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CONJUGATE POINTS IN $\mathcal{D}_\mu(T^2)$

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ABSTRACT. An example of a geodesic in $\mathcal{D}_\mu(T^2)$ with conjugate points is given, thus providing an affirmative answer to a question of V.I. Arnol′d.

1. Introduction

Let $\mathcal{D}_\mu(T^2)$ be the group of smooth volume preserving diffeomorphisms of a 2 torus equipped with a locally euclidean metric $\langle , \rangle$. It is convienient to enlarge it to include all such diffeomorphisms which are of Sobolev class H^s. If $s > 2$ this new set $\mathcal{D}^s_\mu(T^2)$, as demonstrated by Ebin and Marsden [EM], can be given a structure of an infinite dimensional submanifold of a weak Riemannian manifold $\mathcal{D}^s(T^2)$, the group of all H^s diffeomorphisms of T^2. The tangent space to $\mathcal{D}^s(T^2)$ ($\mathcal{D}^s_\mu(T^2)$) at a point η consists of all (divergence free) H^s vector fields on T^2 which cover η. The weak Riemannian structure is given by the L^2 inner product

$$(1.1) \qquad (X, Y)_{L^2} = \int_{T^2} \langle X(x), Y(x)\rangle_{\eta(x)}\, dx$$

where $X, Y \in T_\eta\mathcal{D}^s(T^2)$.

For an arbitrary H^s vector field X on T^2 if Δ denotes the Laplacian of the euclidean metric $\langle , \rangle$, then letting f be a solution of the equation

$$\Delta f = \operatorname{div} X$$

we obtain an L^2 orthogonal decomposition of X into the divergence free and gradient parts

$$X = (X - \operatorname{grad} f) + \operatorname{grad} f.$$

Since for each $\eta \in \mathcal{D}^s_\mu(T^2)$ the right translation $R_\eta(\xi) = \xi \circ \eta$ is an isometry of (1.1), this induces a splitting of $T_\eta\mathcal{D}^s(T^2)$ into a direct sum

$$(1.2) \qquad T_\eta\mathcal{D}^s(T^2) = T_\eta\mathcal{D}^s_\mu(T^2) \oplus_{L^2} \operatorname{grad} H^{s+1}(T^2) \circ \eta.$$

We will denote by P_η and Q_η the projections onto the first and the second summands respectively.

The metric (1.1) induces on $\mathcal{D}^s(T^2)$ and $\mathcal{D}^s_\mu(T^2)$ right invariant Levi Civita connections $\bar\nabla$ and $\tilde\nabla = P_\eta\bar\nabla$ which have smooth exponential maps. The geodesics

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$t \to \xi(t)$ of $\tilde{\nabla}$ correspond to periodic motions of an ideal fluid in R^2 and can be obtained from a variational principle as stationary points of the L^2 action integral

$$E(\xi) = \frac{1}{2}\int_0^l |\dot{\xi}(t)|^2_{L^2} dt \tag{1.3}$$

Using the splitting (1.2) the Euler-Lagrange equations of (1.3) are often written as a nonlinear partial differential equation

$$\begin{gathered} \partial_t V(t) + \nabla_{V(t)} V(t) = \operatorname{grad} p(t), \\ \operatorname{div} V(t) = 0 \end{gathered} \tag{1.4}$$

where $V(t) = \dot{\xi}(t) \circ \xi^{-1}(t)$ is a time dependent vector field on T^2 and $p(t)$ is the pressure function which can be determined from $V(t)$.

The idea that studying the geometry of $\mathcal{D}_\mu(T^2)$ is important for hydrodynamics goes back to Arnol′d [A], where he investigated stability of ideal fluids and presented complete computations of the curvature tensor of $\mathcal{D}_\mu(T^2)$ at the identity e. He showed that in many directions the sectional curvature was negative and gave an example of a two-plane for which it was positive. Arnol′d then asked whether there were any conjugate points in $\mathcal{D}_\mu(T^2)$. In this paper we show that a modification of Arnol′d's example yields a geodesic in $\mathcal{D}_\mu(T^2)$ which provides an affirmative answer to this question.

Namely, on T^2 consider the function $\phi = \frac{-1}{\pi\sqrt{40}} \cos 6x_1 \cos 2x_2$. Let $V = \operatorname{rot}\phi$ be the corresponding divergence free vector field. We will prove the following

Theorem 1. *Let $\eta(t)$ be a geodesic in $\mathcal{D}^s_\mu(T^2)$ emanating from $\eta(0) = e$ in the direction of $\dot{\eta}(0) = V$. There exist $k > 0$ and $t_c \in (0, \pi\sqrt{\frac{2}{k}}]$ such that $\eta(t_c)$ is conjugate to e along η.*

For a general Hilbert Riemannian manifold $\mathcal{M}$ two types of conjugate points are possible. If $\xi(t)$ is a geodesic in $\mathcal{M}$ and $\exp^{\mathcal{M}}$ is the exponential map of the (possibly weak) Riemannian metric, we say that the point $\xi(l)$ is epiconjugate (monoconjugate) to $\xi(0)$ along ξ if the map $\exp^{\mathcal{M}}_{\xi(0)*l\dot{\xi}(0)} : T_{l\dot{\xi}(0)}T_{\xi(0)}\mathcal{M} \to T_{\xi(l)}\mathcal{M}$ is not onto (one to one).[1] We shall say that a point is conjugate if it is either epi- or monoconjugate. In finite dimensions the two types coincide.

Existence of conjugate points in $\mathcal{D}^s_\mu(M)$ is related to stability of fluid flows in M. The method presented in this paper may be used to find conjugate points along geodesics corresponding to flows on arbitrary compact Riemannian manifold M.

2. Outline of the proof

We begin with some basic facts. Let $\xi(s,t) : (-\epsilon, \epsilon) \times [0, l] \to \mathcal{D}^s_\mu(T^2)$ be a two parameter variation of a geodesic $\xi(t)$ with $\xi(s,0) = \xi(0) = e$ and $\xi(s,l) = \xi(l)$, and let $Z(t) = \partial_s \xi(s,t)|_{s=0}$ be the associated vector field. Then the first and the second variations of the L^2 action (1.3) are

$$E'(\xi)^l_0(Z) = (Z, \dot{\xi})|^l_0 - \int_0^l (Z, \tilde{\nabla}_{\dot{\xi}}\dot{\xi}) dt = 0,$$

$$E''(\xi)^l_0(Z,Z) = \int_0^l \{(\tilde{\nabla}_{\dot{\xi}} Z, \tilde{\nabla}_{\dot{\xi}} Z) - (\tilde{R}_\xi(Z,\dot{\xi})\dot{\xi}, Z)\} dt \tag{2.5}$$

[1] This definition was given by Grossman [G].

where $\tilde{R}_\xi$ is the curvature tensor of the L^2 metric on $\mathcal{D}^s_\mu(T^2)$.

We will also need

Gauss Lemma. *If $X \in T_e\mathcal{D}^s_\mu(T^2)$ and $Y \in T_{tX}T_e\mathcal{D}^s_\mu(T^2)$ is such that $(Y, X) = 0$, then $(\tilde{\exp}_{e*tX}X, \tilde{\exp}_{e*tX}Y) = 0$.*

The proof of this lemma follows readily, as in [CE], pp. 8–9, from the first variation formula in (2.5).

Let $V = \operatorname{rot}\phi$ be the vector field in Theorem 1. We first note that $P_e\nabla_V V = 0$. This implies that V is a stationary (that is, independent of time) solution of (1.4) and hence $\eta(t)$ is a one parameter subgroup of $\mathcal{D}^s_\mu(T^2)$.

Next, let $W = \operatorname{rot}\psi$ be a divergence free vector field on T^2 corresponding to the function $\psi = -\cos(mx_1 + x_2) - \cos(mx_1 - 3x_2)$, m an integer. A lengthy but straightforward computation shows that the expression

$$\frac{1}{|W|^2_{L^2}}(\nabla_{[V,W]}V + \nabla_V[V, W], W) \tag{2.6}$$

can be estimated from below by a positive constant k whenever $m = \pm4, \pm5, \pm6$ or ±7. Let $t_k = \pi\sqrt{\frac{2}{k}}$.

Lemma 2. *Let $\eta(t)$ be the geodesic in the theorem. There exists a vector field $\tilde{W}(t)$ along η vanishing at e and $\eta(t_k)$ for which the bilinear form $E''(\eta)^{t_k}_0(\tilde{W}, \tilde{W}) < 0$.*

Proof. Let $f(t)$ be a nonzero function such that $f(0) = f(t_k) = 0$. Right translate $W \in T_e\mathcal{D}_\mu(T^2)$ to a vector field along η and define $\tilde{W}(t) = f(t)\cdot W\circ\eta$. A convienient formula for the connection $\tilde{\nabla}$ can be obtained from (1.4). Applied to $\tilde{W}(t)$ in the direction $\dot{\eta}$, it gives

$$\tilde{\nabla}_{\dot{\eta}}\tilde{W} = P_\eta\bar{\nabla}_{\dot{\eta}}\tilde{W} = P_\eta\Big(\frac{d}{dt}(\tilde{W}\circ\eta^{-1})\circ\eta + (\nabla_{\dot{\eta}\circ\eta^{-1}}\tilde{W}\circ\eta^{-1})\circ\eta\Big)$$

$$= f'\cdot W\circ\eta + f\cdot(P_e\nabla_V W)\circ\eta.$$

Thus we obtain

$$\begin{aligned}(\tilde{\nabla}_{\dot{\eta}}\tilde{W}, \tilde{\nabla}_{\dot{\eta}}\tilde{W}) &= {f'}^2|W|^2_{L^2} + 2ff'(W, P_e\nabla_V W) + f^2|P_e\nabla_V W|^2_{L^2}\\ &= {f'}^2|W|^2_{L^2} + f^2|P_e\nabla_V W\|^2_{L^2}\end{aligned}$$

since

$$(W, P_e\nabla_V W) = (W, \nabla_V W) = \frac{1}{2}(V, \operatorname{grad}\langle W, W\rangle) = 0$$

because both V and W are divergence free.

On the other hand we have $(\tilde{R}_\eta(\tilde{W}, \dot{\eta})\dot{\eta}, \tilde{W}) = f^2(\tilde{R}_e(W, V)V, W)$ by the right invariance of the weak metric and connection $\tilde{\nabla}$ and $\dot{\eta} = V\circ\eta$. Since $\mathcal{D}^s_\mu(T^2)$ is a Riemannian submanifold of the full diffeomorphism group $\mathcal{D}^s(T^2)$, their curvature tensors $\tilde{R}_e$ and $\bar{R}_e$ respectively are related by the Gauss-Codazzi equations. Moreover, for any smooth compact Riemannian manifold M, the curvature tensor of $\mathcal{D}^s(M)$ is completely determined by the curvature tensor R of the manifold M. Thus from (1.2) and the equations of Gauss-Codazzi together with the estimate on

(2.6) and the fact that $P_e\nabla_V V = 0$ we get

$$\begin{aligned}
&E''(\eta)_0^{t_k}(\tilde{W},\tilde{W}) \\
&= \int_0^{t_k}\{{f'}^2|W|_{L^2}^2 + f^2|P_e\nabla_V W|_{L^2}^2 \\
&\qquad - f^2(R(W,V)V,W) - f^2(\nabla_V V,\nabla_W W) + f^2|Q_e\nabla_V W|_{L^2}^2\}dt \\
&= \int_0^{t_k}\{{f'}^2|W|_{L^2}^2 - f^2(\nabla_{[V,W]}V + \nabla_V[V,W],W)\}dt \\
&\le |W|_{L^2}^2\int_0^{t_k}\{{f'}^2 - kf^2\}dt.
\end{aligned}$$

If we now take $f(t) = \sin t\sqrt{\frac{k}{2}}$, then $\tilde{W}(0) = \tilde{W}(t_k) = 0$ and $E''(\eta)_0^{t_k}(\tilde{W},\tilde{W}) \le -\frac{\pi}{2}\sqrt{\frac{k}{2}} < 0$. □

Since $(W,V) = 0$, the field $\tilde{W}(t)$ constructed above is in fact perpendicular to $\dot{\eta}(t)$ in the metric (1.1) by right invariance.

To proceed we need the following

Lemma 3. *Let $\eta(t)$ be as above. Let $Z(t)$ be a smooth vector field on η such that $Z(0) = Z(t_k) = 0$. If there are no points conjugate to e along $\eta(t)$ for $0 \le t \le t_k$, then $E''(\eta)_0^{t_k}(Z,Z) \ge 0$.*

Proof. Let $\tilde{R}_t = \tilde{R}_\eta(\cdot,\dot{\eta})\dot{\eta} : T_\eta\mathcal{D}_\mu^s(T^2) \to T_\eta\mathcal{D}_\mu^s(T^2)$ and τ_t denote the curvature operator and the parallel translation of the weak metric (1.1) along η. Let $U(t)$ be the evolution operator of the Jacobi equation along η written as a first order system in $T_e\mathcal{D}_\mu^s(T^2) \times T_e\mathcal{D}_\mu^s(T^2)$

$$\frac{d}{dt}\begin{pmatrix} Y \\ Y' \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ (\tau_t)^{-1}\circ\tilde{R}_t\circ\tau_t & 0 \end{pmatrix}\begin{pmatrix} Y \\ Y' \end{pmatrix} = 0.$$

Since for any t the map $X \to (\tau_t)^{-1}\circ\tilde{R}_t\circ\tau_t X$ is linear and bounded in the (strong) H^s topology of $T_e\mathcal{D}_\mu^s(T^2)$, $s > 2$ (cf. [M]), the operator $U(t)$ is a topological isomorphism of $T_e\mathcal{D}_\mu^s(T^2) \times T_e\mathcal{D}_\mu^s(T^2)$. Thus the linear map[2]

$$T_{tV}T_e\mathcal{D}_\mu^s(T^2) \ni X \to \tilde{\exp}_{e*tV}tX = \tau_t\circ\pi_1\circ U(t)(0,X) \in T_{\eta(t)}\mathcal{D}_\mu^s(T^2)$$

is bounded. The assumption about conjugate points implies it is one to one and onto and therefore also an isomorphism for each t by the open mapping theorem. It follows that $\tilde{\exp}_e$ is a diffeomorphism on an open neighbourhood $\mathcal{U}(t)$ of each tV in $T_e\mathcal{D}_\mu^s(T^2)$. By compactness of $[0,t_k]$ we can find a finite number of such neighbourhoods $\mathcal{U}(t_1)$, ... , $\mathcal{U}(t_N)$ whose union $\mathcal{U} = \bigcup_{i=1}^N\mathcal{U}(t_i)$ covers the set $\{tV : 0 \le t \le t_k\}$ and such that $\eta([0,t_k]) \subset \bigcup_{i=1}^N \tilde{\exp}_e\mathcal{U}(t_i) = \mathcal{V}$.

Similarly by compactness it is possible to choose a $\delta > 0$ so that each $\tilde{\exp}_{\eta(t)}$ maps a ball $B^s(\delta)$ of radius δ in $T_{\eta(t)}\mathcal{D}_\mu^s(T^2)$ diffeomorphically onto $\tilde{\exp}_{\eta(t)}B^s(\delta) \subset \mathcal{V}$ for $t \in [0,t_k]$.

Let $\xi(s,t) = \tilde{\exp}_{\eta(t)}sZ(t)$. Then for a fixed s, satisfying $s < \frac{\delta}{\max_t|Z(t)|_{H^s}}$, the curve $t \to \xi(s,t)$ lies in $\mathcal{V}$. Consequently $c_s(t) = \tilde{\exp}_e^{-1}\xi(s,t)$ is a curve in $\mathcal{U}$ joining

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[2] Here π_1 denotes the projection onto the first summand of $T_e\mathcal{D}_\mu^s \times T_e\mathcal{D}_\mu^s$.

$0 \in T_e\mathcal{D}^s_\mu(T^2)$ and t_kV. If we set $c_s(t) = r(t)\frac{c(t)}{|c(t)|_{L^2}}$ where $r : [0, t_k] \to R$ is a function satisfying $r(0) = 0$ and $r(t_k) = t_k$, we obtain

$$\dot{c}_s(t) = \dot{r}(t)\frac{c_s(t)}{|c_s(t)|_{L^2}} + r(t)\frac{d}{dt}(\frac{c_s(t)}{|c_s(t)|_{L^2}}).$$

It follows from the Gauss Lemma that

$$|\dot{\xi}_s(t)|^2_{L^2} = |\frac{d}{dt}(\tilde{\exp}_e c_s(t))|^2_{L^2} = |\tilde{\exp}_{e*c_s(t)}\dot{c}_s(t)|^2_{L^2} \geq (\dot{r}(t))^2.$$

Therefore by the Cauchy-Schwartz inequality $E(\xi_s) \geq \frac{1}{2}\int_0^{t_k}(\dot{r}(t))^2 dt \geq \frac{1}{2}t_k = E(\eta)$ for $s < \frac{\delta}{\max|Z(t)|_{H^s}}$, but this implies that $E''(\eta)(Z, Z) \geq 0$. □

The theorem is now a consequence of the above lemmas.

3. Concluding remarks

Remark 1. A simple corollary of the above theorem is that conjugate points can be found on $\mathcal{D}^s_\mu(T^n)$ where T^n is a flat torus of arbitrary dimension n.

Remark 2. More examples of conjugate points in $\mathcal{D}^s_\mu(M)$ can be found if M has positive curvature. For instance, any rotation of a sphere in R^3 yields a stationary solution of the Euler equations (1.4). Sectional curvatures along the corresponding geodesic $\eta(t)$ in $\mathcal{D}^s_\mu(S^2)$ are nonnegative and $\eta(t)$ can be shown to have monoconjugate points.

Remark 3. In the light of Arnol′d's curvature computations mentioned in the Introduction it is natural to expect that there exist in $\mathcal{D}^s_\mu(T^2)$ geodesics without conjugate points. Indeed, this is shown to be true (cf. [M]) of any geodesic in $\mathcal{D}^s_\mu(M)$ which is also a geodesic in $\mathcal{D}^s(M)$ (such geodesics correspond to fluid flows in M with constant pressure) whenever M is a Riemannian manifold of nonpositive sectional curvature.

An example on a flat 2-torus is provided by

$$\eta(t)(x_1, x_2) = (x_1 + th(x_2), x_2)$$

where h is an arbitrary smooth periodic function.

Remark 4. One may apply the method presented here to study stability of specific fluid flows. A particularly interesting example on a flat 3-torus studied by Arnol′d [A] and more recently by Friedlander, Gilbert and Vishik [FGV] and Dombre et al. [DFGHMS] is given by a vector field

$$V = (A\sin x_3 + C\cos x_2, B\sin x_1 + A\cos x_3, C\sin x_2 + B\cos x_1)$$

where A, B and C are arbitrary constants. This flow is stationary and there exist two-planes at the identity of $\mathcal{D}^s_\mu(T^3)$ containing V for which the curvature is positive. It is likely that conjugate points can be found along this flow as well.

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