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PARAMETRIZING MAXIMAL COMPACT SUBVARIETIES

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Abstract. For the Lie group $G = Sp(n, \mathbb{R})$, let D_i be the open $G-$orbit of Lagrangian planes of signature $(i, n-i)$ in the generalized flag variety of Lagrangian planes in $\mathbb{C}^{2n}$. For a suitably chosen maximal compact subgroup K of G and a base point x_i we have that the $K-$orbit of x_i is a maximal compact subvariety of D_i. We show that for $i = 1, \ldots, n-1$ the connected component containing Kx_i in the space of $G_{\mathbb{C}}$ translates of Kx_i which lie in D_i is biholomorphic to $G/K \times \overline{G/K}$, where $\overline{G/K}$ denotes G/K with the opposite complex structure.

1. Introduction

Let G be a real semisimple Lie group and $D = G/H$ an open G-orbit in a generalized flag variety for the complexification $G_{\mathbb{C}}$ of G. Let K be a maximal compact subgroup of G and let x_0 be a basepoint in D chosen so that Kx_0 is a maximal compact subvariety of D. Let $\widetilde{M}_D$ be the set of maximal compact subvarieties in D that are translates of Kx_0 by elements of $G_{\mathbb{C}}$ and let M_D be the connected component of $\widetilde{M}_D$ containing Kx_0. Wells and Wolf [WW] have shown that M_D is a complex manifold and Wolf [W1] has further shown that it is Stein. The purpose of this paper is to determine M_D explicitly for a class of examples in which $G = Sp(n, \mathbb{R})$.

The space M_D is useful in studying representations of G that may be realized as the Dolbeault cohomology of D with coefficients in a homogeneous vector bundle. When a holomorphic double fibration exists between G/H and G/K, we know that M_D is biholomorphic to G/K [WW] and the cohomology space can be realized as the kernel of a differential operator on M_D using a Penrose transform. Wells and Wolf [WW] and Patton and Rossi [PR] give a general description of the Penrose transform (see also [BE]) and Patton and Rossi realize unitary highest weight representations of $SU(p, q)$ in this manner [PR, PR1]. When a holomorphic double fibration does not exist between G/H and G/K, one does exist between G/H and M_D. Once M_D is described explicitly, there is reason to believe that the Penrose transform can be adapted to study unitary representations of G without highest weights.

In this paper, we consider the family of examples where $G = Sp(n, \mathbb{R})$ and D is any open G-orbit in the generalized flag variety of Lagrangian planes in $\mathbb{C}^{2n}$.

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When $\mathbb{C}^{2n}$ carries a Hermitian form of signature (n,n), the open orbits are indexed by signature. Let D_i be the orbit of Lagrangian planes of signature $(i,n-i)$. Let K be a suitable maximal compact subgroup of G. Then a basepoint x_i in D_i can be chosen such that Kx_i is a maximal compact subvariety in D_i. For $1 \leq i \leq n-1$ we shall see that M_{D_i}, the connected component containing Kx_i in the space of $G_{\mathbb{C}}$ translates of Kx_i which lie in D_i, is biholomorphic to a product of bounded symmetric domains, namely $G/K \times \overline{G/K}$ where $\overline{G/K}$ denotes G/K with the opposite complex structure.

There are several other examples for which the parametrization of M_D is known in the nonholomorphic case (when a holomorphic double fibration does not exist between G/H and G/K). The first two examples have the property that G/K is not Hermitian symmetric. Wells [We] has determined M_D for $G = SO(2m,r)$ with compact isotropy subgroup $H = U(m) \times SO(r)$ and Dunne and Zierau [DZ] dealt with the case $G = SO(n,1)$ for $n \geq 4$ with D an open orbit in the generalized flag variety of isotropic lines. In these examples, M_D is an algebraic variety given by equations derived from the geometry of the flags. Although these descriptions are explicit, they do not suggest a general picture nor are they in a form which enables one readily to use the Penrose transform to study representations of G.

In the next example G/K is Hermitian symmetric. For $G = U(p,q)$ and D the open orbit of (r,s)-planes in the generalized flag variety of $(r+s)$-planes, Dunne and Zierau [DZ] have shown that M_D is biholomorphic to $G/K \times \overline{G/K}$. This example and the example in this paper suggest that M_D is biholomorphic to $G/K \times \overline{G/K}$ when G/K is Hermitian symmetric and when there is no double holomorphic fibration between G/H and G/K.

2. NOTATION, DEFINITIONS AND STATEMENT OF RESULTS

This section contains the notation and the definitions we will use and the statement of the main result. Let $\{e_1, \ldots, e_{2n}\}$ be the standard basis of $\mathbb{C}^{2n}$. With respect to this basis, we define a Hermitian form

$$\langle u\,, v\rangle_{H} = \sum_{i=1}^{n} \left(u_i \overline{v}_i - u_{n+i}\, \overline{v}_{n+i}\right)$$

and a symplectic form

$$\omega(u\,, v) = \sum_{i=1}^{n} \left(u_i v_{n+i} - u_{n+i}\, v_i\right) .$$

Definitions. Let y and w be subspaces of $\mathbb{C}^{2n}$.

(1) Two subspaces y and w are *transverse* if $y \cap w = 0$.
(2) A subspace y is *Lagrangian* if $y = y^{\perp_\omega}$.
(3) A subspace y is *isotropic* if $\omega(u\,,v) = 0$ for all $u, v \in y$.
(4) A subspace y is *null* if $\langle u\,, v\rangle_H = 0$ for all $u, v \in y$.
(5) A subspace y is *positive* (respectively *negative*) if $\langle u\,, u\rangle_H > 0$ (respectively < 0) for all nonzero u in y.
(6) The *signature* of a subspace y is $\operatorname{sgn}(y) = (a,b,c)$ if y has a Hermitian orthogonal basis of a positive vectors, b negative vectors and c null vectors. If $c = 0$, then we write $\operatorname{sgn}(y) = (a,b)$.

Let

$$G_{\mathbb{C}} = Sp(n,\mathbb{C}) = \{g \in GL(2n,\mathbb{C}) : \omega(gx\,,gy) = \omega(x\,,y)\}$$

and

$$SU(n,n) = \{g \in GL(2n,\mathbb{C}) : \det g = 1 \text{ and } \langle gx\,,gy\rangle_{H} = \langle x\,,y\rangle_{H}\}$$

Then $G = Sp(n,\mathbb{C}) \cap SU(n,n)$ is a Lie group isomorphic to $Sp(n,\mathbb{R})$. Let X be the generalized flag variety of Lagrangian planes in $\mathbb{C}^{2n}$ and D_i the open G-orbit of Lagrangian $(i, n-i)$-planes in X for $0 \le i \le n$. Let

$$x_i = \operatorname{span}\{e_1\,,\dots\,,e_i\,,e_{n+i+1}\,,\dots\,,e_{2n}\}.$$

Then x_i is in D_i and its stabilizer in G, which we denote H_i, is isomorphic to $U(i, n-i)$. We note that H_i is the set of fixed points of the involution

$$\sigma_i = \operatorname{Ad}\begin{pmatrix} I_i & & & \\ & -I_{n-i} & & \\ & & -I_i & \\ & & & I_{n-i} \end{pmatrix}$$

on G and that $D_i = G/H_i$ is an indefinite Kähler symmetric space. Let K, a maximal compact subgroup of G, be given by

$$K = \left\{ \begin{pmatrix} A & 0 \\ 0 & \overline{A} \end{pmatrix} : A \in U(n) \right\}.$$

Then the K-orbit of x_i is a maximal compact subvariety of D_i [SW] and is biholomorphic to the Grassmanian of i-planes in $\mathbb{C}^n$. In this paper, we will study the space of $G_{\mathbb{C}}$ translates of Kx_i in X.

Definition. Let M_X^i be the set of maximal compact subvarieties that are $G_{\mathbb{C}}$ translates of Kx_i in X. Let $\widetilde{M}_{D_i}$ be the $G_{\mathbb{C}}$ translates of Kx_i that are in D_i. Let M_{D_i} be the connected component of $\widetilde{M}_{D_i}$ containing Kx_i.

If $i = 0$ or n, then $Kx_i = x_i$, the G-orbit D_i is the Stein manifold G/K, and M_{D_i} is D_i. Our main result is the following parametrization of M_{D_i} for $i = 1, \dots, n-1$.

Theorem 1. *M_{D_i} is biholomorphic to $G/K \times \overline{G/K}$, where $\overline{G/K}$ denotes G/K with the opposite complex structure.*

We will see that each point in M_X^i, that is, each $G_{\mathbb{C}}$ translate of Kx_i, is naturally associated to a pair (y, w) of transverse Lagrangian planes in $\mathbb{C}^{2n}$. Once we address the uniqueness of this association in Lemma 2, we use a connectedness argument to show that each point in M_{D_i} can be associated to a pair (y, w) of transverse Lagrangian planes where y is positive and w is negative. It will emerge that G/K is biholomorphic to the set of positive Lagrangian planes in $\mathbb{C}^{2n}$ and $\overline{G/K}$ is biholomorphic to the set of negative Lagrangian planes. This then leads to identifying a point in $G/K \times \overline{G/K}$ with a maximal compact subvariety in M_{D_i}.

3. PARAMETRIZING M_X^i

In this section, we describe how to associate a pair (y, w) of transverse Lagrangian planes in $\mathbb{C}^{2n}$ to a $G_{\mathbb{C}}$ translate of Kx_i, we discuss the uniqueness of this association and we realize M_X^i as a homogeneous space. We will use this information in the next section to parametrize M_{D_i}. A closer look at the maximal compact subvariety Kx_i leads to the observation that Kx_i is a set of Lagrangian n-planes of signature $(i, n-i)$ each of which meets $y_0 = \operatorname{span}\{e_1, \dots, e_n\}$ in an i-plane and $w_0 = \operatorname{span}\{e_{n+1}, \dots, e_{2n}\}$ in an $(n-i)$-plane. In fact, for each i-plane in y_0 there is exactly one n-plane in Kx_i that meets y_0 in that i-plane. Likewise, for each $(n-i)$-plane in w_0 there is exactly one plane in Kx_i that meets w_0 in that $(n-i)$-plane. This is a consequence of the following more general lemma.

Lemma 1. *Suppose y and w are transverse Lagrangian planes in $\mathbb{C}^{2n}$ and that u is a j-dimensional subspace of y. Then there exists a unique isotropic n-plane z, called the isotropic completion of u in w, such that $z \cap y = u$ and $z \cap w$ is an $(n-j)$-plane.*

Proof. The dimension formula

$$(*) \qquad \dim(A+B) = \dim(A) + \dim(B) - \dim(A \cap B)$$

for subspaces A and B and the nondegeneracy of ω imply that $\dim u^{\perp_\omega} \cap w \geq n-j$. The reverse inequality follows since y intersects w trivially. Put $z = u \oplus (u^{\perp_\omega} \cap w)$. □

This leads to the following definition.

Definition. For transverse Lagrangian planes y and w in $\mathbb{C}^{2n}$, let $V^i(y,w)$ be the set of Lagrangian planes z such that z meets y in an i-plane and z is the isotropic completion of $z \cap y$ in w.

In this notation, we see that $Kx_i = V^i(y_0,w_0)$ and that the translate of Kx_i by g in $G_{\mathbb{C}}$ is $V^i(gy_0, gw_0)$. Then M_X^i, the space of all $G_{\mathbb{C}}$ translates of Kx_i, is the set of $V^i(y,w)$ such that y and w are transverse Lagrangian planes in $\mathbb{C}^{2n}$. The following more general lemma will enable us to compute the stabilizer of $V^i(y_0,w_0)$ in $G_{\mathbb{C}}$ so as to realize M_X^i as a homogeneous space.

Lemma 2. *Let (y, w) and (y', w') be pairs of transverse Lagrangian planes.*

1. *If $2i \neq n$, then $V^i(y,w) = V^i(y',w')$ if and only if $(y, w) = (y', w')$.*
2. *If $2i = n$, then $V^i(y,w) = V^i(y',w')$ if and only if $(y, w) = (y', w')$ or $(y, w) = (w', y')$.*

Sketch of Proof. Since $G_{\mathbb{C}}$ acts transitively on M_X^i, it is sufficient to determine when $V^i(y,w) = V^i(y_0,w_0)$. To prove (1), we consider the possibility that $V^i(y,w) = V^i(y_0,w_0)$ when

(a) $y_0 = y$ and $w_0 \neq w$,
(b) $y_0 \neq y$ and $w_0 \neq w$,
(c) $y_0 \neq y$ and $w_0 = w$.

Since the proofs for $i < \frac{n}{2}$ and $i > \frac{n}{2}$ are similar, we assume $i > \frac{n}{2}$. For (b) and (c) we can also assume that $e_1 \in (y_0 \setminus y)$ since K acts on M_X^i and K fixes y_0. Let

$$z_1 = \operatorname{span}\{e_1, \dots, e_i, e_{n+i+1}, \dots, e_{2n}\}$$

and

$$z_2 = \operatorname{span}\{e_1, e_{n-i+2}, \ldots, e_n, e_{n+2}, \ldots, e_{2n-i+1}\}.$$

Then z_1 and z_2 are in $V^i(y_0,w_0)$ and hence in $V^i(y,w)$. To see that (b) and (c) are not possible, we use the dimension formula $(*)$ to estimate the dimension of $(z_1 \cap y + z_2 \cap y)$. Since $e_1 \notin y$, we have that $\dim(y \cap z_1 \cap z_2) \leq 2i - n - 1$ and hence that $\dim(z_1 \cap y + z_2 \cap y) \geq n + 1$. This contradicts the fact that y is an n-plane. Hence $y_0 = y$.

For (a), one can show that the columns of the matrix

$$\begin{pmatrix} A \\ I_n \end{pmatrix}$$

form a basis of w where A is a symmetric $n \times n$ matrix since every element of $V^i(y_0,w_0)$ must meet w in an $(n-i)$-plane. One can then show that A must be the zero matrix. This implies that $w_0 = w$. Hence (1) is true for $i > \frac{n}{2}$.

For (2), the argument to show that (a) and (c) cannot happen is similar to the argument that (a) cannot happen in (1) when $i > \frac{n}{2}$. For (b), it is easy to see that $V^i(y_0,w_0) = V^i(w_0, y_0)$. That this is the only possibility is an involved but elementary linear algebra argument. □

When $2i \neq n$, the stabilizer of $V^i(y_0,w_0)$ in $G_{\mathbb{C}}$ is $K_{\mathbb{C}}$ where

$$K_{\mathbb{C}} = \left\{ \begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix} : A \in GL(n, \mathbb{C}) \right\}$$

and the complex structure on $G_{\mathbb{C}}/K_{\mathbb{C}}$ induces a complex structure on M_X^i. When $2i = n$, the stabilizer of $V^i(y_0,w_0)$ in $G_{\mathbb{C}}$ is the group L generated by $K_{\mathbb{C}}$ and the matrix

$$\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

and the complex structure on $G_{\mathbb{C}}/L$ induces a complex structure on M_X^i. In this case, $G_{\mathbb{C}}/K_{\mathbb{C}}$ is the holomorphic double cover of M_X^i.

4. PARAMETRIZING M_{D_i}

To parametrize M_{D_i}, we must identify which pairs (y, w) of transverse Lagrangian planes are associated to elements of M_{D_i}. Clearly, if y is positive and w is negative, then $V^i(y,w)$ is in D_i and hence in $\widetilde{M}_{D_i}$. We will use a connectivity argument to show that every element in M_{D_i} can be expressed as $V^i(y,w)$ with y positive and w negative. We now prove Theorem 1.

Proof of Theorem 1. Let

$$A = \{(y, w) : y \text{ and } w \text{ are transverse Lagrangian planes}\}$$

and

$$B = \{(y, w) \in A : y \text{ is positive and } w \text{ is negative}\}.$$

Then $G_{\mathbb{C}}/K_{\mathbb{C}}$ induces a holomorphic structure on A. Define a map $\gamma : A \to M_X^i$ by $(y, w) \mapsto V^i(y,w)$. The map γ is holomorphic and $\gamma(B) \subset \widetilde{M}_{D_i}$. Now B is connected and (y_0, w_0) is in B, so $\gamma(B) \subset M_{D_i}$. To get equality, let $V^i(y,w)$ be an element in M^i_X that is in the boundary of $\gamma(B)$. By Lemma 3, which follows below, $V^i(y,w)$ is not in M_{D_i}. Then $M_{D_i} \setminus \overline{\gamma(B)}$ and $\gamma(B)$ are a separation of M_{D_i}. Since M_{D_i} is connected and $\gamma(B)$ is nonempty, $M_{D_i} = \gamma(B)$. Now Lemma 2 implies that $\gamma|_B$ is injective for $i = 1, \ldots, n-1$, so $\gamma|_B$ is a holomorphic bijection and hence a biholomorphism. Thus B is biholomorphic to M_{D_i}.

The set B is bijective with $G/K \times G/K$. We need to identify which holomorphic structure on $G/K \times G/K$ makes this a biholomorphism. The Harish-Chandra embedding (see, for example, [K]) of G/K into $G_{\mathbb{C}}/K_{\mathbb{C}}P_+$ induces a complex structure on G/K where $K_{\mathbb{C}}P_+$ is the stabilizer of the positive Lagrangian plane y_0. For the second factor, the opposite complex structure is needed since $K_{\mathbb{C}}P_+$ is replaced by the opposite parabolic $K_{\mathbb{C}}P_-$ which is the stabilizer of the negative Lagrangian plane w_0. Thus, $G/K \times \overline{G/K}$ is biholomorphic to B and hence to M_{D_i}. $\square$

Lemma 3. *Suppose y and w are transverse Lagrangian planes in $\mathbb{C}^{2n}$. If y is positive semi-definite but not positive definite or w is negative semi-definite but not negative definite, then $V^i(y,w) \not\subset D_i$.*

Proof. Suppose $\operatorname{sgn}(y) = (j, 0,\, n-j)$ for some $0 \leq j \leq n-1$. Since the orbits of G in X are determined by signature [W], there exists $g \in G$ such that $gy = y_1$ where

$$y_1 = \operatorname{span}\{e_1\,, \ldots\,, e_j\,, e_{j+1} + e_{j+1+n}\,, \ldots\,, e_n + e_{2n}\}.$$

Let $w_1 = gw$. We will construct an isotropic n-plane z such that $z \in V^i(y_1,w_1)$ but $z \notin D_i$. This will imply that $V^i(y,w) \not\subset D_i$.

Let $u = u_0 \oplus u_+$ where

$$u_0 = \operatorname{span}\bigl\{e_{j+k} + e_{j+k+n} : 1 \leq k \leq \min\{n-j, i\}\bigr\}$$

and

$$u_+ = \begin{cases} \operatorname{span}\{e_k : 1 \leq k \leq i+j-n\} & \text{if } n-j < i, \\ 0 & \text{otherwise.} \end{cases}$$

Let z be the isotropic completion of u in w_1; thus z is in $V^i(y_1,w_1)$. To see that z is not in D_i, observe that there exists a Hermitian orthogonal basis of z containing null vectors. Such a basis exists because u_0 has been chosen such that $u_0^{\perp_\omega} = u_0^{\perp_H}$ and $u_0 \perp_\omega z$. The argument when w is negative semi-definite is analogous. $\square$

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