

AFFINE AND PROJECTIVE LINES OVER ONE-DIMENSIONAL SEMILOCAL DOMAINS

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ABSTRACT. We characterize those partially ordered sets that can occur as the spectra of polynomial rings over one-dimensional semilocal (Noetherian) domains. We also determine the posets that can occur as projective lines over one-dimensional semilocal domains.

1. INTRODUCTION AND PRELIMINARIES

In [W], R. Wiegand shows that the structure of $\text{Spec}(\mathbb{Z}[x])$ is much simpler than the structure of $\text{Spec}(\mathbb{Q}[y, x])$ even though $\text{Spec}(\mathbb{Z}) \cong \text{Spec}(\mathbb{Q}[y])$. Here we view $\text{Spec}(R)$ as the partially ordered set of prime ideals of R . However, Heinzer and S.Wiegand in [HW] show that for a *countable* one-dimensional semilocal domain R , $\text{Spec}(R[x])$ is one of two types depending on whether or not R is Henselian. In [HLW], Heinzer, Lantz and S.Wiegand extend the work in [HW] to analyze the structure of the projective line $\text{Proj}(R[s, t])$. In §2, we generalize the characterization of $\text{Spec}(R[x])$ in [HW] to include one-dimensional semilocal domains of arbitrary cardinality. In §3, we use the result of §2 to generalize the characterization of $\text{Proj}(R[s, t])$ in [HLW] to one-dimensional semilocal domains of arbitrary cardinality. We note that the proofs in this paper are modifications of proofs in [HW] and [HLW]. For results on the spectra of higher dimensional polynomial rings see [M2,S,MS].

Throughout this paper, R denotes a Noetherian domain. In order to simplify our discussion we use the following terminology:

Definition 1.1. Let P be a prime ideal in an integral domain R and let \mathfrak{p} be a prime ideal of $R[x]$ such that $\mathfrak{p} \cap R = P$. If $\mathfrak{p} \neq PR[x]$, then \mathfrak{p} is called an *upper* to P in $R[x]$. If the upper \mathfrak{p} contains a monic polynomial, then \mathfrak{p} is called an *integral upper* to P . If \mathfrak{p} is of the form $\mathfrak{p} = (P, x - a)$ for some $a \in R$, then \mathfrak{p} is called a *linear integral upper*.

For elementary facts about uppers, see [K, §1–5].

In §3 we discuss the projective line X over R . For convenience we use two interpretations of the projective line:

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1. $X = \text{Proj}(R[s, t])$, the set of relevant homogeneous primes in the polynomial ring in two indeterminates over R , and
2. X is the union of the affine lines $\text{Spec}(R[x])$ and $\text{Spec}(R[1/x])$, where $x = s/t$. Note that the only elements in the second affine line that are not in the first are the height-one prime $(1/x)R[1/x]$ and the height-two maximal $(P_i, 1/x)R[1/x]$, and the two affine lines intersect in $\text{Spec}(R[x, 1/x])$.

We refer to the relevant homogeneous prime ideals of $R[s, t]$ as points of X . We use the following terminology when discussing points of X .

Definition 1.2. Let P be a prime ideal in an integral domain R and let \mathfrak{p} be a point of $X = \text{Proj}(R[s, t])$ such that $\mathfrak{p} \cap R = P$. If $\mathfrak{p} \neq PR[s, t]$, then \mathfrak{p} is called a *projective upper* to P in $R[s, t]$. If the upper \mathfrak{p} contains a homogeneous polynomial f such that the highest power of one of s, t has coefficient 1, then \mathfrak{p} is called a *projective integral upper* to P . If \mathfrak{p} is of the form $\mathfrak{p} = (P, as - bt)$ for $a, b \in R$ with one of $a, b = 1$, then \mathfrak{p} is called a *linear projective integral upper*.

For elementary facts about graded rings and the homogeneous spectrum of $R[s, t]$ see [Ku, §1.5].

2. THE AFFINE LINE $\text{Spec}(R[x])$

We utilize the following notation used by Heinzer, Lantz and Wiegand [HLW] for discussing partially ordered sets:

For u an element and T a subset of a partially ordered set U of finite dimension, let

$$G(u) = \{w \in U \mid w > u\}, \quad L(u) = \{w \in U \mid w < u\},$$

$$L_e(T) = \{x \in U \mid G(x) = T\}.$$

Let $\mathcal{M}(U)$ denote the set of elements of U of maximal height.

Definition 2.1. Given infinite cardinalities $k_1, \dots, k_n \leq r$, a partially ordered set U is called affine of type (r, k_1, \dots, k_n) or $\mathbb{A}(r, k_1, \dots, k_n)$ provided:

- (P0) $|U| = r$.
- (P1) U has a unique minimal element u_0 .
- (P2) U has dimension 2.
- (P3) There exist n height-one elements u_1, \dots, u_n (called *special* elements) satisfying
 - (i) $G(u_1) \cup G(u_2) \cup \dots \cup G(u_n) = \mathcal{M}(U)$.
 - (ii) $G(u_i) \cap G(u_j) = \emptyset$ for $i \neq j$.
 - (iii) $|G(u_i)| = k_i$ for $i = 1, \dots, n$.
- (P4) For each non-special height-one element u , $G(u)$ is finite (possibly \emptyset).
- (P5) For each finite subset T (including \emptyset) of $\mathcal{M}(U)$, $|L_e(T)| = r$.

Remark. If r, k_1, \dots, k_n are countably infinite, then $\mathbb{A}(r, k_1, \dots, k_n)$ is just $C\mathbb{Z}(n)P$ in the terminology used by [HW].

Definition 2.2. Given infinite cardinalities $k_1 \leq r$, a partially ordered set U is called Henselian affine of type (r, k_1) or $H\mathbb{A}(r, k_1)$ provided:

- (P0)–(P4) of $\mathbb{A}(r, k_1)$ hold.
- (P5) If T is a finite subset of $\mathcal{M}(U)$, then $L_e(T) = \emptyset$ if $|T| > 1$, and $|L_e(T)| = r$ if $|T| \leq 1$.

Remark. If r, k_1 are countably infinite, then $H\mathbb{A}(r, k_1)$ is just *CHP* in the terminology used by [HW].

Proposition 2.3. (1) *If U and V are both $\mathbb{A}(r, k_1, \dots, k_n)$, then U is order isomorphic to V .*

(2) *If U and V are both $H\mathbb{A}(r, k_1)$, then $U \cong V$.*

Proof. We define an order-preserving bijection f from U to V . For $i = 0, 1, \dots, n$, let f map u_i to v_i respectively. (Here u_0, v_0 are the unique minimal elements of U, V respectively and u_i, v_i are the special elements of U, V respectively such that $|G(u_i)| = |G(v_i)| = k_i$ for $i = 1, \dots, n$.) For each i , let f take the elements of $G(u_i)$ to elements of $G(v_i)$ (via any set bijection). It remains to define f on non-special height-one elements of U . Note that the non-special height-one elements of U can be partitioned into a collection of sets $\{L_e(T) \mid T \text{ is a finite subset of } \mathcal{M}(U)\}$ where in the Henselian case $|T| \leq 1$. Let f send elements of $L_e(T)$ to $L_e(f(T))$ (again using any set bijection). It follows easily that f gives an order isomorphism from U to V . □

Theorem 2.4. *Let U be a partially ordered set. Let c denote the cardinality of \mathbb{R} and let r, k_1, \dots, k_n be infinite cardinal numbers. Then*

- (i) *$U \cong \text{Spec}(R[x])$ for some one-dimensional semilocal non-Henselian domain (R, P_1, \dots, P_n) with $|R| = r$ and $|\frac{R}{P_i}[x]| = k_i$ for each i if and only if U is $\mathbb{A}(r, k_1, \dots, k_n)$ and either (a) $r \leq c$ or (b) $k_1 = \dots = k_n = r > c$.*
- (ii) *$U \cong \text{Spec}(R[x])$ for some one-dimensional local Henselian domain (R, P_1) where $r = |R|$ and $k_1 = |\frac{R}{P_1}[x]|$ if and only if U is $H\mathbb{A}(r, k_1)$ and either (a) $r \leq c$ or (b) $k_1 = r > c$.*

Remark. If R/P_i is infinite, then $|\frac{R}{P_i}[x]| = |\frac{R}{P_i}|$.

Before we can prove Theorem 2.4, we need the following lemmas and propositions.

Lemma 2.5. *If (R, M) is a local domain, then $|R| \leq \sup(|R/M|, c)$. Thus if $|R| > c$, then $|R/M| = |R|$.*

Proof. For $i \geq 0$, let R_i be a complete set of representatives in R of each residue class of M^i/M^{i+1} . We define a set map $\Psi: R \rightarrow \prod_{i=0}^{\infty} R_i$. For $a \in R$, let $\Psi(a) = (a_0, a_1, \dots)$, where $a_i \in R_i$ is such that $a_i \equiv a - \sum_{j=0}^{i-1} a_j \pmod{M^{i+1}}$. Note that Ψ is one-one since $\bigcap_{i=0}^{\infty} M^i = 0$. Since M^i/M^{i+1} is a finite-dimensional vector space over R/M , $|M^i/M^{i+1}| = |R_i|$ is either finite or equal to $|R/M|$. Thus $|R| \leq |R/M|^{\aleph_0} = \sup(|R/M|, c)$.

Lemma 2.6 [HM, Theorem 1.2]. *For $k \geq 2$, let P_1, \dots, P_k be non-zero primes (not necessarily distinct) of a non-Henselian Noetherian domain R . Then there exists a finitely generated integral extension domain R' of R containing distinct, pairwise comaximal primes P'_1, \dots, P'_k lying over P_1, \dots, P_k respectively.*

Lemma 2.7 [S, Lemma 3.7]. *Let $\mathfrak{p}_1, \dots, \mathfrak{p}_k$ be integral uppers in $R[x]$ (R not necessarily Noetherian). There exists a finitely generated integral extension domain R' of R such that if \mathfrak{p}' is any prime in $R'[x]$ with $\mathfrak{p}' \cap R[x] = \mathfrak{p}_i$, then \mathfrak{p}' is a linear integral upper.*

Proposition 2.8. *Let (R, M) be a local Henselian domain. Let \mathbf{k} be an upper to (0) . Then one of the following holds: (a) \mathbf{k} is contained in $MR[x]$, (b) \mathbf{k} is contained in no upper to M or (c) \mathbf{k} is an integral upper to (0) and is contained in a unique upper to M .*

Proof. Suppose that (a) and (b) are false. We must show that (c) holds. Let F be the quotient field of R . Now \mathbf{k} is contained in at least one upper \mathbf{m} to M . Also $\mathbf{k} = \alpha(x)F[x] \cap R[x]$ where $\alpha(x)$ is a monic irreducible polynomial in $F[x]$. Let u be a root of $\alpha(x)$. Then \mathbf{k} is the kernel of the canonical map $R[x] \rightarrow R[u]$. We show that u is in fact integral over R . Let S be the integral closure of R in $R[u]$. Note that u is in the quotient field of S since u is algebraic over R . Also S has a unique maximal ideal N since S is an integral extension of R [N, 30.5]. We note that $R[u] = S[u]$. Let Q be the image of \mathbf{m}/\mathbf{k} under the isomorphism $R[x]/\mathbf{k} \cong R[u] = S[u]$. Under the canonical embedding $R \subset R[x]/\mathbf{k}$, we see that $\frac{\mathbf{m}}{\mathbf{k}} \cap R = M$; thus $Q \cap R = M$ and so $Q \cap S = N$. Now by assumption, $\mathbf{k} \not\subseteq MR[x]$, so u must satisfy some polynomial in $R[x] \subseteq S[x]$ with at least one coefficient which is not in M and hence not in N . By the u, u^{-1} -Lemma [K, §1-6, Exercise 31] either u or u^{-1} is in S . If $u \notin S$, then $u^{-1} \in N$ which implies $u^{-1} \in Q$, which contradicts that Q is a proper ideal of $S[u]$. Therefore $u \in S$ so that $S = S[u] = R[u]$. Thus we see that u is integral over R and so \mathbf{k} contains a monic polynomial and thus is an integral upper to (0) . Now if $\mathbf{k} \subseteq \mathbf{m}'$ where \mathbf{m}' is an upper to M distinct from \mathbf{m} , then in the integral extension $R \subseteq R[x]/\mathbf{k}$ we would have two distinct maximal ideals \mathbf{m}/\mathbf{k} and \mathbf{m}'/\mathbf{k} which contradicts the fact that $R[x]/\mathbf{k}$ must be local [N, 43.16]. \square

Proposition 2.9. *Let R be a non-Henselian one-dimensional semilocal domain. Let $\mathbf{p}_1, \dots, \mathbf{p}_k$ be height-two primes of $R[x]$. Then there exist $|R|$ height-one primes \mathbf{k} such that $\mathbf{k} \subseteq \mathbf{p}_i$ for each i , but $\mathbf{k} \not\subseteq \mathbf{m}$ for every height-two maximal ideal \mathbf{m} of $R[x]$ with $\mathbf{m} \notin \{\mathbf{p}_1, \dots, \mathbf{p}_k\}$.*

Proof. For each $i = 1, \dots, k$, let $P_i = \mathbf{p}_i \cap R$. Then P_1, \dots, P_k are non-zero primes of R and each \mathbf{p}_i is an integral upper to P_i respectively. By Lemma 2.6, there exists a finitely generated integral extension domain R' of R with distinct (necessarily maximal) primes P'_1, \dots, P'_k lying over P_1, \dots, P_k , respectively. By going up we can find integral uppers \mathbf{p}'_i in $R'[x]$ lying over \mathbf{p}_i respectively. By going to yet another finitely generated integral extension domain S of R' we can find primes \mathbf{q}_i in $S[x]$ lying over \mathbf{p}'_i such that $\mathbf{q}_i = (Q_i, x - a_i)$ for some $a_i \in S$, where $Q_i = \mathbf{q}_i \cap S$ (Lemma 2.7). Since S is a finitely generated integral extension domain of R , we note that S is also a non-Henselian semilocal domain. For convenience, we label the non-zero primes of S as $Q_1, \dots, Q_k, Q_{k+1}, \dots, Q_m$. Let E be the quotient field of S . We find $|S|$ uppers \mathbf{l}_μ to (0) with $\mathbf{l}_\mu \subseteq \mathbf{q}_i$ for each i but $\mathbf{l}_\mu \not\subseteq \mathbf{n}$ for every height-2 maximal $\mathbf{n} \notin \{\mathbf{q}_1, \dots, \mathbf{q}_k\}$.

Using the Chinese Remainder Theorem we can find $a, b \in S$ such that

$$\text{for } 1 \leq i \leq k, a \equiv a_i \text{ and } b \equiv 1 \pmod{Q_i}.$$

$$\text{for } k+1 \leq i \leq m, a \equiv 1 \text{ and } b \equiv 0 \pmod{Q_i}.$$

For each element μ in the Jacobson radical of S , let $c_\mu = a + \mu$.

Remark. If I is a non-zero ideal of S , then since $I = IS$, $|I| = |S|$. Therefore the Jacobson radical of S has cardinality $|S|$.

Note that $c_\mu \equiv a_i \pmod{Q_i}$ for $1 \leq i \leq k$ and $c_\mu \equiv 1 \pmod{Q_i}$ for $k+1 \leq i \leq m$.

Now $\mathbf{l}_\mu = (x - \frac{c_\mu}{b})E[x] \cap S[x]$ is the unique minimal prime divisor of $(bx - c_\mu)S[x]$ since there does not exist any height-one prime in S containing both b, c_μ [M1, Lemma 2]. We claim that $\{\mathbf{l}_\mu\}$ is the desired collection of uppers to (0). Suppose \mathbf{l}_μ is contained in some upper \mathbf{q} to Q_i for some i .

If $1 \leq i \leq k$, then $bx - c_\mu \equiv x - a_i \pmod{Q_i}$; thus $(Q_i, x - a_i) \subseteq \mathbf{q} \Rightarrow \mathbf{q} = \mathbf{q}_i$.

If $k + 1 \leq i \leq m$, then $bx - c_\mu \equiv -1 \pmod{Q_i} \Rightarrow -1 \in \mathbf{q}$ which is a contradiction. We return to $R[x]$. Now the set $\{\mathbf{k}_\mu \mid \mathbf{k}_\mu = \mathbf{l}_\mu \cap R[x]\}$ has cardinality $|R| = |S|$, since S is a finitely generated integral extension of R . Clearly $\mathbf{k}_\mu \subseteq \mathbf{p}_i$ for each i . Now suppose $\mathbf{k}_\mu \subseteq \mathbf{m}$ for some upper \mathbf{m} . Since \mathbf{l}_μ in $S[x]$ lies over \mathbf{k}_μ , by going up there exists a maximal ideal \mathbf{n} of $S[x]$ lying over \mathbf{m} such that $\mathbf{l}_\mu \subseteq \mathbf{n}$. But by construction of \mathbf{l}_μ , $\mathbf{n} = \mathbf{q}_i$ for some i . Therefore $\mathbf{m} = \mathbf{q}_i \cap R[x] = \mathbf{p}_i$. Hence $\{\mathbf{k}_\mu\}$ is our desired collection. \square

Proof of Theorem 2.4. We first show that if (R, P_1, \dots, P_n) is a one-dimensional semilocal domain with $|R| = r$ and $|\frac{R}{P_i}[x]| = k_i$, then $\text{Spec}(R[x])$ satisfies (P0)–(P4) of $\mathbb{A}(r, k_1, \dots, k_n)$.

Since $|R[x]| = |R| = r$, if T is any subset of $\text{Spec}(R[x])$, then $|T| \leq r$. Since $(x - a)R[x]$ is prime in $R[x]$ for every $a \in R$, $\text{Spec}(R[x])$ satisfies (P0). Of course (0) is the unique minimal prime of $R[x]$. (P2) follows from the fact that the Krull dimension of $R[x]$ is 2. For (P3), let each u_i be $P_i R[x]$. If \mathbf{m} is a maximal ideal of height 2, then \mathbf{m} is an upper to P_i for some i so that $\mathbf{m} \in G(u_i)$. Also $G(u_i)$ corresponds to the maximal ideals of $\frac{R}{P_i}[x]$, thus $|G(u_i)| = k_i$. For (P4), note that for u non-special height-one, $G(u)$ is a finite union of all the $G(u) \cap G(u_i)$ which must be finite. (Since in the Noetherian ring $R[x]$, there are only finitely many primes minimal over $u + u_i$.)

For (P5) of (2.1) and (2.2), let T be a finite subset of $\mathcal{M}(\text{Spec}(R[x]))$. Suppose T is empty. For each β in the Jacobson radical of R , $(\beta x - 1)K[x] \cap R[x]$ is a maximal height-one prime of $R[x]$; thus $|L_e(T)| = r$. If T is a non-empty finite collection of height-2 maximal ideals of $R[x]$, Theorem 2.4 breaks up into two cases depending on whether or not R is Henselian:

For (i) of Theorem 2.4, suppose $T = \{\mathbf{p}_1, \dots, \mathbf{p}_k\}$. Then by Proposition 2.9, there are r height-one primes \mathbf{k} such that $\mathbf{k} \subseteq \mathbf{m}$ for \mathbf{m} a height-2 maximal ideal if and only if $\mathbf{m} \in \{\mathbf{p}_1, \dots, \mathbf{p}_k\}$. Thus for each finite subset T of $\mathcal{M}(\text{Spec}(R[x]))$, $|L_e(T)| = r$. Thus (P5) of $\mathbb{A}(r, k_1, \dots, k_n)$ is satisfied.

For (ii), if $|T| > 1$, then by Proposition 2.8, $L_e(T) = \emptyset$. Suppose $T = \{\mathbf{p}\}$. By Lemma 2.7 there exist a finitely generated extension domain S (necessarily Henselian) of R and a prime \mathbf{q} of $S[x]$ lying over \mathbf{p} such that $\mathbf{q} = (Q_1, x - a)$ for some $a \in S$, where Q_1 is the unique maximal ideal of S . For each $\alpha \in Q_1$, let $\mathbf{l}_\alpha = (x - a + \alpha)S[x]$. Now as in the proof of Proposition 2.8, $\{\mathbf{k}_\alpha = \mathbf{l}_\alpha \cap R[x]\}$ is a collection of r integral uppers to (0) with $\mathbf{k}_\alpha \subseteq \mathbf{p}$. Hence axiom (P5) of $HA(r, k_1)$ holds.

We now prove the cardinality restrictions. If $r > c$, we must show that each $k_i = r$. Localize R at P_i to obtain a local domain $(R_{P_i}, P_i R_{P_i})$. Since $|R_{P_i}| = |R| > c$, by Lemma 2.5, $r = |R| = |R_{P_i}| = |R_{P_i}/P_i R_{P_i}| = |R/P_i| = |\frac{R}{P_i}[x]| = k_i$.

For the converse of (i), we construct a one-dimensional normal semilocal domain (R, P_1, \dots, P_n) with $|R| = r$ and $|R/P_i| = k_i$. We first deal with the case $r \leq c$. I am grateful to R. Heitmann for suggesting the technique used in constructing the following example.

Example 2.10. Let A be a set of cardinality r and let B_1, \dots, B_n be subsets of A of cardinality k_1, \dots, k_n respectively (if $k_i = r$, choose $B_i = A$). Let $F = \mathbb{Q}(z_1, \dots, z_n, \{t_\alpha\}_{\alpha \in A})$, where z_j, t_α are indeterminates. Thus F is a field of cardinality r . Now $K_i = \mathbb{Q}(z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_n, \{t_\alpha\}_{\alpha \in B_i})$ is a proper subfield of F of cardinality k_i . Let $K_i^* = K_i[[y]][y^{-1}]$. Either $k_i = r = c$ (in which case $F = K_i(z_i)$) or the transcendence degree of K^* over K_i is c (since $|K^*| = c > k_i$). In any case we can extend the inclusion map of K_i into K_i^* to an isomorphism $\phi_i : F \rightarrow K_i^*$ onto $F_i^* = \phi_i(F)$ such that $\phi_i(z_i) = y$. By [K, Theorem 99], it follows that $F_i^* \cap K_i[[y]]$ is a DVR of F_i^* with maximal ideal $yK_i[[y]] \cap F_i^*$. By pulling back we get that $V_i = \phi_i^{-1}(K_i[[y]])$ is a DVR of F with maximal ideal $N_i = \phi_i^{-1}(yK_i[[y]])$ such that $|V_i/N_i| = |K_i[[y]]/yK_i[[y]]| = k_i$. Since z_i is a unit in V_j if and only if $i \neq j$, $V_i \not\subseteq V_j$ for $i \neq j$. Let $R = V_1 \cap \dots \cap V_n$ and let $P_i = N_i \cap R$. Then by [Ma, Theorem 12.2], (R, P_1, \dots, P_n) has the desired properties.

We now assume $k_1 = \dots = k_n = r > c$.

Example 2.11. Let F be a field of cardinality r . In $T = F[z]$, let $P_i = (z - a_i)F[z]$ where z is an indeterminate and a_1, \dots, a_n are distinct elements of F . Localizing T at the set $S = T - (P_1 \cup \dots \cup P_n)$ gives a normal semilocal non-Henselian domain $(T_S, P_1T_S, \dots, P_nT_S)$ with the required cardinalities. \square

The converse of (ii) follows from the following:

Remark 2.12. Let (R, M) be a normal local domain and let (R^*, M^*) be the Henselization of (R, M) . Then (R^*, M^*) is a local Henselian domain with $|R^*| = |R|$ and $R^*/M^* = R/M$ [N, §43].

3. THE PROJECTIVE LINE $\text{Proj}(R[s, t])$

Let (R, P_1, \dots, P_n) be a one-dimensional semilocal domain. We introduce two axiom systems analogous to the ones introduced in §2.

Definition 3.1. Given infinite cardinalities $k_1, \dots, k_n \leq r$, a partially ordered set U is called projective of type (r, k_1, \dots, k_n) or $\mathbb{P}(r, k_1, \dots, k_n)$ provided:

- (P0) $|U| = r$.
- (P1) U has a unique minimal element u_0 .
- (P2) U has dimension 2.
- (P3) There exist n height-one elements u_1, \dots, u_n (called *special* elements) satisfying
 - (i) $G(u_1) \cup G(u_2) \cup \dots \cup G(u_n) = \mathcal{M}(U)$.
 - (ii) $G(u_i) \cap G(u_j) = \emptyset$ for $i \neq j$.
 - (iii) $|G(u_i)| = k_i$ for $i = 1, \dots, n$.
- (P4) For each non-special height-one element u , $G(u)$ is finite and $G(u) \cap G(u_i) \neq \emptyset$ for each $1 \leq i \leq n$.
- (P5) For each non-empty finite subset T of $\mathcal{M}(U)$ such that $\{u_1, \dots, u_n\} \subseteq \bigcup\{L(t) \mid t \in T\}$, $|L_e(T)| = r$.

Remark. If r, k_1, \dots, k_n are countably infinite, then $\mathbb{P}(r, k_1, \dots, k_n)$ is just $\mathbf{PCZ}(n)P$ in the terminology used by [HLW].

Definition 3.2. Given infinite cardinalities $k_1 \leq r$, a partially ordered set U is called Henselian projective of type (r, k_1) or $H\mathbb{P}(r, k_1)$ provided:

(P0)–(P4) of $\mathbb{P}(r, k_1)$ hold.

(P5) If T is a finite subset of $\mathcal{M}(U)$, then $L_e(T) = \emptyset$ if $|T| \neq 1$, and $|L_e(\{t\})| = r$.

Remark. If r, k_1 are countably infinite, then $H\mathbb{P}(r, k_1)$ is just **PCHP** in the terminology used by [HLW].

Proposition 3.3. (1) *If U and V are both $\mathbb{P}(r, k_1, \dots, k_n)$, then $U \cong V$.*

(2) *If U and V are both $H\mathbb{P}(r, k_1)$, then $U \cong V$.*

Proof. Note that the non-special height-one elements can be partitioned into a collection $\{L_e(T)\}$ for appropriate finite subsets T of $\mathcal{M}(U)$. The proof is analogous to the proof of Proposition 2.3.

Theorem 3.4. *Let U be a partially ordered set. Let r, k_1, \dots, k_n be infinite cardinal numbers. Then*

- (i) *$U \cong \text{Proj}(R[s, t])$ for some one-dimensional semilocal non-Henselian domain (R, P_1, \dots, P_n) with $|R| = r$ and $|\frac{R}{P_i}[x]| = k_i$ for each i if and only if U is $\mathbb{P}(r, k_1, \dots, k_n)$ and either (a) $r \leq c$ or (b) $k_1 = \dots = k_n = r > c$.*
- (ii) *$U \cong \text{Proj}(R[s, t])$ for some one-dimensional local Henselian domain (R, P_1) where $r = |R|$ and $k_1 = |\frac{R}{P_1}[x]|$ if and only if U is $H\mathbb{P}(r, k_1)$ and either (a) $r \leq c$ or (b) $k_1 = r > c$.*

Before we can prove Theorem 3.4, we need the following lemmas and proposition.

Remark 3.5. Let S be an integral extension of R and let \mathfrak{q} be a homogeneous prime in $S[s, t]$. Then $\mathfrak{q} \cap R[s, t]$ is a homogeneous prime in $R[s, t]$. We also note that lying over and going up theorems also hold for homogeneous primes.

Lemma 3.6. *Let $\mathfrak{p}_1, \dots, \mathfrak{p}_k$ be a finite set of height-two points of X . Then there is a finitely generated integral extension R' of R such that if \mathfrak{p}' is any point in $X' = \text{Proj}(R'[s, t])$ lying over \mathfrak{p}_i , then \mathfrak{p}' is a linear projective integral upper.*

Proof. This follows immediately by applying Lemma 2.7, by considering the height-two points as maximal ideals in either $R[s/t]$ or $R[t/s]$. □

Lemma 3.7. *Let \mathfrak{k} be a projective upper to 0 in X . Then for each $i = 1, \dots, n$, there exists a projective upper \mathfrak{p}_i to P_i containing \mathfrak{k} .*

Proof. See [HLW], proof of second part of (P5) of Lemma 2.4.

Proposition 3.8. *Let (R, P_1, \dots, P_n) be a non-Henselian one-dimensional semilocal domain. Let $\mathfrak{p}_{11}, \dots, \mathfrak{p}_{1r_1}, \dots, \mathfrak{p}_{n1}, \dots, \mathfrak{p}_{nr_n}$ be height-two points of X such that each $r_i \geq 1$ and $\mathfrak{p}_{ij} \cap R = P_i$. Then there exist $|R|$ height-one points \mathfrak{k} such that $\mathfrak{k} \subseteq \mathfrak{p}_{ij}$ for each $i = 1, \dots, n, j = 1, \dots, r_i$, but $\mathfrak{k} \not\subseteq \mathfrak{m}$ for any height-two point \mathfrak{m} of X with $\mathfrak{m} \not\subseteq \{\mathfrak{p}_{11}, \dots, \mathfrak{p}_{1r_1}, \dots, \mathfrak{p}_{n1}, \dots, \mathfrak{p}_{nr_n}\}$.*

Proof. Note that the height-two points of X are necessarily integral projective uppers. In light of Lemma 3.6, we replace R by an integral extension of R and get a (possibly larger) collection of all points in the projective line of the integral extension lying over some \mathfrak{p}_{ij} . Thus we assume that each \mathfrak{p}_{ij} is a linear projective integral upper. By Lemma 2.6, there exists a finitely generated integral extension domain S of R with distinct (maximal) primes $P'_{11}, \dots, P'_{nr_n}$ such that P'_{ij} lies over P_i . Relabel the non-zero primes of S as Q_1, \dots, Q_m and for $j = 1, \dots, m$, pick a linear projective integral upper \mathfrak{q}_j in such a way that $\{\mathfrak{q}_1, \dots, \mathfrak{q}_m\} \cap R[s, t] = \{\mathfrak{p}_{11}, \dots, \mathfrak{p}_{nr_n}\}$. Thus we assume Q_1, \dots, Q_m are distinct non-zero primes of S

and for $j = 1, \dots, m$, $\mathfrak{q}_j = (Q_j, a_j s - b_j t)$ where one of a_j, b_j is 1. We find $|S|$ projective uppers \mathfrak{l}_μ to (0) with $\mathfrak{l}_\mu \subset \mathfrak{q}_j$ for each j but $\mathfrak{l}_\mu \not\subset \mathfrak{n}$ for any height-2 maximal $\mathfrak{n} \notin \{\mathfrak{q}_1, \dots, \mathfrak{q}_m\}$.

By the Chinese Remainder Theorem we find elements a, b in S such that for each j , $a \equiv a_j \pmod{Q_j}$ and $b \equiv b_j \pmod{Q_j}$. For each element μ in the Jacobson radical of S , let $c_\mu = b - \mu$. Note that $c_\mu \equiv b \equiv b_j \pmod{Q_j}$. Now $\mathfrak{l}_\mu = (as - c_\mu t)S[s, t]$ is a linear projective upper to (0) , since mod each Q_j , either a or c_μ is congruent to 1. We claim that $\{\mathfrak{l}_\mu\}$ is the desired collection of projective uppers to (0) . Suppose \mathfrak{l}_μ is contained in a projective upper \mathfrak{q} to some Q_j . Then since $as - c_\mu t \equiv a_j s - b_j t \pmod{Q_j}$, \mathfrak{q} must contain $(Q_j, a_j s - b_j t) = \mathfrak{q}_j$ which implies $\mathfrak{q} = \mathfrak{q}_j$.

We return to $R[s, t]$. Now the set $\{\mathfrak{k}_\mu \mid \mathfrak{k}_\mu = \mathfrak{l}_\mu \cap R[s, t]\}$ has cardinality $|R| = |S|$, since S is a finitely generated integral extension of R . It is not difficult to see that \mathfrak{k}_μ is a projective upper to (0) , i.e. a point in X . Clearly $\mathfrak{k}_\mu \subseteq \mathfrak{p}_{ij}$ for each i, j . Now suppose $\mathfrak{k}_\mu \subset \mathfrak{m}$ for some height-two point \mathfrak{m} . Since \mathfrak{l}_μ in $Y = \text{Proj } S[s, t]$ lies over \mathfrak{k}_μ , by going up there exists a height-two point \mathfrak{n} of Y lying over \mathfrak{m} such that $\mathfrak{l}_\mu \subseteq \mathfrak{n}$. But by construction of \mathfrak{l}_μ , $\mathfrak{n} = \mathfrak{q}_j$ for some $j = 1, \dots, m$. Therefore $\mathfrak{m} = \mathfrak{q}_j \cap R[s, t] \in \{\mathfrak{p}_{11}, \dots, \mathfrak{p}_{nr_n}\}$. Hence $\{\mathfrak{k}_\mu\}$ is our desired collection. \square

Proof of Theorem 3.4. In view of the proof of Theorem 2.4, it suffices to show that if (R, P_1, \dots, P_n) is a one-dimensional semilocal domain with $|R| = r$ and $|\frac{R}{P_i}[x]| = k_i$, then $\text{Proj}(R[s, t])$ is (i) $\mathbb{P}(r, k_1, \dots, k_n)$ if R is not Henselian and (ii) $H\mathbb{P}(r, k_1)$ if $n = 1$ and R is Henselian. We first show $\text{Proj}(R[s, t])$ satisfies (P0)–(P4) of $\mathbb{P}(r, k_1, \dots, k_n)$. From the second definition of $X = \text{Proj}(R[s, t])$ and the fact that there are only finitely many points of X not in $R[x] = R[s/t]$, we see that most of the cardinality arguments of Theorem 2.4 carry over to $\text{Proj}(R[s, t])$. Thus (P0) follows immediately. Of course (0) is the unique minimal element of X . Since $\text{Spec}(R[s, t])$ has dimension 3 and the only height-three homogeneous primes in $R[s, t]$ are of the form $(P_i, s, t)R[s, t]$, which are irrelevant, it follows that X has dimension 2. For (P3), we let u_i be the points $P_i R[s, t]$, the rest follow easily from Theorem 2.4. (P4) follows from Lemma 3.7.

Let T be a finite collection of height-two points of X such that each P_i is contained in some point in T . Again (P5) breaks up into two cases. For (i), suppose $T = \{\mathfrak{p}_{11}, \dots, \mathfrak{p}_{nr_n}\}$. Then by Proposition 3.8, there are r height-one points \mathfrak{k} such that $\mathfrak{k} \subset \mathfrak{m}$ for \mathfrak{m} a height-two point of X if and only if $\mathfrak{m} \in \{\mathfrak{p}_{11}, \dots, \mathfrak{p}_{nr_n}\}$. Thus for each finite subset T of $\mathcal{M}(X)$ with $\{u_1, \dots, u_n\} \subseteq \bigcup\{L(t) \mid t \in T\}$, $|L_e(T)| = r$.

For (ii), we first show that if $|T| \neq 1$, then $L_e(T)$ is empty. By Lemma 3.7, if $|T| = 0$, then $L_e(T) = \emptyset$. Suppose \mathfrak{k} is a non-special height-one point of X . Without loss of generality, we view \mathfrak{k} as an upper to (0) in $R[x] = R[s/t]$. If \mathfrak{k} is a maximal ideal in $R[x]$, then \mathfrak{k} is contained a unique height-two point of X , namely $(P_1, 1/x)$. If \mathfrak{k} is not a maximal ideal in $R[x]$, then it follows from Proposition 2.8 that \mathfrak{k} is contained in a unique upper to P_1 and furthermore \mathfrak{k} is an integral upper to (0) . Thus \mathfrak{k} cannot be contained in the point $(P_1, 1/x)$. If $T = \{\mathfrak{p}\}$, then $|L_e(T)| = r$ follows from Theorem 2.4. Thus if (R, P_1) is Henselian, then X satisfies (P5) of $H\mathbb{P}(r, k_1)$. \square

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