

COUNTEREXAMPLE TO A PROBLEM OF GEOGHEGAN-WEST

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ABSTRACT. Let X be a Banach space and $GL(X)$ its general linear group. Let $\|\cdot\|$ denote the operator norm and “ w ” the pointwise convergence topology on $GL(X)$. Is the identity map $(GL(X), \|\cdot\|) \rightarrow (GL(X), w)$ a homotopy equivalence? The answer is negative. One of the possible counterexamples is a well-known James space \mathbb{J} —the “space of counterexamples in Banach spaces theory”.

1.

We start with the problem raised in LS 16 (79 LS 15) (see [1, 2]).

Let X be a Banach space and $GL(X)$ its general linear group. Let $\|\cdot\|$ denote the operator norm and “ w ” the pointwise convergence topology on $GL(X)$. Is the identity map $(GL(X), \|\cdot\|) \rightarrow (GL(X), w)$ a homotopy equivalence?

The answer is negative. One of the possible counterexamples is a well-known James space \mathbb{J} —the “space of counterexamples in Banach spaces theory” [5]:

$$\mathbb{J} = \left\{ x = (x_n)_{n=1}^{\infty} : x_n \in \mathbb{R}, \lim_n x_n = 0, \right. \\ \left. \|x\|^2 = \sup_{p(1) < \dots < p(m)} \sum_{i=1}^{m-1} (x_{p(i+1)} - x_{p(i)})^2 < \infty \right\}$$

where supremum is taken over all finite sets of indices $p = \{p(1), \dots, p(m)\} \subset \mathbb{N}$ and over all $m \in \mathbb{N}$.

The group $(GL(\mathbb{J}), \|\cdot\|)$ is homotopically equivalent to the group $GL(\mathbb{R}) = \mathbb{R} \setminus \{0\}$ (see [3]). In this article we show that $(GL(\mathbb{J}), w)$ is a connected topological space. We would remind the reader that the general linear group is not a topological group in pointwise convergence topology.

Theorem 1. *The topological space $(GL(\mathbb{J}), w)$ is an arcwise connected space.*

This theorem has a generalization for a large class of Banach spaces (see section 3 below).

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2.

The space \mathbb{J} has a Schauder basis:

$$e_1 = (1, 0, 0, \dots), e_2 = (0, 1, 0, 0, \dots), \dots$$

We define projections P_m as follows:

$$P_m \left(\sum_{n=1}^{\infty} x_n e_n \right) = \sum_{n=1}^m x_n e_n, \quad P_m : \mathbb{J} \rightarrow \mathbb{J}.$$

Define Q_m by $Q_m = 1|_{\mathbb{J}} - P_m$. Then $w\text{-}\lim_m P_m = \mathbf{1}$ or, more precisely,

$$\lim_m \|P_m x - x\| = 0$$

for any x from \mathbb{J} .

Next, we define the right R and left L translations:

$$R: (x_1, x_2, x_3, \dots) \mapsto (0, x_1, x_2, x_3, \dots),$$

$$L: (x_1, x_2, x_3, \dots) \mapsto (x_2, x_3, x_4, \dots).$$

Then $\|R^n\| \leq 2$ for any $n \in \mathbb{N}$; $\lim_n L^n x = 0$ for any $x \in \mathbb{J}$ or, equivalently, $w\text{-}\lim_n L^n = 0$.

We begin a proof of Theorem 1 with the following lemma.

Lemma 1. *Let*

$$T_n(e_i) = \begin{cases} e_i & \text{if } i \neq n, i \neq n + 1, \\ e_{n+1} & \text{if } i = n, \\ e_n & \text{if } i = n + 1. \end{cases}$$

Then there exists a continuous mapping

$$\alpha : [0, +\infty] \rightarrow (\text{GL}(\mathbb{J}), w)$$

such that $\alpha_n = \alpha(n) = T_{n+1}$ for any $n \in \mathbb{N}$ and $\alpha(\infty) = \mathbf{1}|_{\mathbb{J}}$.

Proof. Let $t \in [0, 1]$. We define the operator α_t :

- $\alpha_t(e_n) = e_n$ for any $n > 3$;
- on $\text{span}\{e_1, e_2, e_3\}$ the operator α_t has a matrix

$$m_t = \frac{1}{t^2 - t + 1} \begin{pmatrix} t & 1 - t & t(t - 1) \\ 1 - t & t(t - 1) & t \\ t(t - 1) & t & 1 - t \end{pmatrix}.$$

It is easy to check that $\alpha_t \in L(\mathbb{J})$, $m_t^2 = E \in \text{Mat}(3 \times 3)$, i.e., $\alpha_t \in \text{GL}(\mathbb{J})$, α_t is a continuous mapping (in fact, in uniform topology) and

$$m_0 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{i.e., } \alpha_0 = T_1,$$

$$m_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \text{i.e., } \alpha_1 = T_2.$$

In an analogous manner we may connect T_2 with T_3 , T_3 with T_4 , and so on. The analytic expression of such an analogy may be written in the form

$$t \in [n, n + 1) \Rightarrow \alpha_t = P_n + R^n \cdot \alpha_{t-n} \cdot L^n.$$

If $t_0 \in [0, +\infty] \setminus \{0, 1, 2, \dots\}$, then a continuity α at the point t_0 is a consequence of a continuity α at the point $t_0 - [t_0] \in [0, 1)$.

If $t \rightarrow n - 0$, then

$$\alpha_t \rightarrow P_{n-1} + R^{n-1}\alpha_1L^{n-1} = P_{n-1} + R^{n-1}T_2L^{n-1} = T_{n+1}.$$

If $t \rightarrow n + 0$, then

$$\alpha_t \rightarrow P_n + R^n\alpha_0L^n = P_n + R^nT_1L^n = T_{n+1}.$$

Now, if $t_0 = \infty$, then for any $x \in \mathbb{J}$ and for any $t \in [n, n + 1)$

$$\begin{aligned} \|\alpha_\infty(x) - \alpha_t(x)\| &= \|x - P_nx - R^n\alpha_{t-n}L^n x\| \\ &\leq \|x - P_nx\| + \sup_n \|R^n\| \cdot \max_{\tau \in [0,1]} \{\|\alpha_\tau\|\} \cdot \|L^n x\| \xrightarrow{t \rightarrow \infty} 0. \end{aligned}$$

Note that any compact (in w-topology) set of operators is a norm bounded set of operators. Thus Lemma 1 is proved. \square

Lemma 2. *Let $A \in GL(\mathbb{J})$ and $A_1 = P_1 + R \cdot A \cdot L$. Then $A_1 \in GL(\mathbb{J})$ and there exists a continuous mapping $\beta: [0, \infty] \rightarrow (GL(\mathbb{J}), w)$ such that $\beta(0) = A_0 = A$ and $\beta(\infty) = A_1$.*

Proof. If A^{-1} is an inverse of A , then

$$\begin{aligned} &(P_1 + R \cdot A \cdot L) \cdot (P_1 + R \cdot A^{-1} \cdot L) \\ &= P_1^2 + (P_1R)A^{-1}L + RA(LP_1) + RA(LR)A^{-1}L = P_1 + RL = \mathbf{1} \end{aligned}$$

because $P_1R = LP_1 = 0, LR = \mathbf{1}$. Therefore $A_1 \in GL(\mathbb{J})$.

If (a_{ij}) is a matrix of the operator A in the basis e_1, e_2, e_3, \dots , then the operator A_1 has the following matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & a_{11} & a_{12} & a_{13} & \dots \\ 0 & a_{21} & a_{22} & a_{23} & \dots \\ 0 & a_{31} & a_{32} & a_{33} & \dots \\ \cdot & \cdot & \cdot & \cdot & \dots \end{pmatrix}, \quad \mathbb{J} = \text{Im}P_1 \oplus \text{Ker}P_1.$$

Let α be a mapping $\alpha: [0, \infty] \rightarrow GL(\mathbb{J})$ from Lemma 1. Define, for any $t \in [0, 1]$,

$$\beta_t^0 = \alpha_{s(t)} \cdot A_1 \cdot \alpha_{s(t)}$$

where $s(t) = (1 - t)/t, s: [0, 1] \rightarrow [0, \infty]$.

Then β^0 is a continuous mapping because the multiplication in the w-topology is a continuous operation for factors from a norm bounded set of operators. Furthermore $\beta_0^0 = A_1$ and $\beta_1^0 = T_1A_1T_1$.

Next we consider a mapping $\beta^1: [1, 2] \rightarrow GL(\mathbb{J})$,

$$\beta_t^1 = \alpha_{s(t)} \cdot \beta_1^0 \cdot \alpha_{s(t)}$$

where $s(t) = 1 + (2 - t)/(t - 1), s: [1, 2] \rightarrow [1, \infty]$. Then β^1 is a continuous mapping and

$$\begin{aligned} \beta_1^1 &= \alpha_\infty\beta_1^0\alpha_\infty = \beta_1^0, \\ \beta_2^1 &= T_2T_1A_1T_1T_2. \end{aligned}$$

At the n th step define β^n by the formula

$$\beta_t^n = \alpha_{s(t)} \cdot \beta_n^{n-1} \cdot \alpha_{s(t)}$$

where $s(t) = n + (n + 1 - t)/(t - n)$, $s: [n, n + 1] \rightarrow [n, \infty]$. We define a continuous mapping

$$\beta^n: [n, n + 1] \rightarrow \text{GL}(\mathbb{J})$$

with $\beta_n^n = \beta_n^{n-1}$ and $\beta_{n+1}^n = T_{n+1} \cdots T_2 T_1 A_1 T_1 T_2 \cdots T_{n+1}$. It is easy to see that

$$\beta_{n+1}^n = (P_{n+1} - P_n) + P_n A P_n + R Q_n A P_n + P_n A Q_n L + R Q_n A Q_n L$$

or, in the matrix form,

$$\beta_{n+1}^n = \begin{pmatrix} a_{11} & \cdots & a_{1n} & 0 & a_{1,n+1} & a_{1,n+2} & \cdots \\ & \cdots & & \cdots & & \cdots & \\ a_{n1} & \cdots & a_{nn} & 0 & a_{n,n+1} & a_{n,n+2} & \cdots \\ 0 & \cdots & 0 & 1 & 0 & 0 & \cdots \\ a_{n+1,1} & \cdots & a_{n+1,n} & 0 & a_{n+1,n+1} & a_{n+1,n+2} & \cdots \\ a_{n+2,1} & \cdots & a_{n+2,n} & 0 & a_{n+2,n+1} & a_{n+2,n+2} & \cdots \\ & \cdots & & \cdots & & \cdots & \end{pmatrix}.$$

We have $w\text{-}\lim_n \beta_n^{n-1} = A$, $w\text{-}\lim_t \alpha = \mathbf{1}$, and $w\text{-}\lim_t \beta_t^{[t]} = A$ for $n \rightarrow \infty$ and $t \rightarrow \infty$.

To end the proof of Lemma 2 we define

$$\beta: [0, \infty] \rightarrow \text{GL}(\mathbb{J})$$

by the equality $\beta_t = \beta_t^n$ for $n \leq t < n + 1$ and $\beta_\infty = A$. □

Proof of Theorem 1. Let $\beta: [0, \infty] \rightarrow \text{GL}(\mathbb{J})$ be a mapping from Lemma 2. Then we have a continuous mapping $\gamma: [0, 1] \rightarrow \text{GL}(\mathbb{J})$ with $\gamma_0 = A$ and $\gamma_1 = A_1$:

$$\gamma_t = \beta_{s(t)}, \quad \text{where } s(t) = (1 - t)/t, s: [0, 1] \rightarrow [0, \infty].$$

For $t \in [n, n + 1)$ we define

$$\gamma_t = P_n + R^n \cdot \gamma_{t-n} \cdot L^n.$$

The operator γ_n has the following matrix form:

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & & & \\ 0 & 1 & 0 & \cdots & 0 & 0 & & & \\ & & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & & \\ 0 & \cdots & & & 0 & a_{11} & a_{12} & a_{13} & \cdots \\ 0 & \cdots & & & 0 & a_{21} & a_{22} & a_{23} & \cdots \\ 0 & \cdots & & & 0 & a_{31} & a_{32} & a_{33} & \cdots \end{pmatrix}.$$

The proof that we have a continuous mapping

$$\gamma: [0, \infty] \rightarrow \text{GL}(\mathbb{J})$$

with $\gamma_0 = A$ and $\gamma_\infty = \mathbf{1}|_{\mathbb{J}}$ is completely analogous to the proof of Lemma 1. □

3.

The above construction may be generalized.

Definition. Let E and B be Banach spaces. We say that E is a B -divisible space iff:

(a) there exist disjoint projections $F_k: E \rightarrow E$ such that

$$\sum_{k=1}^{\infty} F_k = \mathbf{1}|_E;$$

(b) there exist isomorphisms $\tau_k: \text{Im}F_k \rightarrow B$ such that right and left translations R and L are continuous (in norm topology) operators:

$$R = \sum_{k=1}^{\infty} i_{k+1} \tau_{k+1}^{-1} \tau_k F_k, \quad L = \sum_{k=2}^{\infty} i_{k-1} \tau_{k-1}^{-1} \tau_k F_k$$

($i_k: \text{Im}F_k \subset E$ —identity inclusions);

(c) $\sup_n \|R^n\| < \infty$ and $\text{w-lim}_n L^n = 0$.

Theorem 2. Any norm-bounded subset of the group $\text{GL}(E)$ of isomorphisms of a Banach space E which is B -divisible for some B is contractible in $\text{GL}(E)$ to identity operator $\mathbf{1}|_E$ in pointwise convergence topology. Hence $\text{GL}(E)$ is a weak homotopy trivial set in this topology.

Examples of such Banach spaces E are: $c_0, l_p, L_p, C[0, 1], C^k[0, 1]$, Orlicz spaces l_M and L_M , and so on. In other words, all usual Banach spaces, with the exception of the space l_∞ , which is not known.

The definition of B -divisibility is similar to the definition of infinite divisibility (see [3]). In fact the construction of the proof of Theorem 1 is similar to the construction of Wong [4]. More precisely it is an analogue of the construction of Wong in the category of Banach spaces.

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