexive Banach spaces, well-bounded operators can always be written as an integral with respect to a spectral family of projections. On the other hand, it is easy to find well-bounded operators on most of the classical nonreflexive Banach spaces which do not admit such a representation. Ringrose [R2] later developed a more limited spectral theory for general well-bounded operators, but the technical complications involved have meant that applications of the theory have largely been limited to those well-bounded operators which have a spectral family representation. Examples of these applications may be found in [B] and [BBG].

Well-bounded operators which have a spectral family representation, the so-called well-bounded operators of type (B), were characterized by Berkson and Dowson [BD] and by Spain [Sp] as being those for which the AC-functional calculus is weakly compact. It has been a longstanding open question in the theory as to

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whether there are any nonreflexive spaces on which every well-bounded operator is of type (B).

In earlier work, Doust and deLaubenfels [DdL] showed that if a Banach space X contains a subspace isomorphic to c_0 or a complemented subspace isomorphic to ℓ^1, then there exists a well-bounded operator on X which is not of type (B). In this paper we prove that if X contains any complemented nonreflexive subspace with a basis, then X admits a non-type (B) well-bounded operator.

The situation for well-bounded operators should be compared to that for operators which have a functional calculus for the continuous functions on some compact subset $\Omega \subset \mathbb{C}$. Kluvánek [K] showed that such an operator T can be represented as an integral with respect to a countably additive spectral measure (that is, T is a scalar-type spectral operator) if and only if the $C(\Omega)$ functional calculus is weakly compact. The Banach spaces on which every such functional calculus is weakly compact are those which do not contain a copy of c_0 [DdL].

Section 3 of this paper includes some general results about well-bounded operators with discrete spectra. In particular, it is shown that compact well-bounded operators have a representation theory which is very similar to that for compact self-adjoint operators.

2. Background and notation

In this section we shall give some of the basic definitions regarding well-bounded operators. The theory of well-bounded operators is given in more detail in [Dow].

Throughout X will denote a complex Banach space with dual space X^*. The Banach algebra of all bounded linear operators on X will be denoted by $B(X)$.

An operator $T \in B(X)$ is said to be *well-bounded* if there exist a constant K and a compact interval $[a, b] \subset \mathbb{R}$ such that for all polynomials p,

$$\|p(T)\| \leq K\Big\{|p(a)| + \int_a^b |p'(t)|\ dt\Big\}.$$

Equivalently, T should possess a bounded functional calculus for $AC[a, b]$, the Banach algebra of all absolutely continuous functions on $[a, b]$. That is, there should exist a Banach algebra homomorphism $f \mapsto f(T)$ (extending the natural definition for polynomials) such that

$$\|f(T)\| \leq K\Big\{|f(a)| + \operatorname*{var}_{[a,b]} f\Big\} \equiv K\,\|f\|_{AC}\,.$$

This functional calculus is said to be *weakly compact* if for all $x \in X$, the map $AC[a, b] \to X$, $f \mapsto f(T)x$ is weakly compact. A well-bounded operator whose AC-functional calculus is weakly compact is said to be of type (B). Clearly, if X is reflexive, every well-bounded operator is of type (B).

Definition 2.1. A *spectral family of projections* on a Banach space X is a projection-valued function $E : \mathbb{R} \to B(X)$ such that

(1) E is right continuous in the strong operator topology and has a strong left hand limit at each point in $\mathbb{R}$;
(2) E is uniformly bounded, that is there exists $K < \infty$ such that $\|E(\lambda)\| < K$ for all $\lambda \in \mathbb{R}$;

(3) $E(\lambda)E(\mu) = E(\mu)E(\lambda) = E(\min\{\lambda, \mu\})$ for all $\lambda, \mu \in \mathbb{R}$;
(4) $E(\lambda) \to 0$ (respectively $E(\lambda) \to I$) in the strong operator topology as $\lambda \to -\infty$ (respectively $\lambda \to \infty$).

If $E(\lambda) = 0$ for all $\lambda < a \in \mathbb{R}$ and $E(\lambda) = I$ for all $\lambda \geq b \in \mathbb{R}$, then we say that E is *concentrated on* $[a, b]$, or more loosely, that E is a concentrated spectral family.

The spectral theorem for well-bounded operators of type (B) (see [Dow,DQ]) states that there is a one-to-one correspondence between well-bounded operators of type (B) and concentrated spectral families given by the integral formula

$$T = \int_{[a,b]}^{\oplus} \lambda \, dE(\lambda).$$

General well-bounded operators have an integral representation with respect to a family of projections on X^* known as a decomposition of the identity. The following definition is that given in [DQ] and differs slightly from that given by Ringrose [R2].

Definition 2.2. A *decomposition of the identity* (*for* X) is a family of projections $\{F(\lambda)\}_{\lambda \in \mathbb{R}} \subset B(X^*)$ such that
(1) F is concentrated on some compact interval $[a, b] \subset \mathbb{R}$;
(2) $F(\lambda)F(\mu) = F(\mu)F(\lambda) = F(\min\{\lambda, \mu\})$ for all $\lambda, \mu \in \mathbb{R}$;
(3) F is uniformly bounded;
(4) for all $x \in X$ and $x^* \in X^*$, the function $\lambda \mapsto \langle x, F(\lambda)x^* \rangle$ is Lebesgue measurable;
(5) for all $x \in X$, the map $\gamma_x : X^* \to L^\infty[a, b]$, $x^* \mapsto \langle x, F(\cdot)x^* \rangle$ is continuous when X^* and $L^\infty[a, b]$ are given their weak-$*$ topologies as duals of X and $L^1[a, b]$ respectively;
(6) for all $s \in \mathbb{R}$, if the map $\lambda \mapsto F(\lambda)$ has right weak-$*$ operator topology limit at s, then this limit is $F(s)$.

Given a decomposition of the identity $\{F(\lambda)\}$, there exists a unique well-bounded operator $T \in B(X)$ such that

$$\langle Tx, x^* \rangle = b\langle x, x^* \rangle - \int_a^b \langle x, F(\lambda)x^* \rangle \, d\lambda, \qquad x \in X, \ x^* \in X^*.$$

Every well-bounded operator has such a representation, but in general the decomposition of the identity is not uniquely determined by T. The well-bounded operator T associated with a decomposition of the identity $\{F(\lambda)\}$ is of type (B) if and only if there is a family of projections $\{E(\lambda)\} \subset B(X)$ such that
(1) $F(\lambda) = E(\lambda)^*$ for all $\lambda \in \mathbb{R}$;
(2) E is right continuous in the strong operator topology and has a strong left hand limit at each point in $\mathbb{R}$.

A further concept we shall need is that of a basis of type P. These were introduced by Singer [Si]. A basis $\{x_n\}_{n=1}^\infty$ is of *type P* if
(1) $\inf_n \|x_n\| > 0$;
(2) $\sup_n \left\| \sum_{k=1}^n x_k \right\| < \infty$.

Clearly for such a basis $\sup_n \|x_n\| < \infty$. The canonical example of such a basis is the standard unit basis of c_0.

Let $\{x_n\}_{n=1}^{\infty}$ be a basis for X and let $0 = p_0 < p_1 < p_2 < \dots$ be an increasing sequence of integers. For notational simplicity we shall let $q_n = p_{n-1} + 1$. Then any sequence of nonzero vectors of the form

$$y_n = \sum_{k=q_n}^{p_n} a_k x_k, \qquad n = 1, 2, \dots,$$

where $\{a_k\}$ is a sequence of scalars, is called a *block basic sequence* (with respect to $\{x_n\}$). The span of a block basic sequence is called a block subspace.

If $\{x_n\}_{n\in I}$ is any set of vectors in X, we shall let $[x_n]_{n\in I}$ denote the closed linear span of this set.

3. Some general facts about well-bounded operators

In this section we record some simple results about well-bounded operators with discrete spectra. Dealing with decompositions of the identity is in general rather difficult. If we impose some extra conditions on a well-bounded operator however, then many of the potential complications disappear.

It is, for example, often a nontrivial task to prove that a family of projections is a decomposition of the identity. The next theorem shows that given any uniformly bounded increasing sequence of projections we can construct a corresponding decomposition of the identity and hence a well-bounded operator. A sequence of projections $\{P_n\}_{n=1}^{\infty}$ is said to be increasing if for all $n < m$, $P_n P_m = P_m P_n = P_n$. We shall employ the usual convention that $P_0 = 0$.

Theorem 3.1. *Suppose that $\{\lambda_n\}_{n=1}^{\infty}$ is a strictly increasing sequence of real numbers converging to L. Suppose also that $\{P_n\}_{n=1}^{\infty}$ is a uniformly bounded increasing sequence of projections on X. Define the family of projections $\{F(\lambda)\}_{\lambda\in\mathbb{R}} \subset B(X^*)$ by*

$$F(\lambda) = \begin{cases} 0, & \text{if } \lambda < \lambda_1, \\ P_n^*, & \text{if } \lambda \in [\lambda_n, \lambda_{n+1}), \\ I, & \text{if } \lambda \geq L. \end{cases}$$

Then $\{F(\lambda)\}$ is a decomposition of the identity.

Proof. Most of the conditions are readily verified. The only nontrivial one is (5). We may assume, without loss of generality, that $\lambda_1 = 0$ and $L = 1$. Let $x \in X$ and $x^* \in X^*$. We need to show that if the net $\{x_\alpha^*\}_{\alpha\in\mathcal{A}}$ converges to x^* in the weak-$*$ topology of X^*, then $\{\gamma_x(x_\alpha^*)\}$ converges to $\gamma_x(x^*)$ in the weak-$*$ topology of $L^\infty[0,1]$. Suppose then that for all $u \in X$, $\langle u, x_\alpha^*\rangle \to \langle u, x^*\rangle$. Let $h \in L^1[0,1]$, and fix $\varepsilon > 0$. Then

$$\langle h, \gamma_x(x_\alpha^*) - \gamma_x(x^*)\rangle = \int_0^1 h(\lambda)\langle x, F(\lambda)(x_\alpha^* - x^*)\rangle \, d\lambda.$$

Let $v_n = P_n x$. Then there exist constants K_1 and K_2 such that $\|v_n\| \leq K_1$ $(n = 1, 2, \dots)$ and $\|x_\alpha^* - x^*\| \leq K_2$ $(\alpha \in \mathcal{A})$. Choose N such that $\int_{\lambda_N}^1 |h(\lambda)| \, dt < \varepsilon/(2K_1K_2)$. Then there exists $A \in \mathcal{A}$ such that for all $\alpha > A$ and all $n \leq N$,

$|\langle v_n, x_\alpha^* - x^*\rangle| < \varepsilon/(2\,\|h\|_1)$. Then $\alpha > A$ for

$$\begin{aligned}|\langle h, \gamma_x(x_\alpha^*) - \gamma_x(x^*)\rangle| &= \left|\sum_{n=1}^{\infty}\int_{\lambda_n}^{\lambda_{n+1}} h(\lambda)\langle v_n, x_\alpha^* - x^*\rangle\, d\lambda\right| \\ &\le \int_0^{\lambda_N} |h(\lambda)|\,\varepsilon/(2\,\|h\|_1)\, d\lambda + \int_{\lambda_N}^1 |h(\lambda)|\, K_1K_2\, d\lambda \\ &< \varepsilon.\end{aligned}$$

Thus γ_x has the required continuity property, and $\{F(\lambda)\}$ forms a decomposition of the identity for X.

Of course, once we have constructed a decomposition of the identity in this way, there is a unique well-bounded operator $T \in B(X)$ such that

$$\langle Tx, x^*\rangle = L\langle x, x^*\rangle - \int_{\lambda_1}^{L} \langle x, F(\lambda)x^*\rangle\, d\lambda,$$

for all $x \in X$, $x^* \in X^*$. The well-bounded operator constructed by this method is clearly decomposable in X.

The potential complications of decompositions of the identity also disappear if we know that T is a compact well-bounded operator. This is neither surprising nor difficult, yet we know of no place in the literature where this has been explained. This itself is perhaps surprising given the usefulness of the theory of compact self-adjoint operators in areas such as Sturm-Liouville theory. Recall that if T is a compact self-adjoint operator on a Hilbert space $\mathcal{H}$, with spectrum $\{\lambda_j\}\cup\{0\}$, then there exists a sequence of disjoint orthogonal projections P_j on $\mathcal{H}$ such that

$$T = \sum_{j=1}^{\infty} \lambda_j P_j,$$

where the sum converges unconditionally in the norm topology of $B(\mathcal{H})$. (Here and below we leave it to the reader to make the small changes required for the case when the spectrum of T is finite.) One can obtain a similar representation for compact well-bounded operators. The fundamental lemma is the following.

Lemma 3.2. *Suppose that $T \in B(X)$ is well-bounded and that λ is an isolated point in the spectrum of T. Let E_λ denote the Riesz projection corresponding to the spectral set $\{\lambda\}$. Then for all $x \in E_\lambda X$, $Tx = \lambda x$.*

Proof. First note that $E_\lambda X$ is an invariant subspace for $T - \lambda I$. Now $T - \lambda I$ is well-bounded, so $(T-\lambda I)|E_\lambda X$ is also well-bounded. Note also that $\sigma((T-\lambda I)|E_\lambda X) = \{0\}$ (by, for example, [DS, VII.3.11 and VII.3.20]). But the only quasinilpotent well-bounded operator is 0 [BG, Lemma 5]. Hence $(T-\lambda I)|E_\lambda X = 0$ which proves the result.

An immediate consequence of this is the following generalization of Theorem 4.3(ii) of [BD].

Corollary 3.3. *Let T and λ be as in Lemma 3.2. If $(T-\lambda I)^n x = 0$ for some n, then $(T-\lambda I)x = 0$.*

For the remainder of this section, we shall assume that $T \in B(X)$ is well-bounded and that $\sigma(T) = \{\lambda_j\}_{j=1}^{\infty} \cup \{0\}$, where the sequence λ_j converges to zero. We shall assume that the points λ_j have been ordered so that $|\lambda_1| \geq |\lambda_2| \geq \dots$.

Theorem 3.4. *Suppose that T is as above. Then there exists a uniformly bounded sequence of disjoint projections $P_j \in B(X)$ such that*

$$(*) \qquad T = \sum_{j=1}^{\infty} \lambda_j P_j$$

where the sum converges in the norm topology of $B(X)$.

Proof. Without loss of generality, $\sigma(T) \subset [-a,a]$. For each n let $\sigma_n = \sigma(T) \setminus \{\lambda_1, \dots, \lambda_n\}$. Define functions $f_n \in AC[-a,a]$ having the following properties:

(1) f_n is nondecreasing and piecewise linear;
(2) $f_n \equiv 0$ on an open interval which includes σ_n;
(3) there exists an open interval U_n such that $U_n \cap \sigma(T) = \sigma_n$ and $f_n(x) = x$ for all $x \in [-a,a] \setminus U_n$.

Clearly, if we define $e(x) = x$, $x \in [-a,a]$, then $f_n \to e$ in $AC[-a,a]$ and hence $f_n(T) \to e(T) = T$.

It just remains to prove that $f_n(T)$ is a partial sum of $(*)$. Let E_j denote the Riesz projection associated to the spectral set $\{\lambda_j\}$, and let $X_j = E_j X$. Let E_{σ_n} denote the Riesz projection associated to σ_n, and let $X_{\sigma_n} = E_{\sigma_n}$. Then $X = X_1 \oplus \dots \oplus X_n \oplus X_{\sigma_n}$. Furthermore, each of these subspaces is an invariant subspace for T. It follows that for all $f \in AC[-a,a]$,

$$f(T) = f(T|X_1) \oplus \dots \oplus f(T|X_n) \oplus f(T|X_{\sigma_n}).$$

But by Lemma 3.2, $T|X_j = \lambda_j I$ (on the appropriate space). Moreover $f_n \equiv 0$ on an open neighborhood of $\sigma(T|X_{\sigma_n})$. Thus

$$f_n(T) = \sum_{j=1}^{n} \lambda_j E_j.$$

The projections E_j are clearly disjoint. Let $g_j = (f_j - f_{j-1})/\lambda_j$. Then $\|g_j\|_{AC} \leq 2$ and $g_j(T) = E_j$, so the well-boundedness of T shows that the projections E_j are uniformly bounded.

Corollary 3.5. *Suppose that T is a compact well-bounded operator with spectrum $\{0\} \cup \{\lambda_j\}_{j=1}^{\infty}$ (listed in order of decreasing magnitude). Then there exists a uniformly bounded sequence of disjoint projections $P_j \in B(X)$ such that*

$$T = \sum_{j=1}^{\infty} \lambda_j P_j$$

where the sum converges in the norm topology of $B(X)$.

As should be expected, the sum $(*)$ in Theorem 3.4 may only converge conditionally. That is, in contrast to the situation for compact self-adjoint operators described above, the order in which this sum is taken is important. Of course, if λ_j converges to 0 quickly enough, then the sum will converge absolutely. The following example shows that this conditional convergence may occur.

Example. Let $X = bv$, the space of all sequences of bounded variation. Define the well-bounded operator $T \in B(X)$ by

$$T(x_1, x_2, \dots) = (x_1, x_2/2, \dots, x_n/n, \dots).$$

Clearly $\sigma(T) = \{\lambda_j\} \cup \{0\}$ where $\lambda_j = 1/j$. The appropriate projections P_j are given by

$$P_j(x_1, x_2, \dots) = (0, \dots, 0, x_j, 0, \dots),$$

where the nonzero term on the right-hand-side occurs in the j-th position. Let $S_n = \sum_{j=1}^{n} \lambda_{2j} P_{2j}$. Then

$$S_n(1, 1, \dots) = (0, 1/2, 0, 1/4, 0, \dots, 0, 1/2n, 0, 0, \dots).$$

Thus $\|S_n\| \to \infty$ as $n \to \infty$, proving that the sum $\sum_{j=1}^{\infty} \lambda_j P_j$ does not converge unconditionally.

4. The main result

Theorem 3.1 gives an easy way of constructing well-bounded operators. Testing whether an operator constructed in this way is of type (B) is also easy. The next result follows immediately from the characterization of well-bounded operators of type (B) given in section 2.

Proposition 4.1. *Let $\{\lambda_j\}_{j=1}^{\infty}$ and $\{P_j\}_{j=1}^{\infty}$ be as in Theorem 3.1, and let T be the (unique) well-bounded operator whose decomposition of the identity is the family $\{F(\lambda)\}$ constructed in that theorem. Then T is of type (B) if and only if $\lim_{n\to\infty} P_n$ exists in the strong operator topology.*

Our main result is the following.

Theorem 4.2. *Let X be a nonreflexive Banach space which admits a complemented nonreflexive subspace Y with a basis. Then there exists a well-bounded operator $T \in B(X)$ which is not of type (B).*

The proof requires a number of lemmas. The first, whose proof we shall omit, allows us to just concentrate on finding operators on Y.

Lemma 4.3. *Suppose that T_i is a well-bounded operator on X_i $(i = 1, \dots, n)$. Then $T = T_1 \oplus \cdots \oplus T_n$ is a well-bounded operator on $X = X_1 \oplus \cdots \oplus X_n$. Furthermore T is of type (B) if and only if each T_i is of type (B).*

Proposition 4.4. *Suppose that X has a basis $\{x_n\}_{n=1}^{\infty}$, and that there exists a block basis (with respect to this basis) $\{u_n\}_{n=1}^{\infty}$ which is of type P. Then there exists a well-bounded operator $T \in B(X)$ which is not of type (B).*

Proof. Let $\{x_n^*\}_{n=1}^{\infty}$ be the sequence of coefficient functionals associated to the basis $\{x_n\}$. Without loss of generality we can assume that $u_1 = x_1$. Let $X_1 = [x_1]$. Suppose that for $k > 1$, $u_k \in X_k = [x_{q_k}, \dots, x_{p_k}]$. Let P_k denote the projection onto X_k. Since $\{x_n\}$ is a basis there exists a constant K such that $\sup_m \left\| \sum_{k=1}^{m} P_k \right\| \le K$.

Define the linear transformation Q_1 by

$$Q_1 x = \langle x, x_1^* \rangle u_2 + P_2 x.$$

Clearly Q_1 is a bounded projection on X. For $n > 1$ define

$$Q_n x = \langle x, x_1^* \rangle \sum_{i=2}^{n} u_i + \sum_{i=2}^{n} P_i x.$$

Again it is simple to check that this is a projection. Now

$$\begin{aligned} \|Q_n x\| &\leq |\langle x, x_1^* \rangle| \left\| \sum_{i=1}^{n} u_i - u_1 \right\| + \left\| \sum_{i=2}^{n} P_i \right\| \|x\| \\ &\leq \left(\|x_1^*\| \left(\left\| \sum_{i=1}^{n} u_i \right\| + \|u_1\| \right) + \left\| \sum_{i=1}^{n} P_i \right\| + \|P_1\| \right) \|x\| \\ &\leq M \|x\| \end{aligned}$$

since $\{u_n\}$ is a block basic sequence of type P. An easy calculation shows that this sequence of projections is increasing.

Construct a well-bounded operator $T \in B(X)$ by Proposition 4.1. To show that T is not of type (B) we need to show that Q_n does not converge in the strong operator topology. To see this it suffices to show that $\lim_{n\to\infty} Q_n x_1$ does not exist. From the definition of Q_n we have that $Q_n x_1 = \sum_{i=2}^{n} u_i$. Thus

$$\|Q_n x_1 - Q_{n-1} x_1\| = \|u_n\| \not\to 0$$

since $\{u_n\}$ is a block basis of type P. It follows then that T is a well-bounded operator on X which is not of type (B).

Lemma 4.5 ([Si, Theorem 2])**.** *If X is a nonreflexive Banach space with a basis, then there exists a basis of a block subspace of X which is of type P.*

Proof of Theorem 4.2. Write $X = Y \oplus Z$. By Lemma 4.5, Y has a block basic sequence of type P. By Proposition 4.4, there exists a well-bounded operator $T_0 \in B(Y)$ which is not of type (B). By Lemma 4.3, the operator $T = T_0 \oplus 0$ is a well-bounded operator which is not of type (B) on X.

5. Discussion of the hypotheses

It is appropriate at this point to make some remarks about the scope of Theorem 4.2. The following facts are easy to establish.

(1) If X contains a complemented copy of ℓ^1, then it satisfies the hypotheses of Theorem 4.2.
(2) If X is separable and contains a copy of c_0, then it satisfies the hypotheses of Theorem 4.2.

On the other hand, ℓ^∞ contains a copy of c_0, but does not contain a complemented subspace with a basis. (It it did, it would have to admit a Schauder decomposition; see [Si2]).

It is not difficult to show that there are spaces which are covered by the present theorem, but which do not contain a copy of c_0 or ℓ^1. The James space J is an example [LT, Example 1.d.2].

Perhaps more difficult is producing an example of a nonreflexive space which does not satisfy the hypotheses of Theorem 4.2. One example of such a space can be found in the construction of Pisier [P, p. 201]. Pisier's space is nonreflexive (since it contains a copy of ℓ^1), yet it is easily seen to have no complemented subspace with a basis.

6. Some final thoughts

It might be noted that in all the constructions in [DdL] and in this paper, the well-bounded operator fails to be of type (B) not because its decomposition of the identity is not formed from the adjoints of projections on X, but rather because the projections on X fail to have the appropriate continuity properties. It is possible that on some nonreflexive spaces, every well-bounded operator is decomposable in X. At present, there seems to be no standard way of constructing examples of well-bounded operators which are not decomposable in X; even on ℓ^1 there is no known example.

To construct a nonreflexive Banach space on which every well-bounded operator is of type (B), one might try to find a nonreflexive space X with the following properties:

(1) X is hereditarily indecomposable;
(2) there exists a constant c such that if P is a projection onto a subspace of dimension n, then $\|P\| \geq cn^{1/2}$.

If X has these two properties and $T \in B(X)$, then $\sigma(T)$ is either

(i) $\{\lambda_j\}_{j=1}^{\infty} \cup \{\lambda_0\}$, where $\{\lambda_j\}$ is a sequence whose only accumulation point is λ_0, or;
(ii) a finite set, $\{\lambda_j\}_{j=1}^{N}$.

If T is well-bounded, then its decomposition of the identity $\{F(\lambda)\}$ must be constant between the points λ_j. General theory of well-bounded operators then tells us that for $j \geq 1$, $F(\lambda_j) = P_j^*$ for some projection $P_j \in B(X)$. Since X is hereditarily indecomposable, the range of P_j either has finite dimension, or finite codimension. If case (i) holds, then there must be a subsequence $\{\lambda_{j_k}\}$ of $\{\lambda_j\}$ which is monotone, and for which the ranges of the P_{j_k} are either all finite dimensional, or all finite codimensional. However, (2) above means that this is impossible since $\sup_j \|P_j\| < \infty$. Thus, $\sigma(T) = \{\lambda_j\}_{j=1}^{N}$. But a well-bounded operator with finite spectrum is necessarily of type (B). We do not know if such a construction is possible.

On the other hand, the following simple result provides some evidence that the only space on which every well-bounded operator is of type (B) are the reflexive spaces.

Proposition 6.1. *Every bounded linear operator from $AC[0,1]$ to X is weakly compact if and only if X is reflexive.*

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Department of Mathematics, Jingzhou Teachers College, Jingzhou, Hubei, People's Republic of China

Current address: School of Mathematical and Physical Sciences, Murdoch University, Murdoch, WA 6150, Australia

E-mail address: `cheng@prodigal.murdoch.edu.au`

School of Mathematics, University of New South Wales, Sydney, New South Wales 2052, Australia