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THE ALGEBRA OF ALMOST PERIODIC FUNCTIONS HAS INFINITE TOPOLOGICAL STABLE RANK

FERNANDO DANIEL SUÁREZ

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ABSTRACT. We show that if A is the uniform algebra of almost periodic functions, then the set $U_n(A) = \{(a_1, \dots, a_n) \in A^n\colon \ \sum_{1\le j\le n} Aa_j = A\}$ cannot be dense in A^n for any positive integer n.

By a Banach algebra we mean a commutative complex Banach algebra with unit. Let B be a Banach algebra and n be a positive integer. The set of unimodulars of B is $U_n(B) = \{(b_1, \dots, b_n) \in B^n\colon \ \sum_{1\le j\le n} Bb_j = B\}$, and the topological stable rank of B ($\operatorname{tsr} B$) is the minimum positive integer n such that $U_n(B)$ is dense in B^n. We write $\operatorname{tsr} B = \infty$ if such n does not exist. Since its introduction by Rieffel [5] this concept has been very successful in studying the topological K-theory and some spectral properties of Banach algebras.

For a Banach algebra B, denote by B^* its dual space provided with the weak* topology. The maximal ideal space of B is the compact Hausdorff space

$$X(B) = \{\varphi \in B^*\colon \ \varphi \text{ is multiplicative}, \varphi \neq 0\}.$$

We denote by $C(X(B))$ the uniform algebra of continuous complex-valued functions on $X(B)$. The Gelfand transform $\hat{\ }\colon \ B \to C(X(B))$, defined by $\hat{b}(\varphi) = \varphi(b)$, is a Banach algebras morphism. So, for every positive integer n and every $b \in B^n$ the Gelfand transform induces a continuous function $\hat{b}\colon \ X(B) \to \mathbb{C}^n$. It is an easy exercise to prove that $b \in U_n(B)$ if and only if $0 \notin \hat{b}(X(B))$.

The algebra A of almost periodic functions is the uniform algebra on $\mathbb{R}$ generated by the functions

$$g(t) = \sum_{1\le k\le n} c_k e^{i\lambda_k t} \qquad (t \in \mathbb{R}), \tag{1}$$

where n is a positive integer, $c_k \in \mathbb{C}$ and $\lambda_k \in \mathbb{R}$ for $k = 1, \dots, n$. See [1, pp. 16 and 164] for general background about this algebra. The space $X(A)$ is the so-called Bohr compactification of $\mathbb{R}$, and it is well known that $\mathbb{R}$ is dense in $X(A)$ [1]. So, a necessary and sufficient condition for $(a_1, \dots, a_n) \in A^n$ to be unimodular is that $|a_1(t)|^2 + \cdots + |a_n(t)|^2 \ge \delta > 0$ for all $t \in \mathbb{R}$. As said in the abstract, the purpose of this paper is to show that $\operatorname{tsr} A = \infty$. This problem was posed by I. Spitkovsky

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in a lecture where D. Sarason was present; I am grateful to him for communicating the problem to me.

We begin by establishing some conventions. If X is a compact Hausdorff space and Y is a metric space, then $C(X,Y)$ denotes the space of continuous functions from X into Y with the supremum metric. We simply write $C(X)^n$ when $Y = \mathbb{C}^n$. By a polynomial in $z_1,\dots,z_n$ we mean a finite complex linear combination of $z_1^{p_1}\cdots z_n^{p_n}$, where $p_j \in \mathbb{Z}$ (the integer group). A $\mathbb{C}^n$-valued polynomial means a function with values in $\mathbb{C}^n$ where each coordinate is a polynomial. It is clear that every function g as in (1) can be written as a polynomial $f(e^{i\lambda_1 t},\dots,e^{i\lambda_m t})$, where $\lambda_1,\dots,\lambda_m$ are linearly independent over $\mathbb{Z}$. Therefore Kronecker's theorem [2, Theorem 443] implies that the set $\{(e^{i\lambda_1 t},\dots,e^{i\lambda_m t}):\ t\in\mathbb{R}\}$ is dense in the m-dimensional torus $\mathbb{T}^m$, and we can identify g with a polynomial on the set $\mathbb{T}^m$. It is well known that the set of polynomials on $\mathbb{T}^m$ is dense in the algebra $C(\mathbb{T}^m)$. For s a positive integer we define $\nu_s\colon\ \mathbb{T}^m\to\mathbb{T}^m$ by $\nu_s(\omega_1,\dots,\omega_m) = (\omega_1^s,\dots,\omega_m^s)$. In the sequel $\|\ \|$ denotes the euclidean norm in $\mathbb{C}^n$.

Lemma 1. *Let f be a $\mathbb{C}^n$-valued polynomial on $e^{i\lambda_1 t},\dots,e^{i\lambda_m t}$, where $\lambda_1,\dots,\lambda_m\in\mathbb{R}$ are linearly independent over $\mathbb{Z}$. Then $f\in\overline{U_n(A)}$ if and only if there is a sequence of positive integers $\{s_j\}$ such that*

$$\operatorname{dist}(f\circ\nu_{s_j}, U_n(C(\mathbb{T}^m)))\to 0 \tag{2}$$

when $j\to\infty$.

Proof. If (2) holds, then for any $\varepsilon>0$ there is an integer $s>0$ and a polynomial $F\colon\ \mathbb{T}^m\to\mathbb{C}^n_* = \mathbb{C}^n\backslash\{0\}$ so that

$$\sup_{\omega\in\mathbb{T}^m}\|F(\omega)-f\circ\nu_s(\omega)\|<\varepsilon.$$

Then for every $t\in\mathbb{R}$,

$$\|F(e^{i\frac{\lambda_1}{s}t},\dots,e^{i\frac{\lambda_m}{s}t})-f(e^{i\lambda_1 t},\dots,e^{i\lambda_m t})\|<\varepsilon.$$

Since ε is arbitrary, $f\in\overline{U_n(A)}$. On the other hand, if $f\in\overline{U_n(A)}$, then for $\varepsilon>0$ there are $\mu_1,\dots,\mu_k\in\mathbb{R}$ and a $\mathbb{C}^n_*$-valued polynomial F on $e^{i\lambda_1 t},\dots,e^{i\lambda_m t},e^{i\mu_1 t},\dots,e^{i\mu_k t}$ such that

$$\sup_{t\in\mathbb{R}}\|F(e^{i\lambda_1 t},\dots,e^{i\lambda_m t},e^{i\mu_1 t},\dots,e^{i\mu_k t})-f(e^{i\lambda_1 t},\dots,e^{i\lambda_m t})\|<\varepsilon.$$

We can reduce the number of variables in the writing of F by the following process. If for every linear combination

$$p_1\lambda_1+\cdots+p_m\lambda_m+q_1\mu_1+\cdots+q_k\mu_k=0 \tag{3}$$

with $p_j,q_j\in\mathbb{Z}$ we have that $q_k=0$, then we keep μ_k and we repeat the process with $\lambda_1,\dots,\mu_{k-1}$. If there is a combination (3) with $q_k\neq 0$, then we eliminate μ_k and repeat the process with $\lambda_1/|q_k|,\dots,\mu_{k-1}/|q_k|$. After finite steps we obtain a positive integer s and $\lambda_1/s,\dots,\lambda_m/s,\ \mu_1',\dots,\mu_l'\in\mathbb{R}$ (with $l\le k$) linearly independent over $\mathbb{Z}$, so that F can be written as a $\mathbb{C}^n_*$-valued polynomial in the correspondent exponentials. Consequently,

$$\sup_{t\in\mathbb{R}}\|F(e^{i\lambda_1 t},\dots,e^{i\lambda_m t},e^{is\mu_1' t},\dots,e^{is\mu_l' t})-f(e^{i\lambda_1 st},\dots,e^{i\lambda_m st})\|<\varepsilon.$$

Define $\widetilde{F}\colon\ \mathbb{T}^m\to\mathbb{C}^n_*$ by $\widetilde{F}(\omega_1,\dots,\omega_m)=F(\omega_1,\dots,\omega_m,1,\dots,1)$. Then $\widetilde{F}\in U_n(C(\mathbb{T}^m))$ and $\sup_{\mathbb{T}^m}\|\widetilde{F}-f\circ\nu_s\|<\varepsilon$, as claimed.

Our next lemma requires a classical result of topology due to Borsuk [4, Theorem III.3]. Let X be a compact Hausdorff space and $Z \subset X$ closed. If $f, g \in C(Z, \mathbb{C}_*^n)$ are homotopic and there is an extension $G \in C(X, \mathbb{C}_*^n)$ of g, then there is also an extension $F \in C(X, \mathbb{C}_*^n)$ of f. For $f \in C(X)^n$ put

$$E(f) = \inf\{\delta \geq 0 \; f|_{(\|f\|=\delta)} \text{ admits an extension } F \; X \to \mathbb{C}_*^n\}.$$

Suppose that $\delta > E(f)$. Then by definition of $E(f)$ there are $\delta_0 < \delta$ and an extension $F_0 \, X \to \mathbb{C}_*^n$ of $f|_{(\|f\|=\delta_0)}$. Henceforth, the function $F(x)$ defined as $F_0(x)$ when $\|f(x)\| \leq \delta_0$ and $f(x)$ when $\|f(x)\| > \delta_0$ is an extension of $f|_{(\|f\|=\delta)}$ from X into $\mathbb{C}_*^n$.

Lemma 2. $E(f) \leq \operatorname{dist}(f, U_n(C(X))) \leq 2E(f)$.

Proof. Let $\delta > E(f)$ and let $F \; X \to \mathbb{C}_*^n$ be an extension of $f|_{(\|f\|=\delta)}$. Define

$$\mathcal{F}(x) = \begin{cases} f(x) & \text{if } \|f(x)\| \geq \delta, \\ \delta \frac{F(x)}{\|F(x)\|} & \text{if } \|f(x)\| \leq \delta. \end{cases}$$

Therefore $\mathcal{F} \in U_n(C(X))$ and its distance to f is

$$\sup_{\|f(x)\| \leq \delta} \|\delta F(x)/\|F(x)\| - f(x)\| \leq 2\delta.$$

For the other inequality put $d = \operatorname{dist}(f, U_n(C(X)))$. If $\delta > d$, then there is $G \in C(X, \mathbb{C}_*^n)$ such that $\sup_X \|f - G\| < \delta$. Hence, $f + t(G - f)|_{(\|f\|=\delta)}$ $(0 \leq t \leq 1)$ is a homotopy in $C((\|f\| = \delta), \mathbb{C}_*^n)$ between the restrictions of f and G to $(\|f\| = \delta)$. Consequently Borsuk's theorem assures that there is an extension $F \in C(X, \mathbb{C}_*^n)$ of $f|_{(\|f\|=\delta)}$. That is, $E(f) \leq \delta$. Since $\delta > d$ is arbitrary, the lemma follows.

Theorem 3. *The topological stable rank of A is infinite.*

Proof. Let n be a positive integer. Since the polynomials on $\mathbb{T}^{2n}$ are dense in $C(\mathbb{T}^{2n})$, Lemmas 1 and 2 imply that if there is an $f \in C(\mathbb{T}^{2n})^n$ such that $E(f \circ \nu_s) \geq 1$ for all positive integers s, then $\operatorname{tsr} A \geq n$.

Let $\mathbb{T}_s^{2n}$ be the $2n$-times cartesian product of $\{e^{i\theta} \; |\theta| \leq \pi/2s\}$. Then the map $\tilde{\nu}_s = \nu_s|_{\partial \mathbb{T}_s^{2n}}$ is a homeomorphism from $\partial \mathbb{T}_s^{2n}$ onto $\partial \mathbb{T}_1^{2n}$.

The set $\mathbb{T}_1^{2n}$ is a product of $2n$ closed arc-intervals. Since each arc-interval is homeomorphic to $I = [0, 1]$, then $\mathbb{T}_1^{2n}$ is homeomorphic to I^{2n}, and therefore to $B_n = \{z \in \mathbb{C}^n \; \|z\| \leq 1\}$. Let $\varphi \; \mathbb{T}_1^{2n} \to B_n$ be an onto homeomorphism, and take any $f \in C(\mathbb{T}^{2n}, B_n)$ so that $f|_{\mathbb{T}_1^{2n}} = \varphi$. Clearly

$$f(\nu_s(\partial \mathbb{T}_2^{2n})) = f(\partial \mathbb{T}_1^{2n}) = \varphi(\partial \mathbb{T}_1^{2n}) = \partial B_n.$$

In other words, $\partial \mathbb{T}_s^{2n}$ is contained in $\{\omega \in \mathbb{T}^{2n} \|(f \circ \nu_s)(\omega)\| = 1\}$. So, if $E(f \circ \nu_s) < 1$, then there exists some extension $F_s \in C(\mathbb{T}_s^{2n}, \mathbb{C}_*^n)$ of $f \circ \nu_s|_{\partial \mathbb{T}_s^{2n}} = \varphi \circ \tilde{\nu}_s$, and since $\mathbb{T}_s^{2n}$ is a contractible space, this only happens if $\varphi \circ \tilde{\nu}_s$ is homotopic to some constant function in the space $C(\partial \mathbb{T}_s^{2n}, \partial B_n)$. Put $\varphi_{-1} = (\varphi|_{\partial \mathbb{T}_1^{2n}})^{-1}$ and consider the following string of mappings:

$$\partial B_n \xrightarrow{\varphi_{-1}} \partial \mathbb{T}_1^{2n} \xrightarrow{\tilde{\nu}_s^{-1}} \partial \mathbb{T}_s^{2n} \xrightarrow{\tilde{\nu}_s} \partial \mathbb{T}_1^{2n} \xrightarrow{\varphi} \partial B_n.$$

It is immediate that $\varphi \circ \tilde{\nu}_s \circ \tilde{\nu}_s^{-1} \circ \varphi_{-1} = \mathrm{id}_{\partial B_n}$. Therefore, if $\varphi \circ \tilde{\nu}_s$ is null-homotopic in $C(\partial \mathbb{T}_s^{2n}, \partial B_n)$, then so is $\mathrm{id}_{\partial B_n}$ in $C(\partial B_n, \partial B_n)$, which is clearly false. Thus $E(f \circ \nu_s) \geq 1$.

The Bass stable rank of a Banach algebra B ($\operatorname{bsr} B$) is the minimum positive integer n with the following property. For every $(b_1, \ldots, b_{n+1}) \in U_{n+1}(B)$ there are $c_1, \ldots, c_n \in B$ such that $(b_1 + c_1 b_{n+1}, \ldots, b_n + c_n b_{n+1}) \in U_n(B)$. We put $\operatorname{bsr} B = \infty$ if there is no such n. This notion originates in algebraic K-theory and it is the direct antecessor of the topological stable rank.

It is interesting to notice that although the Bass stable rank is a purely algebraic invariant, it coincides with the topological stable rank in the special case of C^*-algebras (see [3]). Since the algebra A is a C^*-algebra, Theorem 3 also says that $\operatorname{bsr} A = \infty$.

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Department of Mathematics, University of California, Berkeley, California 94720

Current address: Instituto Argentino de Matemática, Vio monte 1636-1° Cpo.-1° Piso, 1055 Buenos Aires, Argentina