THE ALGEBRA OF ALMOST PERIODIC FUNCTIONS HAS INFINITE TOPOLOGICAL STABLE RANK

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Abstract. We show that if \( A \) is the uniform algebra of almost periodic functions, then the set \( U_n(A) = \{ (a_1, \ldots, a_n) \in A^n : \sum_{1 \leq j \leq n} Aa_j = A \} \) cannot be dense in \( A^n \) for any positive integer \( n \).

By a Banach algebra we mean a commutative complex Banach algebra with unit. Let \( B \) be a Banach algebra and \( n \) be a positive integer. The set of unimodulars of \( B \) is \( U_n(B) = \{ (b_1, \ldots, b_n) \in B^n : \sum_{1 \leq j \leq n} Bb_j = B \} \), and the topological stable rank of \( B \) (tsr \( B \)) is the minimum positive integer \( n \) such that \( U_n(B) \) is dense in \( B^n \). We write tsr \( B = \infty \) if such \( n \) does not exist. Since its introduction by Rieffel [5] this concept has been very successful in studying the topological K-theory and some spectral properties of Banach algebras.

For a Banach algebra \( B \), denote by \( B^* \) its dual space provided with the weak* topology. The maximal ideal space of \( B \) is the compact Hausdorff space \( X(B) = \{ \varphi \in B^* : \varphi \text{ is multiplicative}, \varphi \neq 0 \} \). We denote by \( C(X(B)) \) the uniform algebra of continuous complex-valued functions on \( X(B) \). The Gelfand transform \( \hat{\varphi} : B \to C(X(B)) \), defined by \( \hat{\varphi}(b) = \varphi(b) \), is a Banach algebras morphism. So, for every positive integer \( n \) and every \( b \in B^n \) the Gelfand transform induces a continuous function \( \hat{b} : X(B) \to \mathbb{C}^n \). It is an easy exercise to prove that \( b \in U_n(B) \) if and only if \( 0 \notin b(X(B)) \).

The algebra \( A \) of almost periodic functions is the uniform algebra on \( \mathbb{R} \) generated by the functions

\[
g(t) = \sum_{1 \leq k \leq n} c_k e^{i\lambda_k t} \quad (t \in \mathbb{R}),
\]

where \( n \) is a positive integer, \( c_k \in \mathbb{C} \) and \( \lambda_k \in \mathbb{R} \) for \( k = 1, \ldots, n \). See [1, pp. 16 and 164] for general background about this algebra. The space \( X(A) \) is the so-called Bohr compactification of \( \mathbb{R} \), and it is well known that \( \mathbb{R} \) is dense in \( X(A) \) [1]. So, a necessary and sufficient condition for \( (a_1, \ldots, a_n) \in A^n \) to be unimodular is that \( |a_1(t)|^2 + \cdots + |a_n(t)|^2 \geq \delta > 0 \) for all \( t \in \mathbb{R} \). As said in the abstract, the purpose of this paper is to show that tsr \( A = \infty \). This problem was posed by I. Spitkovsky.
in a lecture where D. Sarason was present; I am grateful to him for communicating the problem to me.

We begin by establishing some conventions. If \( X \) is a compact Hausdorff space and \( Y \) is a metric space, then \( C(X,Y) \) denotes the space of continuous functions from \( X \) into \( Y \) with the supremum metric. We simply write \( C(X)^n \) when \( Y = \mathbb{C}^n \). By a polynomial in \( z_1, \ldots, z_n \) we mean a finite complex linear combination of \( z_1^{p_1} \cdots z_n^{p_n} \), where \( p_j \in \mathbb{Z} \) (the integer group). A \( \mathbb{C}^n \)-valued polynomial means a function with values in \( \mathbb{C}^n \) where each coordinate is a polynomial. It is clear that every function \( g \) as in (1) can be written as a polynomial \( f(e^{i\lambda_1 t}, \ldots, e^{i\lambda_m t}) \), where \( \lambda_1, \ldots, \lambda_m \) are linearly independent over \( \mathbb{Z} \). Therefore Kronecker’s theorem [2, Theorem 443] implies that the set \( \{(e^{i\lambda_1 t}, \ldots, e^{i\lambda_m t}) : t \in \mathbb{R} \} \) is dense in the \( m \)-dimensional torus \( \mathbb{T}^m \), and we can identify \( g \) with a polynomial on the set \( \mathbb{T}^m \).

It is well known that the set of polynomials on \( \mathbb{T}^m \) is dense in the algebra \( C(\mathbb{T}^m) \). For \( s \) a positive integer we define \( \nu_s : \mathbb{T}^m \to \mathbb{T}^m \) by \( \nu_s(\omega_1, \ldots, \omega_m) = (\omega_1^s, \ldots, \omega_m^s) \). In the sequel \( \| \| \) denotes the euclidean norm in \( \mathbb{C}^n \).

**Lemma 1.** Let \( f \) be a \( \mathbb{C}^n \)-valued polynomial on \( e^{i\lambda_1 t}, \ldots, e^{i\lambda_m t} \), where \( \lambda_1, \ldots, \lambda_m \in \mathbb{R} \) are linearly independent over \( \mathbb{Z} \). Then \( f \in \overline{U}_n(A) \) if and only if there is a sequence of positive integers \( \{s_j\} \) such that

\[
\text{dist}(f \circ \nu_{s_j}, U_n(C(\mathbb{T}^m))) \to 0
\]

when \( j \to \infty \).

**Proof.** If (2) holds, then for any \( \varepsilon > 0 \) there is an integer \( s > 0 \) and a polynomial \( F : \mathbb{T}^m \to \mathbb{C}_*^n = \mathbb{C}^n \setminus \{0\} \) so that

\[
\sup_{\omega \in \mathbb{T}^m} \| F(\omega) - f \circ \nu_s(\omega) \| < \varepsilon.
\]

Then for every \( t \in \mathbb{R} \),

\[
\| F(e^{i\lambda_1 t}, \ldots, e^{i\lambda_m t}) - f(e^{i\lambda_1 t}, \ldots, e^{i\lambda_m t}) \| < \varepsilon.
\]

Since \( \varepsilon \) is arbitrary, \( f \in \overline{U}_n(A) \). On the other hand, if \( f \in \overline{U}_n(A) \), then for \( \varepsilon > 0 \) there are \( \mu_1, \ldots, \mu_k \in \mathbb{R} \) and a \( \mathbb{C}^n \)-valued polynomial \( F \) on \( e^{i\lambda_1 t}, \ldots, e^{i\lambda_m t}, e^{i\mu_1 t}, \ldots, e^{i\mu_k t} \) such that

\[
\sup_{t \in \mathbb{R}} \| F(e^{i\lambda_1 t}, \ldots, e^{i\lambda_m t}, e^{i\mu_1 t}, \ldots, e^{i\mu_k t}) - f(e^{i\lambda_1 t}, \ldots, e^{i\lambda_m t}) \| < \varepsilon.
\]

We can reduce the number of variables in the writing of \( F \) by the following process. If for every linear combination

\[
p_1\lambda_1 + \cdots + p_m\lambda_m + q_1\mu_1 + \cdots + q_k\mu_k = 0
\]

with \( p_j, q_j \in \mathbb{Z} \) we have that \( q_k = 0 \), then we keep \( \mu_k \) and we repeat the process with \( \lambda_1, \ldots, \mu_{k-1} \). If there is a combination (3) with \( q_k \neq 0 \), then we eliminate \( \mu_k \) and repeat the process with \( \lambda_1/|q_k|, \ldots, \mu_{k-1}/|q_k| \). After finite steps we obtain a positive integer \( s \) and \( \lambda_1/s, \ldots, \lambda_m/s, \mu'_1, \ldots, \mu'_l \in \mathbb{R} \) (with \( l \leq k \)) linearly independent over \( \mathbb{Z} \), so that \( F \) can be written as a \( \mathbb{C}^n \)-valued polynomial in the correspondent exponentials. Consequently,

\[
\sup_{t \in \mathbb{R}} \| F(e^{i\lambda_1 t}, \ldots, e^{i\lambda_m t}, e^{is\mu'_1 t}, \ldots, e^{is\mu'_l t}) - f(e^{i\lambda_1 st}, \ldots, e^{i\lambda_m st}) \| < \varepsilon.
\]

Define \( \tilde{F} : \mathbb{T}^m \to \mathbb{C}^n_\ast \) by \( \tilde{F}(\omega_1, \ldots, \omega_m) = F(\omega_1, \ldots, \omega_m, 1, \ldots, 1) \). Then \( \tilde{F} \in U_n(C(\mathbb{T}^m)) \) and \( \sup_{\mathbb{T}^m} \| \tilde{F} - f \circ \nu_s \| < \varepsilon \), as claimed.
Our next lemma requires a classical result of topology due to Borsuk [4, Theorem III.3]. Let $X$ be a compact Hausdorff space and $Z \subset X$ closed. If $f, g \in C(Z, C^n_\nu)$ are homotopic and there is an extension $G \in C(X, C^n_\nu)$ of $g$, then there is also an extension $F \in C(X, C^n_\nu)$ of $f$. For $f \in C(X)^n$ put

$$E(f) = \inf \{ \delta \geq 0 \ | \ f |(\|f\| = \delta) \mbox{ admits an extension } F : X \to C^n_\nu \}.$$ 

Suppose that $\delta > E(f)$. Then by definition of $E(f)$ there are $\delta_0 < \delta$ and an extension $F_0 : X \to C^n_\nu$ of $f |(\|f\| = \delta_0)$. Henceforth, the function $F(x)$ defined as $F_0(x)$ when $\|f(x)\| \leq \delta_0$ and $f(x)$ when $\|f(x)\| > \delta_0$ is an extension of $f |(\|f\| = \delta)$ from $X$ into $C^n_\nu$.

**Lemma 2.** $E(f) \leq \text{dist}(f, U_n(C(X))) \leq 2E(f)$.

**Proof.** Let $\delta > E(f)$ and let $F : X \to C^n_\nu$ be an extension of $f |(\|f\| = \delta)$. Define

$$\mathcal{F}(x) = \begin{cases} f(x) & \text{if } \|f(x)\| \geq \delta, \\ \delta \frac{F(x)}{\|F(x)\|} & \text{if } \|f(x)\| \leq \delta. \end{cases}$$

Therefore $\mathcal{F} \in U_n(C(X))$ and its distance to $f$ is

$$\sup_{\|f(x)\| \leq \delta} \|\mathcal{F}(x) - f(x)\| = 2\delta.$$

For the other inequality put $d = \text{dist}(f, U_n(C(X)))$. If $\delta > d$, then there is a homotopy $H : I \times X \to C(X)$ such that $H(0, x) = G(x) = f(x)$ and $H(1, x) = f(x)$. Hence, $f(t(G - f)(\|G - f\| = \delta) \ (0 \leq t \leq 1)$ is a homotopy in $C(\|G - f\| = \delta, C^n_\nu)$ between the restrictions of $f$ and $G$ to $\|G - f\| = \delta$. Consequently, Borsuk’s theorem assures that there is an extension $F \in C(X, C^n_\nu)$ of $f |(\|f\| = \delta)$. That is, $E(f) \leq \delta$. Since $\delta > d$ is arbitrary, the lemma follows.

**Theorem 3.** The topological stable rank of $A$ is infinite.

**Proof.** Let $n$ be a positive integer. Since the polynomials on $T^{2n}$ are dense in $C(T^{2n})$, Lemmas 1 and 2 imply that if there is an $f \in C(T^{2n})^n$ such that $E(f \nu_s) \geq 1$ for all positive integers $s$, then $A \geq n$.

Let $T_s$ be the $2n$-times cartesian product of $\{ e^{i\theta} \ | \ \theta \leq \pi/2s \}$. Then the map $\nu_s : T_s \to T^{2n}$ is a homeomorphism from $T_s$ onto $T^{2n}$.

The set $T_s^{2n}$ is a product of $2n$ closed arc-intervals. Since each arc-interval is homeomorphic to $I = [0, 1]$, then $T_s^{2n}$ is homeomorphic to $I^{2n}$, and therefore to $B_n = \{ z \in C^n \ | \ |z| \leq 1 \}$. Let $\varphi : T_s^{2n} \to B_n$ be an onto homeomorphism, and take any $f \in C(T_s^{2n}, B_n)$ so that $f |(\partial T_s^{2n}) = \varphi$. Clearly

$$f(\nu_s(\partial T_s^{2n})) = f(\partial T_s^{2n}) = \varphi(\partial T_s^{2n}) = \partial B_n.$$ 

In other words, $\partial T_s^{2n}$ is contained in $\{ \omega \in T^{2n} \ | \ |f(\varphi_s)(\omega)| = 1 \}$. So, if $E(f \nu_s) < 1$, then there exists some extension $F_s \in C(T_s^{2n}, C^n_\nu)$ of $f \circ \nu_s |(\partial T_s^{2n}) = \varphi \circ \nu_s$, and since $T_s^{2n}$ is a contractible space, this only happens if $\varphi \circ \nu_s$ is homotopic to some constant function in the space $C(\partial T_s^{2n}, \partial B_n)$. Put $\varphi_{-1} = (\varphi |(\partial T_s^{2n}))^{-1}$ and consider the following string of mappings:

$$\partial B_n \xrightarrow{\varphi_{-1}} \partial T_s^{2n} \xrightarrow{\nu_s^{-1}} \partial T_s^{2n} \xrightarrow{\varphi} \partial T_s^{2n} \xrightarrow{\varphi_{-1}} \partial B_n.$$ 

It is immediate that $\varphi \circ \nu_s \circ \nu_s^{-1} \circ \varphi_{-1} = \text{id}_{\partial B_n}$. Therefore, if $\varphi \circ \nu_s$ is null-homotopic in $C(\partial T_s^{2n}, \partial B_n)$, then so is $\text{id}_{\partial B_n}$ in $C(\partial B_n, \partial B_n)$, which is clearly false. Thus $E(f \circ \nu_s) \geq 1$. 

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The Bass stable rank of a Banach algebra $B$ ($\text{bsr } B$) is the minimum positive integer $n$ with the following property. For every $(b_1, \ldots, b_{n+1}) \in U_{n+1}(B)$ there are $c_1, \ldots, c_n \in B$ such that $(b_1 + c_1 b_{n+1}, \ldots, b_n + c_n b_{n+1}) \in U_n(B)$. We put $\text{bsr } B = \infty$ if there is no such $n$. This notion originates in algebraic $K$-theory and it is the direct antecessor of the topological stable rank.

It is interesting to notice that although the Bass stable rank is a purely algebraic invariant, it coincides with the topological stable rank in the special case of $C^*$-algebras (see [3]). Since the algebra $A$ is a $C^*$-algebra, Theorem 3 also says that $\text{bsr } A = \infty$.

REFERENCES