COUNTEREXAMPLE TO A PROBLEM OF GEOGHEGAN-WEST

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Abstract. Let $X$ be a Banach space and $\text{GL}(X)$ its general linear group. Let $\| \cdot \|$ denote the operator norm and “$w$” the pointwise convergence topology on $\text{GL}(X)$. Is the identity map $\left( \text{GL}(X), \| \cdot \| \right) \rightarrow \left( \text{GL}(X), w \right)$ a homotopy equivalence? The answer is negative. One of the possible counterexamples is a well-known James space $\mathcal{J}$—the “space of counterexamples in Banach spaces theory”.

1.

We start with the problem raised in LS 16 (79 LS 15) (see [1, 2]). Let $X$ be a Banach space and $\text{GL}(X)$ its general linear group. Let $\| \cdot \|$ denote the operator norm and “$w$” the pointwise convergence topology on $\text{GL}(X)$. Is the identity map $\left( \text{GL}(X), \| \cdot \| \right) \rightarrow \left( \text{GL}(X), w \right)$ a homotopy equivalence?

The answer is negative. One of the possible counterexamples is a well-known James space $\mathcal{J}$—the “space of counterexamples in Banach spaces theory” [5]:

$$\mathcal{J} = \left\{ x = (x_n)_{n=1}^{\infty}: x_n \in \mathbb{R}, \lim_{n} x_n = 0, \right\}$$

$$\|x\|^2 = \sup_{p(1)<\cdots<p(m)} \sum_{i=1}^{m-1} (x_{p(i+1)} - x_{p(i)})^2 < \infty$$

where supremum is taken over all finite sets of indices $p = \{p(1), \ldots, p(m)\} \subset \mathbb{N}$ and over all $m \in \mathbb{N}$.

The group $\left( \text{GL}(\mathcal{J}), \| \cdot \| \right)$ is homotopically equivalent to the group $\text{GL}(\mathbb{R}) = \mathbb{R} \setminus \{0\}$ (see [3]). In this article we show that $\left( \text{GL}(\mathcal{J}), w \right)$ is a connected topological space. We would remind the reader that the general linear group is not a topological group in pointwise convergence topology.

Theorem 1. The topological space $\left( \text{GL}(\mathcal{J}), w \right)$ is an arcwise connected space.

This theorem has a generalization for a large class of Banach spaces (see section 3 below).
The space $J$ has a Schauder basis:
\[ e_1 = (1, 0, 0, \ldots), e_2 = (0, 1, 0, 0, \ldots), \ldots \]
We define projections $P_m$ as follows:
\[ P_m \left( \sum_{n=1}^{\infty} x_n e_n \right) = \sum_{n=1}^{m} x_n e_n, \quad P_m : J \to J. \]
Define $Q_m$ by $Q_m = 1_{J} - P_m$. Then $w\text{-lim } m P_m = 1$ or, more precisely,
\[ \lim_m \|P_m x - x\| = 0 \]
for any $x$ from $J$.

Next, we define the right $R$ and left $L$ translations:
\[ R : (x_1, x_2, x_3, \ldots) \mapsto (0, x_1, x_2, x_3, \ldots), \]
\[ L : (x_1, x_2, x_3, \ldots) \mapsto (x_2, x_3, x_4, \ldots). \]
Then $\|R^n\| \leq 2$ for any $n \in \mathbb{N}$; $\lim_n L^n x = 0$ for any $x \in J$ or, equivalently, $w\text{-lim} L^n = 0$.

We begin a proof of Theorem 1 with the following lemma.

**Lemma 1.** Let
\[ T_n(e_i) = \begin{cases} e_i & \text{if } i \neq n, i \neq n + 1, \\ e_{n+1} & \text{if } i = n, \\ e_n & \text{if } i = n + 1. \end{cases} \]
Then there exists a continuous mapping
\[ \alpha : [0, +\infty] \to (\text{GL}(J), w) \]
such that $\alpha_n = \alpha(n) = T_{n+1} for any n \in \mathbb{N}$ and $\alpha(\infty) = 1_{J}$.

**Proof.** Let $t \in [0, 1]$. We define the operator $\alpha_t$:
- $\alpha_t(e_n) = e_n$ for any $n > 3$;
- on span$\{e_1, e_2, e_3\}$ the operator $\alpha_t$ has a matrix
\[ m_t = \begin{pmatrix} 1 & 1 & 0 \\ t & 1 - t & t(t-1) \\ t(t-1) & t & 1 - t \end{pmatrix}. \]
It is easy to check that $\alpha_t \in L(J)$, $m_t^2 = E \in \text{Mat}(3 \times 3)$, i.e., $\alpha_t \in \text{GL}(J)$, $\alpha_t$ is a continuous mapping (in fact, in uniform topology) and
\[ m_0 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{i.e., } \alpha_0 = T_1, \]
\[ m_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \text{i.e., } \alpha_1 = T_2. \]
In an analogous manner we may connect $T_2$ with $T_3, T_3$ with $T_4, \ldots$. The analytic expression of such an analogy may be written in the form
\[ t \in [n, n+1] \Rightarrow \alpha_t = P_n + R^n \cdot \alpha_{t-n} \cdot L^n. \]
If $t_0 \in [0, +\infty) \setminus \{0, 1, 2, \ldots\}$, then a continuity $\alpha$ at the point $t_0$ is a consequence of a continuity $\alpha$ at the point $t_0 - [t_0] \in [0, 1)$.

If $t \to n - 0$, then 
\[ \alpha_t \to P_{n-1} + R^{n-1}\alpha_1L^{n-1} = P_{n-1} + R^{n-1}T_2L^{n-1} = T_{n+1}. \]

If $t \to n + 0$, then 
\[ \alpha_t \to P_n + R^n\alpha_0L^n = P_n + R^nT_1L^n = T_{n+1}. \]

Now, if $t_0 = \infty$, then for any $x \in \mathcal{J}$ and for any $t \in [n, n + 1)$ 
\[
\|\alpha_\infty(x) - \alpha_t(x)\| = \|x - P_nx - R^n\alpha_{t-n}L^n x\| \\
\leq \|x - P_nx\| + \sup_n \|R^n\| \cdot \max_{\tau \in [0,1]} \{\|\alpha_\tau\|\} \cdot \|L^n x\| \to 0.
\]

Note that any compact (in w-topology) set of operators is a norm bounded set of operators. Thus Lemma 1 is proved. \(\square\)

**Lemma 2.** Let $A \in \text{GL}(\mathcal{J})$ and $A_1 = P_1 + R \cdot A \cdot L$. Then $A_1 \in \text{GL}(\mathcal{J})$ and there exists a continuous mapping $\beta : [0, \infty] \to (\text{GL}(\mathcal{J}), w)$ such that $\beta(0) = A_0 = A$ and $\beta(\infty) = A_1$.

**Proof.** If $A^{-1}$ is an inverse of $A$, then 
\[
(P_1 + R \cdot A \cdot L) \cdot (P_1 + R \cdot A^{-1} \cdot L) = P_1^2 + (P_1 R)L^{-1}L + RA(LP_1) + RA(LR)A^{-1}L = P_1 + RL = 1
\]
because $P_1 R = LP_1 = 0, LR = 1$. Therefore $A_1 \in \text{GL}(\mathcal{J})$.

If $(a_{ij})$ is a matrix of the operator $A$ in the basis $e_1, e_2, e_3, \ldots$, then the operator $A_1$ has the following matrix:
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & \cdots \\
0 & a_{11} & a_{12} & a_{13} & \cdots \\
0 & a_{21} & a_{22} & a_{23} & \cdots \\
0 & a_{31} & a_{32} & a_{33} & \cdots \\
& \ddots & \ddots & \ddots & \cdots
\end{pmatrix}, \quad \mathcal{J} = \text{Im}P_1 \oplus \text{Ker}P_1.
\]

Let $\alpha$ be a mapping $\alpha : [0, \infty] \to \text{GL}(\mathcal{J})$ from Lemma 1. Define, for any $t \in [0, 1]$,
\[ \beta_t^0 = \alpha_s(t) \cdot A_1 \cdot \alpha_s(t) \]
where $s(t) = (1 - t)/t, s : [0, 1] \to [0, \infty]$.

Then $\beta^0$ is a continuous mapping because the multiplication in the w-topology is a continuous operation for factors from a norm bounded set of operators. Furthermore $\beta_0^0 = A_1$ and $\beta_{1}^0 = T_1 A_1 T_1$.

Next we consider a mapping $\beta^1 : [1, 2] \to \text{GL}(\mathcal{J})$,
\[ \beta_t^1 = \alpha_s(t) \cdot \beta_0^0 \cdot \alpha_s(t) \]
where $s(t) = 1 + (2 - t)/(t - 1), s : [1, 2] \to [1, \infty]$. Then $\beta^1$ is a continuous mapping and
\[ \beta_1^1 = \alpha_\infty \beta_0^0 \alpha_\infty = \beta_1^0, \]
\[ \beta_2^1 = T_2 T_1 A_1 T_1 T_2. \]

At the $n$th step define $\beta^n$ by the formula
\[ \beta_t^n = \alpha_s(t) \cdot \beta_{n-1}^0 \cdot \alpha_s(t) \]
where \( s(t) = n + (n+1-t)/(t-n), \) \( s: [n, n+1] \to [n, \infty] \). We define a continuous mapping

\[
\beta^n: [n, n+1] \to \text{GL}(J)
\]

with \( \beta^n_n = \beta^n_{n-1} \) and \( \beta^n_{n+1} = T_{n+1} \cdots T_2 T_1 A_1 T_1 T_2 \cdots T_{n+1} \). It is easy to see that

\[
\beta^n_{n+1} = (P_{n+1} - P_n) + P_n A P_n + R Q_n A P_n + P_n A Q_n L + R Q_n A Q_n L
\]
or, in the matrix form,

\[
\beta^n_{n+1} = \begin{pmatrix}
  a_{11} & \cdots & a_{1n} & 0 & a_{1,n+1} & a_{1,n+2} & \cdots \\
  \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  a_{n1} & \cdots & a_{nn} & 0 & a_{n,n+1} & a_{n,n+2} & \cdots \\
  0 & \cdots & 0 & 1 & 0 & 0 & \cdots \\
  a_{n+1,1} & \cdots & a_{n+1,n} & 0 & a_{n+1,n+1} & a_{n+1,n+2} & \cdots \\
  a_{n+2,1} & \cdots & a_{n+2,n} & 0 & a_{n+2,n+1} & a_{n+2,n+2} & \cdots
\end{pmatrix}.
\]

We have \( \text{w- lim}_n \beta^n_{n-1} = A, \) \( \text{w- lim}_t \alpha = 1, \) and \( \text{w- lim}_t \beta^n_t = A \) for \( n \to \infty \) and \( t \to \infty \).

To end the proof of Lemma 2 we define

\[
\beta: [0, \infty] \to \text{GL}(J)
\]

by the equality \( \beta_t = \beta^n_t \) for \( n \leq t < n+1 \) and \( \beta_\infty = A \).

**Proof of Theorem 1.** Let \( \beta: [0, \infty] \to \text{GL}(J) \) be a mapping from Lemma 2. Then we have a continuous mapping \( \gamma: [0, 1] \to \text{GL}(J) \) with \( \gamma_0 = A \) and \( \gamma_1 = A_1 \):

\[
\gamma_t = \beta_{s(t)}, \ 	ext{where} \ s(t) = (1-t)/t, \ s: [0, 1] \to [0, \infty].
\]

For \( t \in [n, n+1] \) we define

\[
\gamma_t = P_n + R^n \cdot \gamma_{t-n} \cdot L^n.
\]

The operator \( \gamma_n \) has the following matrix form:

\[
\begin{pmatrix}
  1 & 0 & 0 & \cdots & 0 & 0 \\
  0 & 1 & 0 & \cdots & 0 & 0 \\
  \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & 0 & \cdots & 0 & 1 \\
  0 & \cdots & 0 & a_{11} & a_{12} & a_{13} & \cdots \\
  0 & \cdots & 0 & a_{21} & a_{22} & a_{23} & \cdots \\
  0 & \cdots & 0 & a_{31} & a_{32} & a_{33} & \cdots
\end{pmatrix}.
\]

The proof that we have a continuous mapping

\[
\gamma: [0, \infty] \to \text{GL}(J)
\]

with \( \gamma_0 = A \) and \( \gamma_\infty = 1|J \) is completely analogous to the proof of Lemma 1. \( \square \)
3.

The above construction may be generalized.

**Definition.** Let $E$ and $B$ be Banach spaces. We say that $E$ is a $B$-divisible space iff:

(a) there exist disjoint projections $F_k: E \to E$ such that
$$\sum_{k=1}^{\infty} F_k = 1|_E;$$

(b) there exist isomorphisms $\tau_k : \text{Im} F_k \to B$ such that right and left translations $R$ and $L$ are continuous (in norm topology) operators:
$$R = \sum_{k=1}^{\infty} i_{k+1} \tau_{k+1}^{-1} \tau_k F_k, \quad L = \sum_{k=2}^{\infty} i_{k-1} \tau_{k-1}^{-1} \tau_k F_k$$

($i_k : \text{Im} F_k \subset E$—identity inclusions);

(c) $\sup_n ||R^n|| < \infty$ and $\text{w-lim}_n L^n = 0$.

**Theorem 2.** Any norm-bounded subset of the group $\text{GL}(E)$ of isomorphisms of a Banach space $E$ which is $B$-divisible for some $B$ is contractible in $\text{GL}(E)$ to identity operator $1|_E$ in pointwise convergence topology. Hence $\text{GL}(E)$ is a weak homotopy trivial set in this topology.

Examples of such Banach spaces $E$ are: $c_0, l_p, L_p, C[0, 1], C^k[0, 1],$ Orlicz spaces $l_M$ and $L_M$, and so on. In other words, all usual Banach spaces, with the exception of the space $l_\infty$, which is not known.

The definition of $B$-divisibility is similar to the definition of infinite divisibility (see [3]). In fact the construction of the proof of Theorem 1 is similar to the construction of Wong [4]. More precisely it is an analogue of the construction of Wong in the category of Banach spaces.

**References**


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