A GROUP THEORETIC CRITERION FOR PROPERTY FA

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ABSTRACT. We give group-theoretic conditions on a set of generators of a group G which imply that G admits no non-trivial action on a tree. The criterion applies to several interesting classes of groups, including automorphism groups of most free groups and mapping class groups of most surfaces.

INTRODUCTION

Following Serre [9], a group G is said to have property FA if every action of G on a tree is trivial. (An action on a tree is trivial if there is a point of the tree fixed by every element of G.) Serre proved that $SL(3, \mathbb{Z})$ has property FA. Margulis and Tits showed that subgroups of finite index in simple Chevalley groups of rank ≥ 2 over a ring of integers A also have property FA [10, Remark 4.3(b)]. Later, Bogopolski showed that the same is true for automorphism groups of free groups of rank at least three [4].

In this note we give group-theoretic conditions on a set of generators of a group G which imply that G has property FA. To illustrate our theorem, we give very simple proofs of property FA for several classes of groups, including special linear groups of rank at least 3, groups of automorphisms of free groups of rank at least 3, and mapping class groups of most surfaces.

Serre's arguments actually imply that $SL(3, \mathbb{Z})$ has property F$\mathbb{R}$, which is defined to mean that every action on an $\mathbb{R}$-tree has a fixed point. In proving that a group G has property F$\mathbb{R}$ one typically shows that any non-trivial action must have an invariant line upon which the group acts by translations. The existence of such an invariant line implies that G admits a non-trivial homomorphism to $\mathbb{R}$. If the abelianization of G is finite there are no non-trivial homomorphisms to $\mathbb{R}$, so that any action of G must have a fixed point.

We will say that G has property A$\mathbb{R}$ if every non-trivial action of G on an $\mathbb{R}$-tree has an invariant line upon which G acts by translation. Our main theorem gives a criterion for a group to have property A$\mathbb{R}$. The groups listed above satisfy our conditions and also have finite abelianizations. For $n \geq 5$ the n-strand braid groups satisfy our criterion and hence have property A$\mathbb{R}$, although they do admit non-trivial homomorphisms to $\mathbb{R}$ and hence do not have property FA. G. Levitt has pointed out that results of Bieri-Neumann-Strebel [3] and Brown [5] imply that a

group with property A$\mathbb{R}$ must have a finitely generated commutator subgroup (see Remark 2.6).

1. Characteristic subtrees

We assume throughout this section that G is a group acting by isometries on an $\mathbb{R}$-tree X. For background material on groups acting on $\mathbb{R}$-trees, we refer the reader to [9] and [1] or [6].

Given any element g of G we denote by C_g the characteristic subtree of g. If g is elliptic, C_g is the subtree of X fixed by g; if g is hyperbolic, C_g is the translation axis of g, also called A_g. The following simple observation will be useful.

Lemma 1.1. *Let g and h be commuting elements of G, with h hyperbolic. Then C_g contains A_h.*

Proof. For any g and h in G, we have $C_{hgh^{-1}} = hC_g$. If g and h commute, this says that C_g is invariant under h. Any h-invariant subtree must contain the axis of h.□

Minipotent words. A word in the letters $\{g^{\pm 1}, h^{\pm 1}\}$ is *minipotent* if it has one of the two forms

$$g^{\epsilon_1} h^{\epsilon_2} \cdots g^{\epsilon_{2n-1}} h^{\epsilon_{2n}} \text{ or } h^{\epsilon_1} g^{\epsilon_2} \cdots h^{\epsilon_{2n-1}} g^{\epsilon_{2n}},$$

where $\epsilon_i \in \{\pm 1\}$ for $i = 1, \dots, n$.

Proposition 1.2. *Let g and h be elements of G, and let $w \in G$ be represented by a minipotent word in g and h. If $C_g \cap C_h = \emptyset$, then w is hyperbolic, and its axis contains the bridge from C_g to C_h.*

Proof. Let $[p, q]$ denote the bridge from C_g to C_h. Let P denote the subtree of X consisting of all x such that the geodesic $[x, q]$ contains p, and let Q be the subtree consisting of all x such that $[x, p]$ contains q. Note that P and Q are disjoint, that g and g^{-1} map $X - P$ into P and that h and h^{-1} map $X - Q$ into Q. It follows by induction that if the minipotent word which represents w begins with $g^{\pm 1}$, then w maps $X - Q$ into P, and that if the word begins with $h^{\pm 1}$, then w maps $X - P$ into Q. If m is a point in the interior of $[p, q]$, then the two segments $[w^{-1}m, m]$ and $[m, wm]$ meet only at m and their union contains $[p, q]$. It follows that w is hyperbolic and its axis contains $[p, q]$.   □

Corollary 1.3. *Let g and h be elements of G, and suppose that there exists an element w of G which commutes with either g or h and is represented by a minipotent word in g and h. Then $C_g \cap C_h \neq \emptyset$.*

Proof. Assume that $C_g \cap C_h = \emptyset$. Then Proposition 1.2 implies that w is hyperbolic and that its axis contains the bridge from C_g to C_h. In particular the axis of w is not contained in either C_g or C_h. This contradicts Lemma 1.1.   □

Commutators.

Lemma 1.4. *If h is hyperbolic with axis A_h, and $g \in G$ with $[g, h]A_h = A_h$, then $gA_h = A_h$.*

Proof. We have $ghg^{-1}A_h = ghg^{-1}h^{-1}A_h = A_h$, so A_h is the axis for the hyperbolic element ghg^{-1}. But the axis for ghg^{-1} is gA_h, giving $gA_h = A_h$.   □

Corollary 1.5. *If g and h are hyperbolic, and h commutes with $[g, h]$, then $A_g = A_h$.*

Proof. By Lemma 1.1, A_h is invariant under $[g, h]$. Thus A_h is invariant under g. The only line which is invariant under the action of a hyperbolic isometry is its axis. □

Corollary 1.5 together with Corollary 1.3 are all we need for examples where the generators are all conjugate, e.g. $\mathrm{Aut}(F_n)$, special linear groups, mapping class groups and braid groups. In general we need to consider the possibility that some generators are hyperbolic and some are elliptic. For this, we consider a special class of minipotent words which are iterated commutators.

For $x, y \in G$, we define $[x, y^{(k)}]$ inductively by

$$[x, y^{(0)}] = x \quad \text{and} \quad [x, y^{(k)}] = [[x, y^{(k-1)}], y].$$

Proposition 1.6. *Let h be hyperbolic and $g \in G$. If there exists an integer $r > 0$ such that $[g, h^{(r)}] = 1$, then $gA_h = A_h$. If g is elliptic and there exists an integer $s > 0$ such that $[h, g^{(s)}] = 1$, then g fixes A_h.*

Proof. By Lemma 1.1, $C_{[g, h^{(r-1)}]}$ contains A_h, i.e., A_h is invariant under $[g, h^{(r-1)}]$. Applying Lemma 1.4 inductively shows A_h is invariant under g. If g is elliptic and does not fix A_h, then g must act on A_h as a reflection. The second conclusion follows from the fact that the relation $[h, g^{(s)}] = 1$ does not hold in an infinite dihedral group generated by a reflection g and a translation h; in fact, $[h, g^{(s)}]$ acts on A_h as translation by $2^s |h|$. □

2. Property A$\mathbb{R}$

Our criterion for a group to have property A$\mathbb{R}$ is expressed in terms of a graph associated to a given set of generators for the group. We will regard an edge of a graph as being represented by a pair of oppositely oriented directed edges. We fix a group G and a set of generators $S = \{s_1, \dots, s_n\}$ for G.

The graph Δ. Define $\Delta(G, S)$ to be the graph with vertex set S and an edge between s_i and s_j whenever there is a minipotent word in s_i and s_j which commutes with either s_i or s_j. (In particular, if s_i and s_j commute, they are joined by an edge, since they each commute with their (trivial) commutator.)

We will need the following criterion from [9] for the existence of a global fixed point for an action of a group G on an $\mathbb{R}$-tree X. (Serre's proof is given for trees, but works equally well for $\mathbb{R}$-trees.)

Serre's Lemma. *Let G be generated by the finite set of elements $s_1, \dots, s_m$, so that the s_j and the $s_i s_j$ are all elliptic. Then G has a fixed point.*

Proposition 2.1. *Suppose that G acts on an $\mathbb{R}$-tree so that each s_i is elliptic. If $\Delta(G, S)$ is the complete graph on S, then G has a fixed point.*

Proof. Since $\Delta(G, S)$ is complete, each pair $\{s_i, s_j\}$ is connected by an edge. If s_i and s_j are elliptic, Corollary 1.3 then implies that $s_i s_j$ is also elliptic. Thus the proposition follows from Serre's Lemma. □

The graph Δ'. Recall that the iterated commutators $[s_i, s_j^{(k)}]$ are minipotent, for $k > 0$. If $[s_i, s_j^{(k)}]$ commutes with s_j for some k, we say the directed edge of $\Delta(G, S)$ from s_i to s_j is *distinguished*. Define $\Delta'(G, S)$ to be the subgraph of $\Delta(G, S)$ with vertex set S and edge set consisting of the distinguished edges and their opposites.

A subset V of S is Δ'-*connected* if the subgraph of $\Delta'(G, S)$ spanned by V is connected.

A subset V of S is *dense* if for every $x \in S - V$, there is a distinguished edge from x to some element of V and a distinguished edge from some element of V to x.

The following proposition gives a general criterion for a given action to have an invariant line upon which G acts by translations.

Proposition 2.2. *Suppose that G acts on an $\mathbb{R}$-tree. If S contains a nonempty, Δ'-connected, dense subset consisting of hyperbolic elements, then there is an invariant line upon which G acts by translations.*

Proof. Let V be a non-empty Δ'-connected dense subset of S consisting of hyperbolic elements. Since V is Δ'-connected, Corollary 1.5 implies that all the elements of V have the same axis A. Since V is dense, Proposition 1.6 shows that A is invariant under each hyperbolic element of S and fixed by each elliptic element of S. $\square$

Group-theoretic criteria for property FA. The next theorem gives a group-theoretic criterion, in terms of the generators S, for G to have property A$\mathbb{R}$. By a *conjugacy class* in S we mean a non-empty subset which is the intersection of S with a conjugacy class in G. Note that if one element of a conjugacy class is elliptic (hyperbolic), then so are all the other elements.

Theorem 2.3. *Suppose that $\Delta(G, S)$ is the complete graph on S and that each conjugacy class in S contains a Δ'-connected dense subset. Then G has property A$\mathbb{R}$.*

Proof. If all elements of S are elliptic, we apply Proposition 2.1 to conclude that the action is trivial. Otherwise there is a conjugacy class consisting of hyperbolic elements. Since this conjugacy class contains a connected dense subset, we may apply Proposition 2.2 to conclude that there is an invariant line on which G acts by translations. $\square$

If the generators of G are all conjugate, the criterion is more simply stated as follows:

Corollary 2.4. *Suppose that all of the generators in S are conjugate. If $\Delta(G, S)$ is complete and $\Delta'(G, S)$ is connected, then G has property A$\mathbb{R}$. If, in addition, the abelianization $H_1(G)$ is finite, then G has property F$\mathbb{R}$.* $\square$

If the generators are not all conjugate, the following suffices.

Corollary 2.5. *If $\Delta(G, S)$ is the complete graph on S and every directed edge of $\Delta(G, S)$ is distinguished, then G has property A$\mathbb{R}$. If, in addition, the abelianization $H_1(G)$ is finite, then G has property F$\mathbb{R}$.*

Proof. Under these hypotheses $\Delta(G, S) = \Delta'(G, S)$ is complete, and every vertex $s \in S$ is dense. $\square$

Commutator subgroups and property $A\mathbb{R}$.

Remark 2.6. If G is a finitely generated group, let $S(G)$ be the set of non-trivial homomorphisms from G to $\mathbb{R}$, modulo homotheties, and let $\Sigma(G) \subset S(G)$ be the Bieri-Neumann-Strebel invariant. Theorem B1 of [3] states that the commutator subgroup G' is finitely generated if and only if $\Sigma(G) = S(G)$. By Corollary 7.4 of [5], the complement of $\Sigma(G)$ in $S(G)$ consists of homomorphisms associated to non-trivial actions of G on $\mathbb{R}$-trees with a fixed end but no invariant line. If G has property $A\mathbb{R}$, then every non-trivial action has an invariant line; in particular $\Sigma(G)$ is all of $S(G)$, and G' is finitely generated.

3. EXAMPLES

Automorphisms of free groups. Let F_n be the free group of rank $n \geq 3$, with generators $x_1, \dots, x_n$. We show that the group $\mathrm{Aut}(F_n)$ of automorphisms of F_n has property $\mathbb{FR}$, for $n \geq 3$. We first consider the index two subgroup SA_n consisting of special automorphisms (an automorphism is *special* if the determinant of the induced automorphism of $\mathbb{Z}^n$ is equal to $+1$).

For $i \neq j$ let ρ_{ij} (respectively $\lambda_{i,j}$) be the automorphism which sends x_i to $x_i x_j$ (respectively $x_j x_i$) and fixes x_k for $k \neq i$. The ρ_{ij} and $\lambda_{i,j}$ generate SA_n. If i, j and k are distinct, then

$$(*) \qquad\qquad [\rho_{ij}, \rho_{jk}] = \rho_{ik} \quad \text{and} \quad [\lambda_{ij}, \lambda_{jk}] = \lambda_{ik}.$$

It follows that the automorphisms $r_i = \rho_{i,i+1}, l_i = \lambda_{i,i+1}$ $(1 \leq i \leq n - 1)$ and $r_n = \rho_{n,1}, l_n = \lambda_{n,1}$ generate SA_n. Set $S = \{r_1, l_1, \dots, r_n, l_n\}$.

The group SA_n contains the alternating group on n letters and the automorphisms which send exactly two generators to their inverses. Conjugating by appropriate such automorphisms shows that the elements of S are all conjugate.

Let $s_i = r_i$ or l_i. It is easily checked that s_i commutes with $[r_j, s_i]$ and $[l_j, s_i]$ unless $j \equiv i + 1 \pmod{n}$. Thus $\Delta'(G, S)$ is the complete graph on S (although not all directed edges are distinguished). In addition the s_i are all commutators, so the abelianization $H_1(SA_n)$ is trivial. We now apply Corollary 2.4 to conclude that SA_n has property $\mathbb{FR}$.

If a group has property $\mathbb{FR}$, then finite extensions and quotient groups also have property $\mathbb{FR}$. It follows that the full automorphism group $\mathrm{Aut}(F_n)$ has property $\mathbb{FR}$, as do the groups $\mathrm{Out}(F_n)$ of outer automorphisms of F_n, $GL(n, \mathbb{Z})$ and $SL(n, \mathbb{Z})$. Property FA for SA_n was first proved by O. V. Bogopolski [4], by a quite complicated argument. For $SL(n, \mathbb{Z})$, a proof is given in Serre's book on trees [9].

Special and general linear groups. For any finitely-generated commutative ring A and $n \geq 3$, let $E_n(A)$ denote the subgroup of $SL(n, A)$ generated by the elementary matrices $e_{i,j}(a)$, where a is a generator of A. If we set $S = \{e_{i,i+1}(a) | 1 \leq i \leq n - 1\} \cup \{e_{n,1}(a)\}$, then $\Delta'(E_n(A), S)$ is complete, all edges are distinguished, and all the generators are commutators, so by Corollary 2.5, $E_n(A)$ has property $\mathbb{FR}$. If A satisfies Bass's stable range condition S_d (see [2]) and $SK_1(A)$ is finite, then $E_n(A)$ has finite index in $SL(n, A)$ for $n \geq d + 2$, so we may conclude that $SL(n, A)$ has property $\mathbb{FR}$ under these conditions.

Now consider the subgroup G_N of $SL(n, \mathbb{Z})$ generated by $S = \{s_1, \dots, s_n\}$, where $s_i = e_{i,i+1}^N, 1 \leq i \leq n - 1$, and $s_n = e_{n,1}^N$.

Note that s_i commutes with s_j unless $i - j \equiv \pm 1 (\mathrm{mod}\ n)$, and in that case, $[s_i, s_j]$ commutes with both s_i and s_j, since $[e_{i,j}^r, e_{j,k}^s] = e_{i,k}^{rs}$. Thus $\Delta'(G_N, S)$ is the complete graph on S, and each directed edge is distinguished. In addition, $H_1(G_N)$ is finite, since a power of each generator can be expressed as a commutator. Thus, by Corollary 2.5, G_N has property FR.

As remarked by Serre [9], it follows from a theorem of Tits that G_N has finite index in $SL(n, \mathbb{Z})$; since any subgroup of finite index in $SL(n, \mathbb{Z})$ contains G_N for some N, this shows that every subgroup of finite index in $SL(n, \mathbb{Z})$ has property FR.

Mapping class groups of surfaces of positive genus. Let $M_{g,r}$ be the mapping class group of a surface of genus $g > 0$ with $r \geq 0$ boundary components. For a survey of results on mapping class groups, see [7].

For any surface of genus at least one, the group $M_{g,r}$ is generated by a finite set S of Dehn twists along non-separating simple closed curves, such that any two of these curves are either disjoint or intersect transversally in a single point.

Dehn twists along disjoint curves commute. If a and b are curves which intersect transversally in a single point and if the associated Dehn twists are t_a and t_b, then we have the relation $t_a t_b t_a = t_b t_a t_b$. Thus t_a and t_b commute with the minipotent word $(t_a t_b)^3$. This shows that $\Delta(M_{g,r}, S)$ is the complete graph on S.

Since any non-separating simple closed curve can be taken to any other by a homeomorphism of the surface, the Dehn twists in S are all conjugate in $M_{g,r}$.

If the genus g is equal to 1, the mapping class group $M_{g,r}$ has a non-trivial action on a tree with no invariant line. If g is at least two, enough generators commute to make $\Delta'(M_{g,r}, S)$ connected. In addition, $H_1(M_{g,r})$ is finite [8]. Thus by Corollary 2.4, $M_{g,r}$ has property FR for $g \geq 2$.

Braid groups and mapping class groups of punctured spheres. The classical braid group B_n has a generating set $S = \{\sigma_1, \dots, \sigma_{n-1}\}$. These generators are all conjugate. The generators σ_i and σ_j commute when $0 \leq i, j \leq n - 1$ and $|i - j| \geq 2$ and the relation

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

holds for $1 \leq i \leq n - 2$. As in the preceding example of mapping class groups, we have that $\Delta(B_n, S)$ is the complete graph on S, and $\Delta'(B_n, S)$ is connected if $n \geq 5$. However B_n does admit non-trivial homomorphisms onto $\mathbb{Z}$. So we can only conclude that B_n has property AR, for $n \geq 5$. It can be seen from the presentation that each of the groups B_3 and B_4 admits a non-trivial action on a tree with no invariant line.

The mapping class group M_0^s of a sphere with s punctures (which may be permuted) is a quotient of the braid group B_s, so inherits property AR for $s \geq 5$. In addition, $H_1(M_0^s)$ is finite for all s, so M_0^s has property FR for $s \geq 5$. For $s = 0$ or 1, M_0^s is trivial, and $M_0^2 \cong \mathbb{Z}/2\mathbb{Z}$; M_0^3 is generated by two elements of order two, whose product has order 3, so that Serre's lemma applies directly to show that M_0^3 has property FR. The group M_0^4 has a non-trivial action on a tree.

Nilpotent groups. Consider any finite set S of generators of a nilpotent group G. It is immediate that $\Delta'(G, S)$ is the complete graph on S, and that all directed edges in $\Delta'(G, S)$ are distinguished. By Corollary 2.5, this gives another proof that nilpotent groups have Property AR.

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