AFFINE AND PROJECTIVE LINES
OVER ONE-DIMENSIONAL SEMILOCAL DOMAINS

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Abstract. We characterize those partially ordered sets that can occur as the spectra of polynomial rings over one-dimensional semilocal (Noetherian) domains. We also determine the posets that can occur as projective lines over one-dimensional semilocal domains.

1. Introduction and preliminaries

In [W], R. Wiegand shows that the structure of Spec(\(\mathbb{Z}[x]\)) is much simpler than the structure of Spec(\(\mathbb{Q}[y,x]\)) even though Spec(\(\mathbb{Z}\)) \(\cong\) Spec(\(\mathbb{Q}[y]\)). Here we view Spec(\(R\)) as the partially ordered set of prime ideals of \(R\). However, Heinzer and S.Wiegand in [HW] show that for a countable one-dimensional semilocal domain \(R\), Spec(\(R[x]\)) is one of two types depending on whether or not \(R\) is Henselian. In [HLW], Heinzer, Lantz and S.Wiegand extend the work in [HW] to analyze the structure of the projective line Proj(\(R[s,t]\)). In §2, we generalize the characterization of Spec(\(R[x]\)) in [HW] to include one-dimensional semilocal domains of arbitrary cardinality. In §3, we use the result of §2 to generalize the characterization of Proj(\(R[s,t]\)) in [HLW] to one-dimensional semilocal domains of arbitrary cardinality. We note that the proofs in this paper are modifications of proofs in [HW] and [HLW]. For results on the spectra of higher dimensional polynomial rings see [M2, S, MS].

Throughout this paper, \(R\) denotes a Noetherian domain. In order to simplify our discussion we use the following terminology:

Definition 1.1. Let \(P\) be a prime ideal in an integral domain \(R\) and let \(p\) be a prime ideal of \(R[x]\) such that \(p \cap R = P\). If \(p \neq PR[x]\), then \(p\) is called an upper to \(P\) in \(R[x]\). If the upper \(p\) contains a monic polynomial, then \(p\) is called an integral upper to \(P\). If \(p\) is of the form \(p = (P, x-a)\) for some \(a \in R\), then \(p\) is called a linear integral upper.

For elementary facts about uppers, see [K, §1-5].

In §3 we discuss the projective line \(X\) over \(R\). For convenience we use two interpretations of the projective line:
1. $X = \text{Proj}(R[s,t])$, the set of relevant homogeneous primes in the polynomial ring in two indeterminates over $R$, and

2. $X$ is the union of the affine lines $\text{Spec}(R[x])$ and $\text{Spec}(R[1/x])$, where $x = s/t$.

Note that the only elements in the second affine line that are not in the first are the height-one prime $(1/x)R[1/x]$ and the height-two maximals $(P_i, 1/x)R[1/x]$, and the two affine lines intersect in $\text{Spec}(R[x, 1/x])$.

We refer to the relevant homogeneous prime ideals of $R[s,t]$ as points of $X$. We use the following terminology when discussing points of $X$.

**Definition 1.2.** Let $P$ be a prime ideal in an integral domain $R$ and let $p$ be a point of $X = \text{Proj}(R[s,t])$ such that $p \cap R = P$. If $p \neq P R[s,t]$, then $p$ is called a projective upper to $P$ in $R[s,t]$. If the upper $p$ contains a homogeneous polynomial $f$ such that the highest power of one of $s,t$ has coefficient 1, then $p$ is called a projective integral upper to $P$. If $p$ is of the form $p = (P, as - bt)$ for $a, b \in R$ with one of $a, b = 1$, then $p$ is called a linear projective integral upper.

For elementary facts about graded rings and the homogeneous spectrum of $R[s,t]$ see [Ku, §1.5].

2. **The affine line $\text{Spec}(R[x])$**

We utilize the following notation used by Heinzer, Lantz and Wiegand [HLW] for discussing partially ordered sets:

For $u$ an element and $T$ a subset of a partially ordered set $U$ of finite dimension, let

$$G(u) = \{w \in U | w > u\}, \quad L(u) = \{w \in U | w < u\},$$

$$L_u(T) = \{x \in U | G(x) = T \}.$$ 

Let $\mathcal{M}(U)$ denote the set of elements of $U$ of maximal height.

**Definition 2.1.** Given infinite cardinalities $k_1, \ldots, k_n \leq r$, a partially ordered set $U$ is called affine of type $(r, k_1, \ldots, k_n)$ or $\mathcal{A}(r, k_1, \ldots, k_n)$ provided:

(P0) $|U| = r$.

(P1) $U$ has a unique minimal element $u_0$.

(P2) $U$ has dimension 2.

(P3) There exist $n$ height-one elements $u_1, \ldots, u_n$ (called special elements) satisfying

(i) $G(u_1) \cup G(u_2) \cup \cdots \cup G(u_n) = \mathcal{M}(U)$.

(ii) $G(u_i) \cap G(u_j) = \emptyset$ for $i \neq j$.

(iii) $|G(u_i)| = k_i$ for $i = 1, \ldots, n$.

(P4) For each non-special height-one element $u$, $G(u)$ is finite (possibly $\emptyset$).

(P5) For each finite subset $T$ (including $\emptyset$) of $\mathcal{M}(U)$, $|L_u(T)| = r$.

**Remark.** If $r, k_1, \ldots, k_n$ are countably infinite, then $\mathcal{A}(r, k_1, \ldots, k_n)$ is just $CZ(n)P$ in the terminology used by [HW].

**Definition 2.2.** Given infinite cardinalities $k_1 \leq r$, a partially ordered set $U$ is called Henselian affine of type $(r, k_1)$ or $H\mathcal{A}(r, k_1)$ provided:

(P0)–(P4) of $\mathcal{A}(r, k_1)$ hold.

(P5) If $T$ is a finite subset of $\mathcal{M}(U)$, then $L_u(T) = \emptyset$ if $|T| > 1$, and $|L_u(T)| = r$ if $|T| \leq 1$. 

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Remark. If \( r, k_1 \) are countably infinite, then \( \mathcal{H} \mathcal{A}(r, k_1) \) is just CHP in the terminology used by [HW].

**Proposition 2.3.** (1) If \( U \) and \( V \) are both \( \mathcal{A}(r, k_1, \ldots, k_n) \), then \( U \) is order isomorphic to \( V \).

(2) If \( U \) and \( V \) are both \( \mathcal{H} \mathcal{A}(r, k_1) \), then \( U \cong V \).

**Proof.** We define an order-preserving bijection \( f \) from \( U \) to \( V \). For \( i = 0, 1, \ldots, n \), let \( f \) map \( u_i \) to \( v_i \) respectively. (Here \( u_0, v_0 \) are the unique minimal elements of \( U, V \) respectively and \( u_i, v_i \) are the special elements of \( U, V \) respectively such that \( |G(u_i)| = |G(v_i)| = k_i \) for \( i = 1, \ldots, n \).) For each \( i \), let \( f \) take the elements of \( G(u_i) \) to elements of \( G(v_i) \). It remains to define \( f \) on non-special height-one elements of \( U \). Note that the non-special height-one elements of \( U \) can be partitioned into a collection of sets \( \{ L_i(T) \} \) where in the Henselian case \( |T| \leq 1 \). Let \( f \) send elements of \( L_i(T) \) to \( L_i(f(T)) \) (again using any set bijection). It follows easily that \( f \) gives an order isomorphism from \( U \) to \( V \). \( \square \)

**Theorem 2.4.** Let \( U \) be a partially ordered set. Let \( c \) denote the cardinality of \( \mathbb{R} \) and let \( r, k_1, \ldots, k_n \) be infinite cardinal numbers. Then

(i) \( U \cong \text{Spec}(R[x]) \) for some one-dimensional semilocal non-Henselian domain \((R, P_1, \ldots, P_n)\) with \( |R| = r \) and \( \frac{R}{P_i} \) is \( k_i \) for each \( i \) if and only if \( U \) is \( \mathcal{A}(r, k_1, \ldots, k_n) \) and either (a) \( r \leq c \) or (b) \( k_1 = \cdots = k_n = r > c \).

(ii) \( U \cong \text{Spec}(R[x]) \) for some one-dimensional local Henselian domain \((R, P_1)\) where \( r = |R| \) and \( k_1 = \left| \frac{R}{P_1} \right| \) if and only if \( U \) is \( \mathcal{H} \mathcal{A}(r, k_1) \) and either (a) \( r \leq c \) or (b) \( k_1 = r > c \).

**Remark.** If \( R/P_i \) is infinite, then \( \left| \frac{R}{P_1} \right| = \left| \frac{R}{P_i} \right| \).

Before we can prove Theorem 2.4, we need the following lemmas and propositions.

**Lemma 2.5.** If \((R, M)\) is a local domain, then \( |R| \leq \text{sup}(|R/M|, c) \). Thus if \( |R| > c \), then \( |R/M| = |R| \).

**Proof.** For \( i \geq 0 \), let \( R_i \) be a complete set of representatives in \( R \) of each residue class of \( M^i/M^{i+1} \). We define a set map \( \Psi: R \to \prod_{i=0}^{\infty} R_i \). For \( a \in R \), let \( \Psi(a) = (a_0, a_1, \ldots) \), where \( a_i \in R_i \) is such that \( a \equiv a_i \mod M^{i+1} \). Note that \( \Psi \) is one-one since \( \bigcap_{i=0}^{\infty} M^i = 0 \). Since \( M^i/M^{i+1} \) is a finite-dimensional vector space over \( R/M \), \( |M^i/M^{i+1}| = |R| \) is either finite or equal to \( |R/M| \). Thus \( |R| \leq |R/M|^{\text{fin}} \leq \text{sup}(|R/M|, c) \).

**Lemma 2.6** [HM, Theorem 1.2]. For \( k \geq 2 \), let \( P_1, \ldots, P_k \) be non-zero primes (not necessarily distinct) of a non-Henselian Noetherian domain \( R \). Then there exists a finitely generated integral extension domain \( R' \) of \( R \) containing distinct, pairwise comaximal primes \( P_1', \ldots, P_k' \) lying over \( P_1, \ldots, P_k \) respectively.

**Lemma 2.7** [S, Lemma 3.7]. Let \( p_1, \ldots, p_k \) be integral proper primes in \( R[x] \) (not necessarily Noetherian). There exists a finitely generated integral extension domain \( R' \) of \( R \) such that if \( p' \) is any prime in \( R'[x] \) with \( p' \cap R[x] = p_i \), then \( p' \) is a linear integral upper.
Proposition 2.8. Let \((R, M)\) be a local Henselian domain. Let \(k\) be an upper to \((0)\). Then one of the following holds: (a) \(k\) is contained in \(MR[x]\), (b) \(k\) is contained in no upper to \(M\) or (c) \(k\) is an integral upper to \((0)\) and is contained in a unique upper to \(M\).

Proof. Suppose that (a) and (b) are false. We must show that (c) holds. Let \(F\) be the quotient field of \(R\). Now \(k\) is contained in at least one upper \(m\) to \(M\). Also \(k = \alpha(x)F[x] \cap R[x]\) where \(\alpha(x)\) is a monic irreducible polynomial in \(F[x]\).

Let \(u\) be a root of \(\alpha(x)\). Then \(k\) is the kernel of the canonical map \(R[x] \to R[u]\). We show that \(u\) is in fact integral over \(R\). Let \(S\) be the integral closure of \(R\) in \(R[u]\). Note that \(u\) is in the quotient field of \(S\) since \(u\) is algebraic over \(R\). Also \(S\) has a unique maximal ideal \(N\) since \(S\) is an integral extension of \(R\) [N, 30.5].

We note that \(R[u] = S[u]\). Let \(Q\) be the image of \(m/k\) under the isomorphism \(R[x]/k \cong R[u] = S[u]\). Under the canonical embedding \(R \subseteq R[x]/k\), we see that \(\overline{m} \cap R = M\); thus \(Q \cap R = M\) and so \(Q \cap S = N\). Now by assumption, \(k \nsubseteq MR[x]\), so \(u\) must satisfy some polynomial in \(S[x] \subseteq S[x]\) with at least one coefficient which is not in \(M\) and hence not in \(N\). By the \(u, u^{-1}−\)Lemma [K, §1-6, Exercise 31] either \(u\) or \(u^{-1}\) is in \(S\). If \(u \notin S\), then \(u^{-1} \in N\) which implies \(u^{-1} \in Q\), which contradicts that \(Q\) is a proper ideal of \(S[u]\). Therefore \(u \in S\) so that \(S = S[u] = R[u]\). Thus we see that \(u\) is integral over \(R\) and so \(k\) contains a monic polynomial and thus is an integral upper to \((0)\). Now if \(k \subseteq m'\) where \(m'\) is an upper to \(M\) distinct from \(m\), then in the integral extension \(R \subseteq R[x]/k\) we would have two distinct maximal ideals \(m/k\) and \(m'/k\) which contradicts the fact that \(R[x]/k\) must be local [N, 43.16].

Proposition 2.9. Let \(R\) be a non-Henselian one-dimensional semilocal domain. Let \(p_1, \ldots, p_k\) be height-two primes of \(R[x]\). Then there exist \([R]\) height-one primes \(k\) such that \(k \subseteq p_i\) for each \(i\), but \(k \nsubseteq m\) for every height-two maximal ideal \(m\) of \(R[x]\) with \(m \notin \{p_1, \ldots, p_k\}\).

Proof. For each \(i = 1, \ldots, k\), let \(P_i = p_i \cap R\). Then \(P_1, \ldots, P_k\) are non-zero primes of \(R\) and each \(p_i\) is an integral upper to \(P_i\), respectively. By Lemma 2.6, there exists a finitely generated integral extension domain \(R'\) of \(R\) with distinct (necessarily maximal) primes \(P'_1, \ldots, P'_k\) lying over \(P_1, \ldots, P_k\), respectively. By going up we can find integral upper \(p'_i\) in \(R'[x]\) lying over \(p_i\), respectively. By going to yet another finitely generated integral extension domain \(R''\) of \(R'\) we can find primes \(q_i\) in \(S[x]\) lying over \(p'_i\) such that \(q_i = (Q_i, x - a_i)\) for some \(a_i \in S\), where \(Q_i = (q_i) \cap R\) (Lemma 2.7).

Since \(S\) is a finitely generated integral extension domain of \(R\), we note that \(S\) is also a non-Henselian semilocal domain. For convenience, we label the non-zero primes of \(S\) as \(Q_1, \ldots, Q_k, Q_{k+1}, \ldots, Q_m\). Let \(E\) be the quotient field of \(S\). We find \([S]\) uppers \(I_{\mu}\) to \((0)\) with \(I_{\mu} \subseteq q_i\) for each \(i\) but \(I_{\mu} \nsubseteq n\) for every height-2 maximal \(n \notin \{q_1, \ldots, q_k\}\).

Using the Chinese Remainder Theorem we can find \(a, b \in S\) such that

- for \(1 \leq i \leq k\), \(a \equiv a_i\) and \(b \equiv 1 \mod Q_i\),
- for \(k + 1 \leq i \leq m\), \(a \equiv 1\) and \(b \equiv 0 \mod Q_i\).

For each element \(\mu\) in the Jacobson radical of \(S\), let \(c_{\mu} = a + \mu\).

Remark. If \(I\) is a non-zero ideal of \(S\), then since \(I = IS, |I| = |S|\). Therefore the Jacobson radical of \(S\) has cardinality \(|S|\).

Note that \(c_{\mu} \equiv a_i \mod Q_i\) for \(1 \leq i \leq k\) and \(c_{\mu} \equiv 1 \mod Q_i\) for \(k + 1 \leq i \leq m\).
Now \( I_\alpha = (x - \frac{c_\alpha}{k_\alpha})E[x] \cap S[x] \) is the unique minimal prime divisor of \((bx - c_\alpha)S[x]\) since there does not exist any height-one prime in \( S \) containing both \( b, c_\alpha \) [M1, Lemma 2]. We claim that \( \{ I_\alpha \} \) is the desired collection of uppers to \((0)\). Suppose \( I_\alpha \) is contained in some upper \( q \) to \( Q_i \) for some \( i \).

If \( 1 \leq i \leq k \), then \( bx - c_\alpha \equiv x - a_i \mod Q_i \); thus \((Q_i, x - a_i) \subseteq q \Rightarrow q = q_i \).

If \( k + 1 \leq i \leq m \), then \( bx - c_\alpha \equiv -1 \mod Q_i \Rightarrow -1 \in q \) which is a contradiction. We return to \( R[x] \). Now the set \( \{ k_\alpha \mid k_\alpha = I_\alpha \cap R[x] \} \) has cardinality \( |R| = |S| \), since \( S \) is a finitely generated integral extension of \( R \). Clearly \( k_\alpha \subseteq p_i \) for each \( i \). Now suppose \( k_\alpha \not\subseteq m \) for some upper \( m \). Since \( I_\alpha \) in \( S[x] \) lies over \( k_\alpha \), by going up there exists a maximal ideal \( n \) of \( S[x] \) lying over \( m \) such that \( I_\alpha \subseteq n \). But by construction of \( I_\alpha \), \( n = q_i \) for some \( i \). Therefore \( m = q_i \cap R[x] = p_i \). Hence \( \{ k_\alpha \} \) is our desired collection.

\[ \square \]

**Proof of Theorem 2.4.** We first show that if \((R, P_1, \ldots, P_n)\) is a one-dimensional semilocal domain with \( |R| = r \) and \( |R(x)| = k_i \), then \( \text{Spec}(R[x]) \) satisfies (P0)–(P4) of \( H(k(r, k_1, \ldots, k_n)) \).

Since \( |R(x)| = |R| = r \), if \( T \) is any subset of \( \text{Spec}(R[x]) \), then \( |T| \leq r \). Since \((x - a)R[x] \) is prime in \( R[x] \) for every \( a \in R \), \( \text{Spec}(R[x]) \) satisfies (P0). Of course \((0) \) is the unique minimal prime of \( R[x] \). (P2) follows from the fact that the Krull dimension of \( R[x] \) is 2. For (P3), let each \( u_i \) be \( R[x] \). If \( m \) is a maximal ideal of height 2, then \( m \) is an upper to \( P_i \) for some \( i \) so that \( m \in G(u_i) \). Also \( G(u_i) \) corresponds to the maximal ideals of \( R[x] \), thus \( |G(u_i)| = k_i \). For (P4), note that for \( u \) non-special height-one, \( G(u) \) is a finite union of all the \( G(u) \cap G(u_i) \) which must be finite. (Since in the Noetherian ring \( R[x] \), there are only finitely many primes minimal over \( u + a_i \).)

For (P5) of (2.1) and (2.2), let \( T \) be a finite subset of \( \mathcal{M}(\text{Spec}(R[x])) \). Suppose \( T \) is empty. For each \( \beta \) in the Jacobson radical of \( R \), \((\beta x - 1)R[x] \cap R[x] \) is a maximal height-one prime of \( R[x] \); thus \( |L_\alpha(T)| = |R| \). If \( T \) is a non-empty finite collection of height-2 maximal ideals of \( R[x] \), Theorem 2.4 breaks up into two cases depending on whether or not \( R \) is Henselian:

For (i) of Theorem 2.4, suppose \( T = \{ p_1, \ldots, p_k \} \). Then by Proposition 2.9, there are \( r \) height-one primes \( k \) such that \( k \subseteq m \) for \( m \) a height-2 maximal ideal if and only if \( m \in \{ p_1, \ldots, p_k \} \). Thus for each finite subset \( T \) of \( \mathcal{M}(\text{Spec}(R[x])) \), \( |L_\alpha(T)| = r \). Thus (P5) of \( H(k(r, k_1, \ldots, k_n)) \) is satisfied.

For (ii), if \( |T| > 1 \), then by Proposition 2.8, \( L_\alpha(T) = \emptyset \). Suppose \( T = \{ p \} \).

By Lemma 2.7 there exist a finitely generated extension domain \( S \) (necessarily Henselian) of \( R \) and a prime \( q \) of \( S[x] \) lying over \( p \) such that \( q = (Q_1, x - a) \) for some \( a \in S \), where \( Q_1 \) is the unique maximal ideal of \( S \). For each \( a \in Q_1 \), let \( I_\alpha = (x - a + a)S[x] \). Now as in the proof of Proposition 2.8, \( \{ k_\alpha = I_\alpha \cap R[x] \} \) is a collection of \( r \) integral uppers to \( (0) \) with \( k_\alpha \subseteq m \). Hence axiom (P5) of \( H(k(r, k_1)) \) holds.

We now prove the cardinality restrictions. If \( r > c \), we must show that each \( k_i = r \). Localize \( R \) at \( P_i \) to obtain a local domain \( (R_{P_i}, P_iR_{P_i}) \). Since \( |R_{P_i}| = |R| > c \), by Lemma 2.5, \( r = |R| = |R_{P_i}| = |R_{P_i}/P_iR_{P_i}| = |R/P_i| = |R/P_i| = k_i \).

For the converse of (i), we construct a one-dimensional normal semilocal domain \((R, P_1, \ldots, P_n)\) with \( |R| = r \) and \( |R/P_i| = k_i \). We first deal with the case \( r \leq c \). I am grateful to R. Heitmann for suggesting the technique used in constructing the following example.
Theorem 12.2, \( r, k \)

Given infinite cardinalities \( \emptyset \)

Definition 3.2.

Remark 2.12.

Let \( i \)

Example 2.11.

In any case we can extend the inclusion map of \( K \)

Now \( K_i = \mathbb{Q}(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n, \{t_{\alpha}\}_{\alpha \in B_i}) \) is a proper subfield of \( F \) of cardinality \( k_i \). Let \( K_i^* = K_i[[y]][y^{-1}] \). Either \( k_i = r = c \) (in which case \( F = K_i(z_i) \)) or the transcendence degree of \( K^* \) over \( K_i \) is \( c \) (since \( |K^*| = c > k_i \)).

In any case we can extend the inclusion map of \( K_i \) into \( K_i^* \) to an isomorphism \( \phi_i : F \to K_i^* \) onto \( F \) such that \( \phi_i(z_i) = y \). By [K, Theorem 99], it follows that \( F_i^* \cap K_i[[y]] \) is a DVR of \( F_i^* \) with maximal ideal \( yK_i[[y]] \). By pulling back we get that \( V_i = \phi_i^{-1}(K_i[[y]]) \) is a DVR of \( F \) with maximal ideal \( N_i = \phi_i^{-1}(yK_i[[y]]) \) such that \( |V_i/N_i| = |K_i[[y]]/yK_i[[y]]| = k_i \). Since \( z_i \) is a unit in \( V_j \) if and only if \( i \neq j \), \( V_i \not\subseteq V_j \) for \( i \neq j \). Let \( R = V_1 \setminus \cdots \setminus V_n \) and let \( P_i = N_i \cap R \). Then by [Ma, Theorem 12.2], \( (R, P_1, \ldots, P_n) \) has the desired properties.

We now assume \( k_1 = \cdots = k_n = r > c \).

Example 2.10. Let \( A \) be a set of cardinality \( r \) and let \( B_1, \ldots, B_n \) be subsets of \( A \) of cardinality \( k_1, \ldots, k_n \) respectively (if \( k_i = r \), choose \( B_i = A \)). Let \( F = \mathbb{Q}(z_1, \ldots, z_n, \{t_{\alpha}\}_{\alpha \in A}) \), where \( z_j, t_{\alpha} \) are indeterminates. Thus \( F \) is a field of cardinality \( r \).

Now \( K_i = \mathbb{Q}(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n, \{t_{\alpha}\}_{\alpha \in B_i}) \) is a proper subfield of \( F \) of cardinality \( k_i \). Let \( K_i^* = K_i[[y]][y^{-1}] \). Either \( k_i = r = c \) (in which case \( F = K_i(z_i) \)) or the transcendence degree of \( K^* \) over \( K_i \) is \( c \) (since \( |K^*| = c > k_i \)).

In any case we can extend the inclusion map of \( K_i \) into \( K_i^* \) to an isomorphism \( \phi_i : F \to K_i^* \) onto \( F \) such that \( \phi_i(z_i) = y \). By [K, Theorem 99], it follows that \( F_i^* \cap K_i[[y]] \) is a DVR of \( F_i^* \) with maximal ideal \( yK_i[[y]] \). By pulling back we get that \( V_i = \phi_i^{-1}(K_i[[y]]) \) is a DVR of \( F \) with maximal ideal \( N_i = \phi_i^{-1}(yK_i[[y]]) \) such that \( |V_i/N_i| = |K_i[[y]]/yK_i[[y]]| = k_i \). Since \( z_i \) is a unit in \( V_j \) if and only if \( i \neq j \), \( V_i \not\subseteq V_j \) for \( i \neq j \). Let \( R = V_1 \setminus \cdots \setminus V_n \) and let \( P_i = N_i \cap R \). Then by [Ma, Theorem 12.2], \( (R, P_1, \ldots, P_n) \) has the desired properties.

The converse of (ii) follows from the following:

Remark. Let \( (R, M) \) be a normal local domain and let \( (R^*, M^*) \) be the Henselianization of \( (R, M) \). Then \( (R^*, M^*) \) is a local Henselian domain with \( |R^*| = |R| \) and \( R^*/M^* = R/M \) [N, §43].

3. The projective line \( \text{Proj}(R[s, t]) \)

Let \( (R, P_1, \ldots, P_n) \) be a one-dimensional semilocal domain. We introduce two axiom systems analogous to the ones introduced in §2.

Definition 3.1. Given infinite cardinalities \( k_1, \ldots, k_n \leq r \), a partially ordered set \( U \) is called projective of type \( (r, k_1, \ldots, k_n) \) if \( P(r, k_1, \ldots, k_n) \) provided:

(P0) \( |U| = r \).
(P1) \( U \) has a unique minimal element \( u_0 \).
(P2) \( U \) has dimension 2.
(P3) There exist \( n \) height-one elements \( u_1, \ldots, u_n \) (called special elements) satisfying

(i) \( G(u_1) \cup G(u_2) \cup \cdots \cup G(u_n) = \mathcal{M}(U) \).
(ii) \( G(u_i) \cap G(u_j) = \emptyset \) for \( i \neq j \).
(iii) \( |G(u_i)| = k_i \) for \( i = 1, \ldots, n \).

(P4) For each non-special height-one element \( u \), \( G(u) \) is finite and \( G(u) \cap G(u_i) \neq \emptyset \) for each \( 1 \leq i \leq n \).

(P5) For each non-empty finite subset \( T \) of \( \mathcal{M}(U) \) such that \( \{u_1, \ldots, u_n\} \subseteq \bigcup \{L(t) | t \in T \} \), \( |L_0(T)| = r \).

Remark. If \( r, k_1, \ldots, k_n \) are countably infinite, then \( P(r, k_1, \ldots, k_n) \) is just \( \text{PCZ}(n)P \) in the terminology used by [HLW].

Definition 3.2. Given infinite cardinalities \( k_1 \leq r \), a partially ordered set \( U \) is called Henselian projective of type \( (r, k_1) \) or \( HP(r, k_1) \) provided:
(P0)–(P4) of $P(r, k_1)$ hold.

(P5) If $T$ is a finite subset of $M(U)$, then $L_n(T) = \emptyset$ if $|T| \neq 1$, and $|L_n(T)| = r$.

Remark. If $r, k_1$ are countably infinite, then $HP(r, k_1)$ is just $PCHP$ in the terminology used by [HLW].

Proposition 3.3. (1) If $U$ and $V$ are both $P(r, k_1, \ldots, k_n)$, then $U \cong V$.

(2) If $U$ and $V$ are both $HP(r, k_1)$, then $U \cong V$.

Proof. Note that the non-special height-one elements can be partitioned into a collection $\{L_n(T)\}$ for appropriate finite subsets $T$ of $M(U)$. The proof is analogous to the proof of Proposition 2.3.

Theorem 3.4. Let $U$ be a partially ordered set. Let $r, k_1, \ldots, k_n$ be infinite cardinal numbers. Then

(i) $U \cong \text{Proj}(R[s, t])$ for some one-dimensional semilocal non-Henselian domain $(R, P_1, \ldots, P_n)$ with $|R| = r$ and $\frac{|R[x]|}{|R|} = k_i$ for each $i$ if and only if $U$ is $P(r, k_1, \ldots, k_n)$ and either (a) $r \leq c$ or (b) $k_1 = \cdots = k_n = r > c$.

(ii) $U \cong \text{Proj}(R[s, t])$ for some one-dimensional local Henselian domain $(R, P_1)$ where $r = |R|$ and $k_1 = \frac{|R[x]|}{|R|}$ if and only if $U$ is $HP(r, k_1)$ and either (a) $r \leq c$ or (b) $k_1 = r > c$.

Before we can prove Theorem 3.4, we need the following lemmas and proposition.

Remark 3.5. Let $S$ be an integral extension of $R$ and let $q$ be a homogeneous prime in $S[s, t]$. Then $q \cap R[s, t]$ is a homogeneous prime in $R[s, t]$. We also note that lying over and going up theorems also hold for homogeneous primes.

Lemma 3.6. Let $p_1, \ldots, p_n$ be a finite set of height-two points of $X$. Then there is a finitely generated integral extension $R'$ of $R$ such that $p_i$ is any point in $X' = \text{Proj}(R'[s, t])$ lying over $p_i$, then $p_i$ is a linear projective integral upper.

Proof. This follows immediately by applying Lemma 2.7, by considering the height-two points as maximal ideals in either $R[s/t]$ or $R[t/s]$.

Lemma 3.7. Let $k$ be a projective upper to 0 in $X$. Then for each $i = 1, \ldots, n$, there exists a projective upper $p_i$ to $P_i$, containing $k$.

Proof. See [HLW], proof of second part of (P5) of Lemma 2.4.

Proposition 3.8. Let $(R, P_1, \ldots, P_n)$ be a non-Henselian one-dimensional semilocal domain. Let $p_{11}, \ldots, p_{1r_1}, \ldots, p_{n1}, \ldots, p_{nr_n}$ be height-two points of $X$ such that each $r_i \geq 1$ and $p_{ij} \cap R = P_i$ Then there exist $R$ height-one points $k$ such that $k \subseteq p_{ij}$ for each $i = 1, \ldots, n, j = 1, \ldots, r_i$, but $k \not\subseteq m$ for any height-two point $m$ of $X$ with $m \notin \{p_{11}, \ldots, p_{1r_1}, \ldots, p_{n1}, \ldots, p_{nr_n}\}$.

Proof. Note that the height-two points of $X$ are necessarily integral projective uppers. In light of Lemma 3.6, we replace $R$ by an integral extension of $R$ and get a (possibly larger) collection of all points in the projective line of the integral extension lying over some $p_{ij}$. Thus we assume that each $p_{ij}$ is a linear projective integral upper. By Lemma 2.6, there exists a finitely generated integral extension domain $S$ of $R$ with distinct (maximal) primes $P_{11}', \ldots, P_{nr_n}'$ such that $P_{ij}'$ lies over $P_i$. Relabel the non-zero primes of $S$ as $Q_1, \ldots, Q_m$ and for $j = 1, \ldots, m$, pick a linear projective integral upper $q_j$ in such a way that $\{q_1, \ldots, q_m\} \cap R[s/t] = \{p_{11}, \ldots, p_{nr_n}\}$. Thus we assume $Q_1, \ldots, Q_m$ are distinct non-zero primes of $S$.
and for \( j = 1, \ldots, m \), \( q_j = (Q_j, a_j s - b_j t) \) where one of \( a_j, b_j \) is 1. We find \(|S|\) projective uppers \( l_\mu \) to \((0)\) with \( l_\mu \subset q_j \) for each \( j \) but \( l_\mu \not\subset n \) for any height-2 maximal \( n \not\subset q_1, \ldots, q_m \).

By the Chinese Remainder Theorem we find elements \( a, b \) in \( S \) such that for each \( j \), \( a \equiv a_j \mod Q_j \) and \( b \equiv b_j \mod Q_j \). For each element \( \mu \) in the Jacobson radical of \( S \), let \( c_\mu = b - \mu \). Note that \( c_\mu \equiv b \mod Q_j \). Now \( l_\mu = (as - c_\mu t)S[s, t] \) is a linear projective upper to \((0)\), since mod each \( Q_j \), either \( a \) or \( c_\mu \) is congruent to 1. We claim that \( \{ l_\mu \} \) is the desired collection of projective uppers to \((0)\). Suppose \( l_\mu \) is contained in a projective upper \( q \) to some \( Q_j \). Then since \( as - c_\mu t \equiv a_j s - b_j t \mod Q_j \), \( q \) must contain \((Q_j, a_j s - b_j t) = q_j \) which implies \( q = q_j \).

We return to \( R[s, t] \). Now the set \( \{ k_\mu | k_\mu = l_\mu \cap R[s, t] \} \) has cardinality \(|R| = |S|\), since \( S \) is a finitely generated integral extension of \( R \). It is not difficult to see that \( k_\mu \) is a projective upper to \((0)\), i.e. a point in \( X \). Clearly \( k_\mu \subseteq p_{ij} \) for each \( i, j \). Now suppose \( k_\mu \subseteq m \) for some height-two point \( m \). Since \( l_\mu \in Y = \text{Proj}(S[s, t]) \) lies over \( k_\mu \), by going up there exists a height-two point \( n \) of \( Y \) lying over \( m \) such that \( l_\mu \subseteq n \). But by construction of \( l_\mu \), \( n = q_j \) for some \( j = 1, \ldots, m \). Therefore \( m = q_j \cap R[s, t] \subset \{ p_1, \ldots, p_{nr} \} \). Hence \( \{ k_\mu \} \) is our desired collection. \( \square \)

**Proof of Theorem 3.4.** In view of the proof of Theorem 2.4, it suffices to show that if \( (R, P_1, \ldots, P_n) \) is a one-dimensional semilocal domain with \(|R| = r \) and \(|R[x] = k_1 \), then \( \text{Proj}(R[s, t]) \) is (i) \( P(r, k_1, \ldots, k_n) \) if \( R \) is not Henselian and (ii) \( H/P(r, k_1) \) if \( n = 1 \) and \( R \) is Henselian. We first show \( \text{Proj}(R[s, t]) \) satisfies (P0)–(P4) of \( P(r, k_1, \ldots, k_n) \). From the second definition of \( X = \text{Proj}(R[s, t]) \) and the fact that there are only finitely many points of \( X \) not in \( R[x] = R[s/t] \), we see that most of the cardinality arguments of Theorem 2.4 carry over to \( \text{Proj}(R[s, t]) \).

Thus (P0) follows immediately. Of course (0) is the unique minimal element of \( X \). Since \( \text{Spec}(R[s, t]) \) has dimension 3 and the only height-three homogeneous primes in \( R[s, t] \) are of the form \( (P_1, s, t)R[s, t] \), which are irrelevant, it follows that \( X \) has dimension 2. For (P3), we let \( u_0 \) be the points \( P_i R[s, t] \), the rest follow easily from Theorem 2.4. (P4) follows from Lemma 3.7.

Let \( T \) be a finite collection of height-two points of \( X \) such that each \( P_i \) is contained in some point in \( T \). Again (P5) breaks up into two cases. For (i), suppose \( T = \{ p_{11}, \ldots, p_{nr} \} \). Then by Proposition 3.8, there are \( r \) height-one points \( k \) such that \( k \subset m \) for \( m \) a height-two point of \( X \) if and only if \( m \in \{ p_{11}, \ldots, p_{nr} \} \). Thus for each finite subset \( T \) of \( M(X) \) with \( \{ u_1, \ldots, u_n \} \subseteq \bigcup\{ L(t) \mid t \in T \} \), \( |L_0(T)| = r \).

For (ii), we first show that if \( |T| \neq 1 \), then \( L_0(T) \) is empty. By Lemma 3.7, if \( |T| = 0 \), then \( L_0(T) = \emptyset \). Suppose \( k \) is a non-special height-one point of \( X \). Without loss of generality, we view \( k \) as an upper to \((0)\) in \( R[x] = R[s/t] \). If \( k \) is a maximal ideal in \( R[x] \), then \( k \) is contained in a unique height-two point of \( X \), namely \((P_1, 1/x) \). If \( k \) is not a maximal ideal in \( R[x] \), then it follows from Proposition 2.8 that \( k \) is contained in a unique upper to \( P_1 \) and furthermore \( k \) is an integral upper to \((0)\). Thus \( k \) cannot be contained in the point \((P_1, 1/x) \). If \( T = \{ p \} \), then \( |L_0(T)| = r \) follows from Theorem 2.4. Thus if \( (R, P_1) \) is Henselian, then \( X \) satisfies (P5) of \( H/P(r, k_1) \). \( \square \)

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