THE $p^n$ THEOREM FOR SEMISIMPLE HOPF ALGEBRAS

AKIRA MASUOKA

(Communicated by Ken Goodearl)

ABSTRACT. We give an algebraic version of a result of G. I. Kac, showing that a semisimple Hopf algebra $A$ of dimension $p^n$, where $p$ is a prime and $n > 0$, over an algebraically closed field of characteristic 0 contains a non-trivial central group-like. As an application we prove that, if $n = 2$, $A$ is isomorphic to a group algebra.

INTRODUCTION

Throughout the paper we work over an algebraically closed field $k$ of characteristic 0.

Let $p$ be a prime. Recently Y. Zhu [Z] proved that a Hopf algebra of dimension $p$ is isomorphic to the group algebra $kC_p$ of the cyclic group $C_p$ of order $p$. For this, he reformulated G. I. Kac’s Theorem [K, Theorem 2] on ‘ring groups’. In this paper, first we give an algebraic version of another result [K, Corollary 2] of Kac to show that a semisimple Hopf algebra of dimension $p^n$ with a positive integer $n$ contains a non-trivial central group-like. Secondly, applying the first result, we classify all semisimple Hopf algebras of dimension $p^2$. Namely we prove that such a Hopf algebra is isomorphic to the group algebra $kC_{p^2}$ or $k(C_p \times C_p)$.

THE $p^n$ THEOREM

Let $A$ be a finite-dimensional Hopf algebra with antipode $S$. Suppose that $A$ is semisimple as an algebra, or equivalently cosemisimple as a coalgebra, or equivalently involutory, namely $S \circ S = \text{id}$ (See [LR1, Theorem 3.3]; [LR2, Theorems 1, 3]).

Denote by $A^*$ the dual Hopf algebra $\text{Hom}_k(A, k)$ of $A$. Let $\lambda$ be an integral in $A^*$ such that $\lambda(1) = (\dim A)1$. Then it follows by [LR2, Proposition 1] that $\lambda$ is the character of the regular representation of $A$, that is, $\lambda(a)$ for $a \in A$ equals the trace of the right (or left) multiplication $b \mapsto ba$ ($b \in A$) by $a$. Regard $A^*$ as a right $A$-module with the action $\cdot$ determined by

$$(f \cdot a)(b) = f(bS(a)) \quad (f \in A^*, a, b \in A).$$

Then one sees from [Sw, Theorem 5.1.3] that

$$\alpha: A \to A^*, \quad \alpha(a) = \lambda \cdot a$$

Received by the editors April 18, 1994 and, in revised form, September 29, 1994.

1991 Mathematics Subject Classification. Primary 16W30.
gives a right $A$-linear and left $A^*$-linear isomorphism, where $A$ has the left $A^*$-module structure arising from the natural right $A$-comodule structure.

**Lemma.** $\alpha$ gives linear isomorphisms between each pair in (a), (b) below.

(a) The subalgebra $kG(A)$ of $A$ spanned by the group-likes $G(A)$ and the sum of the 1-dimensional ideals of $A^*$.

(b) The center $Z(A)$ of $A$ and the subalgebra $C_k(A)$ of $A^*$ spanned by the characters of $A$.

**Proof.** For (a), one has only to see that $\alpha$ gives a 1-1 correspondence between the 1-dimensional left $A^*$-submodules of $A$ and of $A^*$. (Note that a 1-dimensional left ideal of $A^*$ is actually two-sided since $A^*$ is semisimple.)

For (b), we see first that the antipode $S$ gives a permutation of the primitive central idempotents in $A$, which form a $k$-basis of $Z(A)$. Let $e$ be a primitive central idempotent in $A$ and let $\chi$ be the irreducible character corresponding to $e$ (that is, the character of the representation $A \to eA$). Then it follows that $\alpha(S(e)) = \chi(1)\chi(\neq 0)$, where $\chi(1)$ equals the degree of $\chi$. From these we have that $Z(A) \simeq C_k(A)$. \hfill $\square$

**Corollary.** There is a 1-1 correspondence $g \mapsto I$ between the central group-likes $g$ in $A$ and the 1-dimensional ideals $I$ of $A^*$ included in $C_k(A)$, such that $1 \mapsto k\lambda$.

**Theorem 1.** Suppose that the dimension $\dim A = p^n$, where $p$ is a prime and $n$ is a positive integer. Then there is a non-trivial central group-like in $A$.

**Proof.** By the Corollary it suffices to prove that there is a 1-dimensional ideal $I \neq k\lambda$ which is included in $C_k(A)$.

Let $e_1 = \frac{1}{\dim A} \lambda$, $e_2, \ldots, e_m$ be orthogonal primitive idempotents in $C_k(A)$ whose sum is 1. Then we have

$$A^* = k\lambda \oplus e_2 A^* \oplus \cdots \oplus e_m A^*.$$

Since each $\dim e_i A^*$ divides $p^n$ by [Z, Theorem 1], it follows by counting dimensions that there is $2 \leq i \leq m$ such that $\dim e_i A^* = 1$. This $e_i A^*$ is the required $I$. \hfill $\square$

This theorem is an algebraic (and hopefully accessible) version of a result [K, Corollary 2] of G. I. Kac on ‘ring groups’.

**The $p^2$ theorem**

**Theorem 2.** A semisimple Hopf algebra of dimension $p^2$ with a prime $p$ is isomorphic to the group algebra $kC_{p^2}$ or $k(C_p \times C_p)$, where $C_n$ is the cyclic group of order $n$.

**Proof.** Let $A$ be a semisimple Hopf algebra of dimension $p^2$. It suffices to show that $A$ is commutative and cocommutative.

It follows from Theorem 1 and [NZ, Theorem 7] that there is a group $G$ of group-likes in $A$ such that $G \subset Z(A)$ and the order $|G| = p$. Since $G \subset Z(A)$, the Hopf subalgebra $K = kG$ is normal, so that we have an extension [M1, Definition 1.3]

$$1 \to K \to A \to H \to 1$$

of finite-dimensional Hopf algebras. Since this is cleft (roughly $A \simeq K \otimes_k H$) by [S, Theorem 2.4], it follows that $\dim H = p$, so that $H \simeq kC_p$ by Zhu’s Theorem [Z, Theorem 2] cited in the Introduction. One sees immediately from [DT, Theorem 11] that, as an algebra including $K$, $A$ is a crossed product of $H$ over $K$. In the
The terminology of group theory, $A$ is a crossed product $K \ast C_p$ [P, p. 2] of $C_p$ over $K$. Here the associated action of $C_p$ on $K$ is trivial, since $K$ is central in $A$. Hence $A$ is a twisted group ring $K^t[C_p]$ [P, p. 4], a cyclic extension of a central subalgebra. This is trivially commutative. Thus $A$ is commutative. By applying the result to $A^*$, it follows that $A$ is cocommutative.

Using this result we classify all semisimple Hopf algebras of dimension $p^3$ with an odd prime $p$ in the final version of [M3], while such Hopf algebras of dimension $8 = 2^3$ are classified in [M2].

**Acknowledgment**

I would like to thank the referee for his or her suggestions to improve the exposition of the paper.

**References**


Department of Mathematics, Shimane University, Matsue, Shimane 690, Japan