

EXISTENCE RESULTS FOR SUPERLINEAR SEMIPOSITONE BVP'S

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ABSTRACT. We consider the existence of positive solutions to the BVP

$$\begin{aligned}(p(t)u')' + \lambda f(t, u) &= 0, & r < t < R, \\ au(r) - bp(r)u'(r) &= 0, \\ cu(R) + dp(R)u'(R) &= 0,\end{aligned}$$

where $\lambda > 0$. Our results extend some of the existing literature on superlinear semipositone problems and singular BVPs. Our proofs are quite simple and are based on fixed point theorems in a cone.

1. INTRODUCTION

We consider the existence of positive solutions for the Sturm-Liouville boundary value problem

$$(1.1) \quad \begin{aligned}(p(t)u')' + \lambda f(t, u) &= 0, & r < t < R, \\ au(r) - bp(r)u'(r) &= 0, \\ cu(R) + dp(R)u'(R) &= 0,\end{aligned}$$

where $f(t, 0)$ need not be non-negative. The problem (1.1) with Dirichlet boundary conditions was treated by Garaizar [5] and Castro-Shivaji [3]. In [5], the existence of a positive solution to (1.1) for $\lambda > 0$ small was established with $p(t) = t^{n-1}$, $f(t, u) = t^{n-1}g(u)$, $g(0) < 0$ and $g(u) = O(u^k)$ for some $k > 1$. The existence of a unique positive solution to (1.1) for $\lambda > 0$ small was proved in [3] with $p(t) = 1$, f independent of t , $f(0) < 0$, $f' > 0$, $f'' > 0$ and f superlinear. Further, Anuradha-Shivaji [1] extended this existence result for the Robin boundary condition case. In this paper, we shall establish the existence of a positive solution to (1.1) for $\lambda > 0$ small under the following general conditions:

$$f(t, u) \geq -M$$

for some $M > 0$, and

$$\lim_{u \rightarrow \infty} \frac{f(t, u)}{u} = \infty$$

uniformly on a compact subinterval of (r, R) .

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We also study the problem (1.1) with f possibly singular. Here we do not assume that $f(t, u)$ is decreasing in u , thus extending the corresponding results in [2, 4, 6, 9]. Our proofs are quite simple and are based on fixed point theorems in a cone.

2. EXISTENCE WITH f REGULAR

We make the following assumptions:

- (A.1) $p \in C[r, R]$, $p(t) > 0$ for every $t \in [r, R]$.
- (A.2) $a, b, c, d \geq 0$ and $ac + ad + bc > 0$.
- (A.3) $f : [r, R] \times [0, \infty) \rightarrow \mathbb{R}$ is continuous and there exists an $M > 0$ such that $f(t, u) \geq -M$ for every $t \in [r, R]$, $u \geq 0$.
- (A.4) $\lim_{u \rightarrow \infty} \frac{f(t, u)}{u} = \infty$ uniformly on a compact subinterval $[\alpha, \beta]$ of (r, R) .

Then we have

Theorem 2.1. *Let (A.1)–(A.4) hold. Then the problem (1.1) has a positive solution for $\lambda > 0$ sufficiently small.*

In order to prove Theorem 2.1, we first recall:

Theorem A ([7]–[8]). *Let \mathbb{K} be a cone in a Banach space E and let $A : \mathbb{K} \rightarrow \mathbb{K}$ be a completely continuous operator. Let $0 < r < R$ be such that*

- (i) $u \leq Au \Rightarrow \|u\| \neq r$,
- (ii) $u \geq Au \Rightarrow \|u\| \neq R$.

Here $u \leq v$ iff $v - u \in \mathbb{K}$. Then A has a fixed point u with $r < \|u\| < R$.

We further need the following lemmas.

Lemma 2.1. *Let (A.1), (A.2) hold and let u satisfy*

$$\begin{aligned} (p(t)u')' &= -v, & r < t < R, \\ au(r) - bp(r)u'(r) &= 0, & cu(R) + dp(R)u'(R) = 0, \end{aligned}$$

where $v \in L^1(r, R)$, $v \geq 0$. Then

$$u(t) \geq \|u\|_0 q(t), \quad t \in [r, R],$$

where

$$q(t) = \min \left(\frac{b + a \int_r^t \frac{1}{p}}{b + a \int_r^R \frac{1}{p}}, \frac{d + c \int_t^R \frac{1}{p}}{d + c \int_r^R \frac{1}{p}} \right).$$

Here $\|\cdot\|_0$ stands for the sup norm.

Proof of Lemma 2.1. It can be verified that

$$u(t) = \int_r^R K(t, s)v(s) ds, \quad t \in [r, R],$$

where

$$(2.1) \quad K(t, s) = \begin{cases} \alpha^{-1} \left(b + a \int_r^s \frac{1}{p} \right) \left(d + c \int_t^R \frac{1}{p} \right) & \text{if } s \leq t, \\ \alpha^{-1} \left(b + a \int_r^t \frac{1}{p} \right) \left(d + c \int_s^R \frac{1}{p} \right) & \text{if } s \geq t, \end{cases}$$

where $\alpha = ad + ac \int_r^R \frac{1}{p} + bc$.

Let $|u|_0 = u(t_0)$ for some $t_0 \in [r, R]$. We verify that

$$\frac{K(t, s)}{K(t_0, s)} \geq q(t), \quad s, t, t_0 \in (r, R).$$

Indeed, if $t, t_0 \leq s$,

$$\frac{K(t, s)}{K(t_0, s)} = \frac{b + a \int_r^t \frac{1}{p}}{b + a \int_r^{t_0} \frac{1}{p}} \geq \frac{b + a \int_r^t \frac{1}{p}}{b + a \int_r^R \frac{1}{p}},$$

if $t \leq s \leq t_0$,

$$\frac{K(t, s)}{K(t_0, s)} = \frac{\left(b + a \int_r^t \frac{1}{p}\right) \left(d + c \int_s^R \frac{1}{p}\right)}{\left(b + a \int_r^s \frac{1}{p}\right) \left(d + c \int_{t_0}^R \frac{1}{p}\right)} \geq \frac{\left(b + a \int_r^t \frac{1}{p}\right)}{\left(b + a \int_r^R \frac{1}{p}\right)}.$$

The other cases are treated similarly. Thus

$$u(t) = \int_r^R \frac{K(t, s)}{K(t_0, s)} K(t_0, s) v(s) ds \geq |u|_0 q(t), \quad t \in [r, R].$$

Lemma 2.2. *Let (A.1), (A.2) hold and let \bar{w} be the solution of*

$$\begin{aligned} (p(t)u')' &= -1, & r < t < R, \\ au(r) - bp(r)u'(r) &= 0, \\ cu(R) + dp(R)u'(R) &= 0. \end{aligned}$$

Then there exists a positive number C such that $\bar{w}(t) \leq Cq(t)$ for every $t \in [r, R]$.

Proof of Lemma 2.2. We have

$$\begin{aligned} \bar{w}(t) &= \alpha^{-1} \left[\left(d + c \int_t^R \frac{1}{p} \right) \left(\int_r^t \left(b + a \int_r^s \frac{1}{p} \right) ds \right) \right. \\ &\quad \left. + \left(b + a \int_r^t \frac{1}{p} \right) \left(\int_t^R \left(d + c \int_s^R \frac{1}{p} \right) ds \right) \right] \\ &\leq \alpha^{-1} (R - r) \left(b + a \int_r^t \frac{1}{p} \right) \left(d + c \int_t^R \frac{1}{p} \right) \leq Cq(t), \end{aligned}$$

where $C = \alpha^{-1} (b + a \int_r^R \frac{1}{p}) (d + c \int_r^R \frac{1}{p}) (R - r)$.

Proof of Theorem 2.1. Let λ satisfy

$$(2.2) \quad 0 < \lambda < \min \left(\frac{1}{C_1 |\bar{w}|_0}, \frac{1}{CM} \right),$$

where $C_1 = \sup_{r \leq s \leq R, 0 \leq t \leq 1} g(s, t)$, $g(s, t) = f(s, t) + M$ and C is the constant defined in Lemma 2.2.

Let $w = \lambda M \bar{w}$. Then u is a positive solution of (1.1) iff $\tilde{u} = u + w$ is a solution of

$$\begin{aligned}(p(t)u')' &= -\lambda \tilde{g}(t, u - w), \\ au(r) - bp(r)u'(r) &= 0, \\ cu(R) + dp(R)u'(R) &= 0\end{aligned}$$

with $\tilde{u}(t) > w(t)$ on (r, R) . Here $\tilde{g}(t, u) = g(t, u)$ for $u \geq 0$, and $g(t, u) = g(t, 0)$ for $u < 0$.

Let $\mathbb{K} = \{u \in C[r, R] : u(t) \geq |u|_0 q(t), t \in [r, R]\}$, where q is defined by Lemma 2.1. For each $v \in \mathbb{K}$, let $u = Av$ be the solution of

$$\begin{aligned}(p(t)u')' &= -\lambda \tilde{g}(t, v - w), \\ au(r) - bp(r)u'(r) &= 0, \\ cu(R) + dp(R)u'(R) &= 0.\end{aligned}$$

By Lemma 2.1, $A : \mathbb{K} \rightarrow \mathbb{K}$ and it can be verified that A is completely continuous. We shall prove that A has a fixed point in \mathbb{K} by using Theorem A [7]. Let $u \in \mathbb{K}$ be such that $u \leq Au$. We claim that $|u|_0 \neq 1$. Indeed, if $|u|_0 = 1$, then we have

$$u(t) \leq \lambda \int_r^R K(t, s) \tilde{g}(s, u - w) ds \leq \lambda C_1 \bar{w}(t), \quad t \in [r, R],$$

where $K(t, s)$ is given by (2.1). This implies

$$1 \leq \lambda C_1 |\bar{w}|_0,$$

a contradiction to (2.2), proving the claim.

Now, let $u \in \mathbb{K}$ with $u \geq Au$. Then we have

$$(2.3) \quad u(t) \geq \lambda \int_\alpha^\beta K(t, s) \tilde{g}(s, u - w) ds.$$

Let $\widetilde{M} > 0$ and let $|u|_0 = \bar{R}$. Since

$$w(s) = \lambda M \bar{w}(s) \leq \frac{\lambda CM}{\bar{R}} u(s),$$

it follows that

$$u(s) - w(s) \geq \left(1 - \frac{\lambda CM}{\bar{R}}\right) u(s).$$

Therefore if \bar{R} is sufficiently large, we have

$$(2.4) \quad u(s) - w(s) \geq \frac{1}{2} u(s) \geq \frac{1}{2} \bar{R} \delta, \quad s \in [\alpha, \beta],$$

where $\delta = \min_{\alpha \leq s \leq \beta} q(s)$, and

$$(2.5) \quad \tilde{g}(s, u - w) = g(s, u - w) \geq \widetilde{M}(u(s) - w(s)) \geq \frac{\widetilde{M} \bar{R} \delta}{2}$$

by (A.4).

Combining (2.3)–(2.5), we obtain

$$\bar{R} \geq \frac{\lambda \widetilde{M} \bar{R} \delta}{2} \left(\sup_{r \leq t \leq R} \int_\alpha^\beta K(t, s) ds \right),$$

which is a contradiction if \widetilde{M} is sufficiently large. So there exists an $\overline{R} > 1$ such that $|u|_0 \neq \overline{R}$. By Theorem A (expansion theorem), A has a fixed point \tilde{u} with $1 \leq |\tilde{u}|_0 \leq \overline{R}$.

It follows that $\tilde{u}(t) \geq q(t) \geq \lambda CMq(t) \geq w(t)$, and so $u = \tilde{u} - w$ is a positive solution to (1.1), completing the proof of Theorem 2.1.

3. EXISTENCE WITH f SINGULAR

We now turn our attention to the problem (1.1) with f possibly singular. We make the following assumptions:

(A.5) $f : (r, R) \times (0, \infty) \rightarrow (0, \infty)$ is continuous.

(A.6) There exist positive constants C, α, β with $r < \alpha < \beta < R$, and $h \in L^1(\alpha, \beta)$, $h \geq 0, h \not\equiv 0$ such that

$$f(t, u) \geq h(t)$$

for $t \in (\alpha, \beta), u \leq C$.

(A.7) For each $\theta > 0$, there exists $p_\theta \in L^1(r, R)$ such that

$$f(t, u) \leq p_\theta(t), \quad t \in (r, R),$$

for every $u \in C[r, R]$ with $u(t) \geq \theta q(t)$, where $q(t)$ is given by Lemma 2.1.

We then have

Theorem 3.1. *Let (A.1), (A.2), (A.5)–(A.7) hold, and let $\lambda > 0$; then the problem (1.1) has a positive solution $u \in C^1[r, R] \cap C^2(r, R)$.*

In order to prove Theorem 3.1 we first recall:

Theorem B ([7]–[8]). *Let \mathbb{K} be a cone in a Banach space E ,*

$$D = \{u \in \mathbb{K} : r \leq \|u\| \leq R\}$$

and $A : D \rightarrow \mathbb{K}$ be a completely continuous operator such that

- (i) $u \in D, \lambda \in (0, 1), u = \lambda Au \Rightarrow \|u\| \neq R,$
- (ii) $u \in D, \lambda > 1, u = \lambda Au \Rightarrow \|u\| \neq r,$
- (iii) $\inf_{\|u\|=r} \|Au\| > 0.$

Then A has a fixed point in D .

Proof of Theorem 2.1. Let \mathbb{K} be the cone as in the proof of Theorem 2.1 and let

$$r_0 = \frac{1}{2} \min \left(\lambda \sup_{r \leq t \leq R} \int_\alpha^\beta K(t, s)h(s) ds, C, 1 \right),$$

$$R_0 = 2 \max \left(\lambda \sup_{r \leq t \leq R} \int_r^R K(t, s)p_1(s) ds, 1 \right),$$

where $K(t, s)$ is defined by (2.1).

Let $D = \{u \in \mathbb{K} : r_0 \leq |u|_0 \leq R_0\}$. For each $v \in D$, let $u = Av$ be the solution of

$$(p(t)u')' = -\lambda f(t, v),$$

$$au(r) - bp(r)u'(r) = 0,$$

$$cu(R) + dp(R)u'(R) = 0.$$

Note that u exists since $f(t, v) \leq p_{r_0}(t)$, by (A.7). By Lemma 2.1, $A : D \rightarrow \mathbb{K}$ and it can be verified that A is completely continuous. To apply Theorem B, it is sufficient to verify that

$$\begin{aligned} |Au|_0 &> r_0 && \text{for } u \in D \text{ with } |u|_0 = r_0, \\ |Au|_0 &< R_0 && \text{for } u \in D \text{ with } |u|_0 = R_0. \end{aligned}$$

Let $u \in D$ with $|u|_0 = r_0$. Then by (A.6)

$$Au(t) = \lambda \int_r^R K(t, s) f(s, u) ds \geq \lambda \int_\alpha^\beta K(t, s) h(s) ds, \quad t \in [r, R],$$

and so $|Au|_0 > r_0$.

Next, let $u \in D$ with $|u|_0 = R_0$. Then $u(t) \geq q(t)$ and by (A.7) we have

$$Au(t) \leq \lambda \int_r^R K(t, s) p_1(s) ds$$

and so $|Au|_0 < R_0$.

Thus A has a fixed point u which is a $C^1[r, R] \cap C^2(r, R)$ positive solution to (1.1), completing the proof of Theorem 3.1.

Remark 1. Condition (A.7) is satisfied if

$$(A.8) \quad \int_r^R f(t, \theta s(t)) dt < \infty$$

for every $\theta > 0$, where $s(t) = \min(t - r, R - t)$. In the case where $b, d > 0$, (A.7) is equivalent to

$$(A.9) \quad \int_r^R f(t, u) dt < \infty$$

for every $u > 0$.

Remark 2. In the case where $p(t) = 1$, the existence of a positive $C^1[r, R] \cap C^2(r, R)$ solution to (1.1) was studied in [4, 6, 9]. The result in [6], which extends the one in [9], requires that f satisfy (A.2), (A.5), (A.8), (A.9), $f(t, u) \rightarrow 0$ as $u \rightarrow \infty$ and $f(t, u) \rightarrow \infty$ as $u \rightarrow 0$ uniformly on compact subsets of (r, R) , and $f(t, u)$ be decreasing in u for each t . In [4], (A.8) and the limiting conditions of [6] were removed, provided a, b, c , and d are positive. Also, the result in [2] when applied to the problem (1.1) with $p(t) = 1$ requires that $f(t, u)$ be decreasing in u and f satisfy (A.7). Thus Theorem 3.1 unifies and extends the corresponding results in [2, 4, 6, 9].

REFERENCES

1. V. Anuradha and R. Shivaji, *A quadrature method for classes of multi-parameter two point boundary value problems*, *Applicable Anal.* (to appear).
2. J. V. Baxley, *Some singular nonlinear boundary value problems*, *Siam J. Math. Anal.* **22** (1991), 463–479. MR **92f**:34017
3. A. Castro and R. Shivaji, *Nonnegative solutions for a class of nonpositone problems*, *Proc. Roy. Soc. Edinburgh Sect. A* **108** (1988), 291–302. MR **90m**:34040
4. A. M. Fink, J. A. Gatica, G. E. Hernandez, and P. Waltman, *Approximation of solutions of singular second order boundary value problems*, *Siam J. Math. Anal.* **22** (1991), 440–462. MR **92f**:34016
5. X. Garaizar, *Existence of positive radical solutions for semilinear elliptic equations in the annulus*, *J. Differential Equations* **70** (1987), 69–92. MR **89f**:35019

6. J. A. Gatica, V. Oliver, and P. Waltman, *Singular nonlinear boundary value problems for second order ordinary differential equations*, J. Differential Equations **79** (1989), 62–78. MR **90f**:34030
7. G. B. Gustafson and K. Schmitt, *Method of nonlinear analysis in the theory of differential equations*, Lecture notes, University of Utah, 1975.
8. M. A. Krasnosel'skii, *Positive solutions of operator equations*, Noordhoff, Gronigen, the Netherlands, 1964. MR **31**:6107
9. S. Taliaferro, *A nonlinear singular boundary value problem*, Nonlinear Anal. **3** (1979), 897–904. MR **81i**:34011

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