SHARP MAXIMAL INEQUALITIES
FOR STOCHASTIC INTEGRALS
IN WHICH THE INTEGRATOR IS A SUBMARTINGALE

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(Communicated by Richard T. Durrett)

Abstract. We obtain sharp maximal inequalities for strong subordinates of real-valued submartingales. Analogous inequalities also hold for stochastic integrals in which the integrator is a submartingale. The impossibility of general moment inequalities is also demonstrated.

1. Introduction

Let \((\Omega, \mathcal{F}, P)\) be a complete probability space with a right-continuous filtration \((\mathcal{F}_t)_{t \geq 0}\) where \(\mathcal{F}_0\) contains all \(P\)-null sets. Suppose \(X\) is an adapted right-continuous real-valued submartingale with left limits and \(H\) is a predictable process with values in the closed unit ball of \(\mathbb{R}^\nu\), where \(\nu\) is a positive integer. Define an adapted right-continuous process \(Y\) with left limits by

\[ Y_t = H_0X_0 + \int_{[0,t]} H_s \, dX_s. \]

We will compare the size of \(Y\) with that of \(X\) by finding constants \(\beta\) such that for all \(\lambda > 0\),

\[ \lambda P(Y^* \geq \lambda) \leq \beta \|X\|_1 \]

where \(\|X\|_1 = \sup_{t \geq 0} \|X_t\|_1\) and \(Y^* = \sup_{t \geq 0} |Y_t|\). In this paper we will denote the Euclidean norm of \(y \in \mathbb{R}^\nu\) by \(|y|\) and the inner product of \(y, k \in \mathbb{R}^\nu\) by \(y \cdot k\).

If we restrict \(X\) to the class of martingales, it is known that the best constant satisfying (1.1) is \(\beta = 2\) \([2, 3]\). By the best constant we mean that for any \(\beta < 2\) there exist a martingale \(X\), a predictable process \(H\), and a \(\lambda > 0\) such that \(\lambda P(Y^* \geq \lambda) > \beta \|X\|_1\). It is also known \([5]\) that if we restrict \(X\) to the class of nonnegative submartingales, then the best constant satisfying (1.1) is \(\beta = 3\).

In this paper we will show that for the class of real-valued submartingales, the best constant in (1.1) is \(\beta = 6\). To do this we shall first prove the analogous inequality and more for discrete-time submartingales. In the last section of this
paper we shall show that there are no moment inequalities of the form $\|Y\|_p \leq \beta \|X\|_p$ where $1 < p < \infty$ and $\beta$ is finite and depends only on $p$. In fact, we shall show that for any $p \in [1, \infty)$, there is no finite $\beta$ such that $\|Y\|_1 \leq \beta \|X\|_p$. For the case $p = \infty$, see [7] where it is shown that if $\|X\|_\infty = 1$, then there is a constant $\gamma$ such that for $\lambda > 4$, $P(Y^* \geq \lambda) \leq \gamma \exp(-\lambda/4)$, so, for any $r \in [1, \infty)$, $\|Y\|_r$ is bounded by some constant depending only on $r$.

2. A MAXIMAL INEQUALITY FOR SUBMARTINGALES

Let $f_0, f_1, \ldots$ be a real-valued submartingale relative to a filtration $(\mathcal{F}_n)_{n \geq 0}$ on a probability space $(\Omega, \mathcal{F}, P)$ with difference sequence $d_0, d_1, \ldots$ and $g_0, g_1, \ldots$ an $\mathbb{R}^\nu$-valued process adapted to $(\mathcal{F}_n)_{n \geq 0}$ with difference sequence $e_0, e_1, \ldots$, where $\nu$ is a positive integer. We say that $g$ is strongly subordinate to $f$ if $g$ is both differentially subordinate and conditionally differentially subordinate to $f$, i.e. for all $n \geq 0$, $|e_n| \leq |d_n|$ and $|E(e_n | \mathcal{F}_n)| \leq |E(d_n | \mathcal{F}_n)|$. Note that if for $k \geq 0$, $e_k = h_k d_k$ where $h_k : \Omega \rightarrow [-1,1]$ is $\mathcal{F}_{k-1}$-measurable, then $g$ is strongly subordinate to $f$. In particular, if $g$ is a $\pm$-1-transform of $f$, i.e. $e_k = \epsilon_k d_k$ where $\epsilon_k \in \{-1,1\}$, then $g$ is strongly subordinate to $f$.

Theorem 2.1. If $f = (f_n)_{n \geq 0}$ is a submartingale relative to a filtration $(\mathcal{F}_n)_{n \geq 0}$ and $g = (g_n)_{n \geq 0}$ is strongly subordinate to $f$, then for all $\lambda > 0$,

$$\lambda P(g^* \geq \lambda) \leq \lambda P(\sup_{n \geq 0} (|f_n| + |g_n|) \geq \lambda) \leq 4 \sup_{n \geq 0} E[f_n^+] - 2E f_0$$

where $g^* = \sup_{n \geq 0} |g_n|$.

Remarks. If $f$ is a martingale, then $E[f_n^+]$ and $E[f_n^-]$ are nondecreasing sequences. It then follows from $E[f_0] = E[f_n^+] - E[f_n^-]$ that $\|f\|_1 = 2 \sup_{n \geq 0} E[f_n^+] - E[f_0]$, where $\|f\|_1 = \sup_{n \geq 0} \|f_n\|_1$. Thus in the martingale case, (2.1) implies that

$$\lambda P(g^* \geq \lambda) \leq \lambda P(\sup_{n \geq 0} (|f_n| + |g_n|) \geq \lambda) \leq 2 \|f\|_1$$

which is Theorem 4.1 of [4]. If $f$ is a nonnegative supermartingale, (2.1) implies

$$\lambda P(g^* \geq \lambda) \leq \lambda P(\sup_{n \geq 0} (|f_n| + |g_n|) \geq \lambda) \leq 2E f_0$$

which is Theorem 8.1 of [5]. Both results are shown to be sharp in the articles quoted. If $f$ is a nonnegative submartingale with $f_0 = 0$, the resulting inequality is not sharp in the case $f_0 = 0$, as can be seen from Theorem 4.1 of [5] which shows in this case

$$\lambda P(g^* \geq \lambda) \leq \lambda P(\sup_{n \geq 0} (|f_n| + |g_n|) \geq \lambda) \leq 3 \|f\|_1.$$

Proof. We will assume $\|f\|_1$ is finite. This is equivalent to saying $\sup_{n \geq 0} E[f_n^+]$ is finite, as for all $n \geq 0$, $E[f_n^+] \leq \|f_n\|_1 \leq 2E[f_n^+] - E[f_0]$. The first inequality is obvious, the second follows from $E[f_0] \leq E[f_n] = E[f_n^+] - E[f_n^-]$. The first inequality is obvious, the second follows from $E[f_0] \leq E[f_n] = E[f_n^+] - E[f_n^-]$. To show (2.1), it suffices to show that for $n \geq 0$,

$$\lambda P(|f_n| + |g_n| \geq \lambda) \leq 4E[f_n^+] - 2E f_0$$

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Then (2.2) is equivalent to
\[ \lambda P(\sup_{m \leq n} (|f_m| + |g_m|) \geq \lambda) = \lambda P(|f_{\tau \wedge n}| + |g_{\tau \wedge n}| \geq \lambda) \leq 4\mathbb{E}f_{\tau \wedge n}^+ - 2\mathbb{E}f_0. \]

Since \((f_n^+)_{n \geq 0}\) is a submartingale, it follows by Doob’s optional sampling theorem that \(\mathbb{E}f_{\tau \wedge n}^+ \leq \mathbb{E}f_n^+\), thus implying (2.1).

By dividing by \(\lambda\) throughout in (2.2), we may assume \(\lambda = 1\). Using the methods developed by Burkholder [2], we define \(V\) on \(\mathbb{R} \times \mathbb{R}^\nu\) by
\[
V(x, y) = \begin{cases} 
1 - 4x^+, & \text{if } |x| + |y| \geq 1, \\
-4x^+, & \text{if } |x| + |y| < 1.
\end{cases}
\]

Then (2.2) is equivalent to \(\mathbb{E}V(f_n, g_n) \leq -2\mathbb{E}f_0\). Define \(U\) on \(\mathbb{R} \times \mathbb{R}^\nu\) by
\[
U(x, y) = \begin{cases} 
1 - 4x^+, & \text{if } |x| + |y| \geq 1, \\
|y|^2 - x^2 - 2x, & \text{if } |x| + |y| < 1.
\end{cases}
\]

Then \(V \leq U\) (in the case of \(|x| + |y| < 1\) this follows from \(-4x^+ \leq -x^2 - 2x\) for \(|x| < 1\) and \(U(f_0, g_0) \leq -2f_0\) recall that by assumption \(|f_0| \geq |g_0|\)).

Thus \(\mathbb{E}V(f_0, g_0) \leq \mathbb{E}U(f_0, g_n)\) and \(\mathbb{E}U(f_0, g_0) \leq -2\mathbb{E}f_0\). To show (2.2), it will suffice to show that for \(1 \leq j \leq n\),
\[\mathbb{E}U(f_j, g_j) \leq \mathbb{E}U(f_{j-1}, g_{j-1}).\]

Define \(\phi, \psi\) on \(\mathbb{R} \times \mathbb{R}^\nu\) by
\[
\phi(x, y) = \begin{cases} 
-4, & \text{if } |x| + |y| \geq 1 \text{ and } x \geq 0, \\
0, & \text{if } |x| + |y| \geq 1 \text{ and } x < 0, \\
-2x - 2, & \text{if } |x| + |y| < 1,
\end{cases}
\]
\[
\psi(x, y) = \begin{cases} 
0, & \text{if } |x| + |y| \geq 1, \\
2y, & \text{if } |x| + |y| < 1.
\end{cases}
\]

Then \(U_x(x, y) = \phi(x, y)\) and \(U_y(x, y) = \psi(x, y)\) for \(|x| + |y| \neq 1, y \neq 0,\) and \(x \neq 0\) where \(U_x(x, y)\) and \(U_y(x, y)\) are the partials of \(U\) with respect to \(x\) and \(y\) respectively. Note that \(|\psi| \leq -\phi\).

Claim: Given \(h \in \mathbb{R}\) and \(k \in \mathbb{R}^\nu\) with \(|k| \leq |h|\), then for all \(x \in \mathbb{R}\) and \(y \in \mathbb{R}^\nu\)
\[U(x + h, y + k) \leq U(x, y) + \phi(x, y)h + \psi(x, y) \cdot k.\]

This can be verified by checking the various cases:

For \(|x| + |y| \geq 1\) and \(x \geq 0\), we need to show \(U(x + h, y + k) \leq 1 - 4(x + h)\). For \(|x + h| + |y + k| \geq 1\) this is clear. For \(|x + h| + |y + k| < 1\) it follows from
\[|y + k|^2 < (1 - |x + h|)^2 \leq 1 - 2(x + h) + (x + h)^2.\]

For \(|x| + |y| \geq 1\) and \(x < 0\), we need to show \(U(x + h, y + k) \leq 1\). However \(U(x, y) \leq 1\) for all \(x, y\), this being obvious for \(|x|^2 + |y| \geq 1\). In the region \(|x| + |y| < 1\), since \(U_x(x, y) \leq 0\), it follows that \(U(x, y) \leq |y|^2 - (|y| - 1)^2 - 2(|y| - 1) = 1.\)
Theorem 3.1. If 
\[
\begin{align*}
\lambda P(g^* \geq \lambda) & \leq \lambda P(\sup_{n \geq 0} (|f_n| + |g_n|) \geq \lambda) 
\leq 4 \|f\|_1 - 2E f_0. 
\end{align*}
\]

Thus if \( f_0 \equiv 0 \), then
\[
\lambda P(g^* \geq \lambda) \leq \lambda P(\sup_{n \geq 0} (|f_n| + |g_n|) \geq \lambda) \leq 4 \|f\|_1,
\]

while in general
\[
\lambda P(g^* \geq \lambda) \leq \lambda P(\sup_{n \geq 0} (|f_n| + |g_n|) \geq \lambda) \leq 6 \|f\|_1.
\]

The constants 4 and 6 are the best possible in (3.2) and (3.3) respectively, even in the case \( \nu = 1 \) and \( g \) is a \pm 1-transform of \( f \).

Proof. The inequalities follow immediately from Theorem 2.1. For the sharpness, first consider the following example:
Since we are assuming a strict inequality, there exists an $P$ from both the ($\tilde{\lambda}$-$g_k$) sequence of double or nothings we have that for $j \geq 3$, and $s \in [0,1]$, let

\[
\tilde{f}_{3+j+k}(s) = \tilde{f}_{3+j}(s) + 1_{[1-2^{-j+1}+2^{-j}]1]}(s)\tilde{f}_k(2^{j+1}(s-1+2^{-j})),
\]
\[
\tilde{g}_{3+j+k}(s) = \tilde{g}_{3+j}(s) + 1_{[1-2^{-j+1}+2^{-j}]}(s)g_k(2^{j+1}(s-1+2^{-j})).
\]

By induction on $j \geq 0$, we have

\[
P(\tilde{f}_{3+j} = 1, \tilde{g}_{3+j} = 1) = (1-2^{-j})\alpha, \quad P(\tilde{f}_{3+j} = 0, \tilde{g}_{3+j} = 2) = (1-2^{-j})(\alpha - \alpha^2),
\]
\[
P(\tilde{f}_{3+j} = 0, \tilde{g}_{3+j} = -2\alpha = 1-2^{-j}) = (1-2^{-j})(1-\alpha)^2,
\]

and, for $k \leq 3j$, $\sup \tilde{f}_k \subseteq [0,1-2^{-j}]$. It follows that $\tilde{f}$ is a submartingale, $\tilde{g}$ is a $\pm 1$-transform of $f$, and, for $j \geq 0$, $1 \leq k \leq 3$, $\|\tilde{f}_{3+j+k}\|_1 = \|\tilde{f}_{3+j}\|_1 + 2^{-j-1}\|f_k\|_1$. Since $\|\tilde{f}_{3+j}\|_1 = (1-2^{-j})\alpha$ and $\|f_k\|_1 = \|f_2\|_1 = 2\alpha$, we have that $\|\tilde{f}_{3+j+k}\|_1 \leq \alpha = E\tilde{f}_3$. Thus, with $\lambda = 2$,

\[
\lim_{j \to \infty} \lambda P(\tilde{f}_{3+j} + \tilde{g}_{3+j} \geq \lambda) = \lambda P(f_3 + g_3 \geq \lambda) > \beta E\tilde{f}_3 \geq \beta \sup_{k \geq 0} \|\tilde{f}_k\|_1.
\]

Since we are assuming a strict inequality, there exists an $n$ such

\[
\lambda P(\tilde{f}_n + \tilde{g}_n \geq \lambda) > \beta \sup_{j \geq 0} \|\tilde{f}_j\|_1 \geq \beta \sup_{0 \leq j \leq n} \|\tilde{f}_j\|_1.
\]

Now let $(r_j)_{j \geq 1}$ be a sequence of independent identically distributed random variables such that $P(r_1 = 1) = P(r_1 = -1) = \frac{1}{2}$ and the $(r_j)$ are independent from both the $(\tilde{f}_j)$ and the $(\tilde{g}_j)$. For $j \geq 0$, let $\tilde{f}_{n+j+1} = \tilde{f}_{n+j} + \tilde{f}_{n+j} r_{j+1} + 1$ and $\tilde{g}_{n+j+1} = \tilde{g}_{n+j} + \tilde{g}_{n+j} r_{j+1} + 1$. By this sequence of double or nothings we have that for $j \geq n$, $\|\tilde{f}_j\|_1 = \|\tilde{f}_n\|_1$, yet

\[
\lim_{m \to \infty} \lambda P(\tilde{g}_m \geq \lambda) = \lambda P(\tilde{f}_n + \tilde{g}_n \geq \lambda) > \beta \|\tilde{f}_n\|_1.
\]
and since we are assuming a strict inequality, we can choose an \( m > n \) that satisfies

\[
\lambda P(\tilde{g}_m \geq \lambda) > \beta \|\tilde{f}\|_1.
\]

This immediately implies the sharpness in (3.2). To show the sharpness in (3.3), it suffices to use \( \tilde{f} \) and \( \tilde{g} \) to construct a submartingale \( F \) with a \( \pm 1 \)-transform \( G \) such that

\[
\lambda P(\sup_{j \geq 0} G_j \geq \lambda) > \frac{3}{2} \beta \|F\|_1.
\]

(3.6)

Let \( \alpha = P(\sup_{0 \leq j \leq n} \tilde{g}_j \geq \lambda) \) so that \( \alpha > 0 \) and let \( \delta = (4\|\tilde{f}\|_1 - \lambda\alpha)/(6 - 6\alpha) \) (in the case \( \alpha = 1 \), let \( \delta = 0 \)). By (3.2), \( \lambda \alpha \leq 4\|\tilde{f}\|_1 \), hence \( \delta \geq 0 \).

Let \( s \) and \( t \) be independent random variables, independent from the \( \tilde{f}_j \) such that \( P(s = \lambda/6) = \alpha \) and \( P(s = \delta) = 1 - \alpha \), while \( P(t = -1) = 2/3 \) and \( P(t = 2) = 1/3 \). Note that \( E s \leq 2\|\tilde{f}\|_1/3 \).

Let \( F_0 = -s \), \( G_0 = s \), \( F_1 = F_0 + tF_0 \), and \( G_1 = G_0 - tF_0 \). We then have that \( \|F_1\|_1 = \|F_0\|_1 = E s \).

Let \( F_2 = F_1 - F_1 \) and \( G_2 = G_1 - F_1 \). Thus \( F_2 = 0 \) a.s. while \( G_2 = 6s \) on the set \( \{t = 2\} \) and \( G_2 = 0 \) on the set \( \{t = -1\} \). We then have that

\[
P(F_2 = 0, G_2 = \lambda) = \alpha/3, \quad P(F_2 = 0, G_2 = 6\delta) = (1 - \alpha)/3,
\]

\[
P(F_2 = 0, G_2 = 0) = 2/3.
\]

Let \( A = \{G_2 = 0\} \) and, for \( j \geq 1 \), let \( F_{2+j} = 1_A \tilde{f}_j \) and \( G_{2+j} = G_2 + 1_A \tilde{g}_j \). Then by the independence of \( t \) and the \( \tilde{f}_j \), \( F \) is a submartingale, \( G \) is a \( \pm 1 \)-transform of \( F \), and for \( j \geq 1 \) we have that \( \|F_{2+j}\|_1 = 2\|\tilde{f}_j\|_1/3 \) while

\[
P(\sup_{0 \leq j \leq m+2} G_j \geq \lambda) = P(\sup_{0 \leq j \leq 2} G_j \geq \lambda) + \frac{2}{3} P(\sup_{0 < j \leq m} \tilde{g}_j \geq \lambda)
\]

\[
\geq \frac{1}{3} \alpha + \frac{2}{3} P(\sup_{0 < j \leq m} \tilde{g}_j \geq \lambda) = P(\sup_{0 \leq j \leq m} \tilde{g}_j \geq \lambda),
\]

so that

\[
\lambda P(\sup_{0 \leq j \leq m+2} G_j \geq \lambda) \geq \lambda P(\sup_{0 \leq j \leq m} \tilde{g}_j \geq \lambda) > \beta \|\tilde{f}\|_1 \geq \frac{3\beta}{2} \|F\|_1.
\]

4. Applications to stochastic integrals

**Theorem 4.1.** With \( (\Omega, \mathcal{F}, P) \) and \( (\mathcal{F}_t)_{t \geq 0} \) as in Section 1, suppose \( X \) is an adapted right-continuous submartingale with left limits such that \( E X_0 \) is finite and \( H \) is a predictable process with values in the closed unit ball of \( \mathbb{R}^n \). Then with \( Y \) defined by \( Y_t = H_0 X_0 + \int_{(0,t]} H_s dX_s \), we have that, for \( \lambda > 0 \),

\[
\lambda P(Y^*_t \geq \lambda) \leq 4 \sup_{t \geq 0} E X_t^+ - 2E X_0.
\]
so that

\( \lambda P(Y^* \geq \lambda) \leq 6 \|X\|_1 \)

and if \( X_0 \equiv 0 \), then

\( \lambda P(Y^* \geq \lambda) \leq 4 \|X\|_1 \).

The constants 6 and 4 in (4.2) and (4.3) respectively are the best possible.

**Proof.** As in Theorem 2.1 we have

\[ E X_{t+} + t \leq \|X_t\|_1 \leq 2 E X_{t+} - E X_0, \]

hence we can assume the finiteness of \( \|X\|_1 \). The proof follows in the same way as the proof of Theorem 5.1 of [5], except that we use Theorem 2.1 above to show that \( X \) is an \( L^{1,\infty} \)-integrator in the sense of [1].

The sharpness in (4.2) and (4.3) follow from the sharpness in (3.8) and (3.7) holding even for \( \pm 1 \)-transforms.

### 5. Lack of \( L^p \) inequalities

Fix \( p \in [1, \infty) \) and \( \beta > 1 \). We shall construct a discrete time submartingale \( F = (F_0, F_1, \ldots) \) with \( F_0 = 0 \) and a \( \pm 1 \)-transform of \( F \), \( G = (G_0, G_1, \ldots) \) such that

\[ \|G\|_1 > \beta \|F\|_p. \]

To do this, we will first construct a finite length submartingale \( f = (f_0, f_1, \ldots, f_N) \) with \( f_0 = 0 \) and \( f_N \geq 0 \) together with a \( \pm 1 \)-transform of \( f \), \( g = (g_0, \ldots, g_N) \) such that

\[ \|g\|_1 > \beta \|f^+\|_p \]

where \( \|f^+\|_p = \sup_{0 \leq n \leq N} E(f^+_n)^p \). Let \( M > 4 \beta \) and let \((r_1, \ldots, r_{2M})\) be a sequence of independent random variables such that for \( j = 1, 2 \), \( P(r_j = 1) = P(r_j = -1) = 1/2 \) and for \( 2 \leq j \leq M \),

\[ P(r_{2j} = 1) = P(r_{2j-1} = -1) = \frac{1}{2}, \]

\[ P(r_{2j} = -1) = \frac{1}{3}, \quad P(r_{2j-1} = 1) = \frac{2}{3}. \]

Let \( f_j = \sum_{k=0}^j d_k \) where \( d_0 = 0 \), \( d_1 = r_1/2 \), \( d_2 = r_2/2 \), and \( d_j = 1_{(f_{j-1} \leq 0)} r_j f_{j-1} \) for \( j > 2 \). By the independence of the \( r_j \), \( (f_j)_{j \leq 2M} \) forms a martingale. Note that for \( j \geq 1 \),

\[ P(f_{2j} = 1) = \frac{1}{3}, \quad P(f_{2j} = -3^{j-1}) = \frac{1}{4} \left( \frac{1}{3} \right)^{j-1}, \]

\[ P(f_{2j} = 0) = \frac{3}{4} - \frac{1}{4} \left( \frac{1}{3} \right)^{j-1}. \]
For $0 \leq j \leq 2M$, let $g_j = \sum_{k=0}^{j} (-1)^k d_k$. Then for $2 \leq j \leq 2M$, $\|f_j^+\|_p = \|f_j^-\|_1 = 1/4$, while for $j \geq 1$,

$$\|g_{2j+2}\|_1 = \|g_{2j+1}\|_1 = \|g_{2j}\|_1 + 1/4.$$ 

Since $\|g_2\|_1 = 1/2$, it follows that $\|g_{2M}\|_1 = (M + 1)/4$. Now let $N = 2M + 1$, $f_N = f_{2M}$, and $g_N = g_{2M} + 1_{\{f_{2M} < 0\}} |f_{2M}|$. Then $f = (f_0, \ldots, f_N)$ forms a submartingale; $\|f^+\|_p = 1/4$, and, since $f_{2M} < 0$ implies $g_{2M} = 0$, $\|g_N\|_1 \geq \|g_{2M}\|_1 - \|f_{2M}\|_1 = M/4 > \beta \|f^+\|_p$ by our choice of $M$.

To construct $F$ and $G$, we will work with only a small portion of the probability space at a time in order to keep $\|F\|_p$, close to that of $\|f^+\|_p$. More explicitly, by enriching the probability space if necessary, let $A_1, \ldots, A_K$ be a partition of the space such that $\text{Pr}(A_1, \ldots, A_K)$ is independent of $\sigma(f_0, \ldots, f_N)$ and, for $1 \leq j \leq k$, $P(A_j) \leq \epsilon/3^{M^p}$, where $\epsilon$ satisfies $3^p(\|f^+\|_p^p + \epsilon) < \|g_N\|_p^p$.

Let $F_0 = G_0 = 0$ and for $1 \leq k \leq K$ and $1 \leq n \leq N$, let

$$F_{(k-1)N+n} = F_{(k-1)N} + 1_{A_k} f_n, \quad G_{(k-1)N+n} = G_{(k-1)N} + 1_{A_k} g_n.$$ 

Then $F$ is a submartingale and $G$ is a $\pm 1$-transform of $F$. Since $A_1, \ldots, A_N$ partition the space, $G_{KN} = g_N$ and for $1 \leq k \leq K$ and $1 \leq n \leq N$, the disjointness of the $A_j$ gives us

$$\|F_{(k-1)N+n}\|_p^p = \|f_n 1_{\left( \bigcup_{j=1}^{k-1} A_j \right)}\|_p^p + \|f_n 1_{A_k}\|_p^p.$$ 

Since $f_N \geq 0$ a.s. and the $f_j$ are bounded in absolute value by $3^M$, we have that

$$\|F_{(k-1)N+n}\|_p^p \leq \|f_N\|_p^p + 3^{M^p} P(A_k) \leq \|f^+\|_p^p + \epsilon$$ 

which gives us (5.1) by our choice of $\epsilon$.

ACKNOWLEDGMENT

The author is grateful to Donald Burkholder for his advice and criticisms.

REFERENCES


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