COUNTEREXAMPLE TO A PROBLEM OF GEOGHEGAN-WEST

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(Communicated by James E. West)

Abstract. Let \( X \) be a Banach space and \( \text{GL}(X) \) its general linear group. Let \( \| \cdot \| \) denote the operator norm and “\( w \)” the pointwise convergence topology on \( \text{GL}(X) \). Is the identity map \( \text{GL}(X), \| \cdot \| \to \text{GL}(X), w \) a homotopy equivalence? The answer is negative. One of the possible counterexamples is a well-known James space \( J \)—the “space of counterexamples in Banach spaces theory”.

1.

We start with the problem raised in LS 16 (79 LS 15) (see [1, 2]). Let \( X \) be a Banach space and \( \text{GL}(X) \) its general linear group. Let \( \| \cdot \| \) denote the operator norm and “\( w \)” the pointwise convergence topology on \( \text{GL}(X) \). Is the identity map \( \text{GL}(X), \| \cdot \| \to \text{GL}(X), w \) a homotopy equivalence?

The answer is negative. One of the possible counterexamples is a well-known James space \( J \)—the “space of counterexamples in Banach spaces theory” [5]:

\[
J = \left\{ x = (x_n)_{n=1}^\infty : x_n \in \mathbb{R}, \lim_n x_n = 0, \|x\|^2 = \sup_{p(1)<\cdots<p(m)} \sum_{i=1}^{m-1} (x_{p(i+1)} - x_{p(i)})^2 < \infty \right\}
\]

where supremum is taken over all finite sets of indices \( p = \{p(1), \ldots, p(m)\} \subset \mathbb{N} \) and over all \( m \in \mathbb{N} \).

The group \( \text{GL}(J), \| \cdot \| \) is homotopically equivalent to the group \( \text{GL}(\mathbb{R}) = \mathbb{R}\setminus\{0\} \) (see [3]). In this article we show that \( \text{GL}(J), w \) is a connected topological space. We would remind the reader that the general linear group is not a topological group in pointwise convergence topology.

**Theorem 1.** The topological space \( \text{GL}(J), w \) is an arcwise connected space.

This theorem has a generalization for a large class of Banach spaces (see section 3 below).
2.

The space $J$ has a Schauder basis:

$$e_1 = (1, 0, 0, \ldots), e_2 = (0, 1, 0, 0, \ldots), \ldots$$

We define projections $P_m$ as follows:

$$P_m \left( \sum_{n=1}^{\infty} x_n e_n \right) = \sum_{n=1}^{m} x_n e_n, \quad P_m : J \to J.$$ 

Define $Q_m$ by

$$Q_m = 1_{J} - P_m.$$ 

Then $w\text{-lim}_m P_m = 1$ or, more precisely,

$$\lim_m \|P_m x - x\| = 0$$

for any $x$ from $J$.

Next, we define the right $R$ and left $L$ translations:

$$R : (x_1, x_2, x_3, \ldots) \mapsto (0, x_1, x_2, x_3, \ldots),$$

$$L : (x_1, x_2, x_3, \ldots) \mapsto (x_2, x_3, x_4, \ldots).$$

Then $\|R^n\| \leq 2$ for any $n \in \mathbb{N}$; $\lim_n L^n x = 0$ for any $x \in J$ or, equivalently, $w\text{-lim}_n L^n = 0$.

We begin a proof of Theorem 1 with the following lemma.

**Lemma 1.** Let

$$T_n(e_i) = \begin{cases} e_i & \text{if } i \neq n, i \neq n + 1, \\ e_{n+1} & \text{if } i = n, \\ e_n & \text{if } i = n + 1. \end{cases}$$

Then there exists a continuous mapping

$$\alpha : [0, +\infty] \to (GL(J), w)$$

such that $\alpha_n = \alpha(n) = T_{n+1}$ for any $n \in \mathbb{N}$ and $\alpha(\infty) = 1_{J}$.

**Proof.** Let $t \in [0, 1]$. We define the operator $\alpha_t$:

- $\alpha_t(e_n) = e_n$ for any $n > 3$;
- on span{$e_1, e_2, e_3$} the operator $\alpha_t$ has a matrix

$$m_t = \frac{1}{t^2 - t + 1} \begin{pmatrix} t & 1-t & t(t-1) \\ 1-t & t(t-1) & t \\ t(t-1) & t & 1-t \end{pmatrix}.$$ 

It is easy to check that $\alpha_t \in L(J)$, $m_t^2 = E \in \text{Mat}(3 \times 3)$, i.e., $\alpha_t \in GL(J)$, $\alpha_t$ is a continuous mapping (in fact, in uniform topology) and

$$m_0 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{i.e., } \alpha_0 = T_1,$$

$$m_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \text{i.e., } \alpha_1 = T_2.$$ 

In an analogous manner we may connect $T_2$ with $T_3$, $T_3$ with $T_4$, and so on. The analytic expression of such an analogy may be written in the form

$$t \in [n, n+1] \Rightarrow \alpha_t = P_n + R^n \cdot \alpha_{t-n} \cdot L^n.$$
If \( t_0 \in [0, +\infty) \setminus \{0, 1, 2, \ldots\} \), then a continuity \( \alpha \) at the point \( t_0 \) is a consequence of a continuity \( \alpha \) at the point \( t_0 - [t_0] \in [0, 1) \).

If \( t \to n - 0 \), then

\[
\alpha_t \to P_{n-1} + R^{n-1} \alpha_1 L^{n-1} = P_{n-1} + R^{n-1} T_2 L^{n-1} = T_{n+1}.
\]

If \( t \to n + 0 \), then

\[
\alpha_t \to P_n + R^n \alpha_0 L^n = P_n + R^n T_1 L^n = T_{n+1}.
\]

Now, if \( t_0 = \infty \), then for any \( x \in J \) and for any \( t \in [n, n + 1) \)

\[
\| \alpha_\infty(x) - \alpha_t(x) \| = \| x - P_n x - R^n \alpha_{t-n} L^n x \|
\leq \| x - P_n x \| + \sup_n \| R^n \| \cdot \max_{\tau \in [0,1]} \{ \| \alpha_\tau \| \} \cdot \| L^n x \| \to 0.
\]

Note that any compact (in w-topology) set of operators is a norm bounded set of operators. Thus Lemma 1 is proved. \( \square \)

**Lemma 2.** Let \( A \in \text{GL}(J) \) and \( A_1 = P_1 + R \cdot A \cdot L \). Then \( A_1 \in \text{GL}(J) \) and there exists a continuous mapping \( \beta : [0, \infty) \to (\text{GL}(J), w) \) such that \( \beta(0) = A_0 = A \) and \( \beta(\infty) = A_1 \).

**Proof.** If \( A^{-1} \) is an inverse of \( A \), then

\[
(P_1 + R \cdot A \cdot L) \cdot (P_1 + R \cdot A^{-1} \cdot L)
= P_1^2 + (P_1 R) A^{-1} L + RA(LP_1) + RA(LR) A^{-1} L = P_1 + RL = I
\]

because \( P_1 R = LP_1 = 0, LR = 1 \). Therefore \( A_1 \in \text{GL}(J) \).

If \((a_{ij})\) is a matrix of the operator \( A \) in the basis \( e_1, e_2, e_3, \ldots \), then the operator \( A_1 \) has the following matrix:

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & \cdots \\
0 & a_{11} & a_{12} & a_{13} & \cdots \\
0 & a_{21} & a_{22} & a_{23} & \cdots \\
0 & a_{31} & a_{32} & a_{33} & \cdots \\
& \ddots & \ddots & \ddots & \ddots
\end{pmatrix}, \quad J = \text{Im}P_1 \oplus \text{Ker}P_1.
\]

Let \( \alpha \) be a mapping \( \alpha : [0, \infty) \to \text{GL}(J) \) from Lemma 1. Define, for any \( t \in [0, 1] \),

\[
\beta^0_t = \alpha_{s(t)} \cdot A_1 \cdot \alpha_{s(t)}
\]

where \( s(t) = (1 - t)/t, s : [0, 1] \to [0, \infty] \).

Then \( \beta^0 \) is a continuous mapping because the multiplication in the w-topology is a continuous operation for factors from a norm bounded set of operators. Furthermore \( \beta^0_0 = A_1 \) and \( \beta^0_1 = T_1 A_1 T_1 \).

Next we consider a mapping \( \beta^1 : [1, 2] \to \text{GL}(J) \),

\[
\beta^1_t = \alpha_{s(t)} \cdot \beta^0_t \cdot \alpha_{s(t)}
\]

where \( s(t) = 1 + (2 - t)/(t - 1), s : [1, 2] \to [1, \infty] \). Then \( \beta^1 \) is a continuous mapping and

\[
\beta^1_1 = \alpha_\infty \beta^0_1 \alpha_\infty = \beta^0_1,
\]

\[
\beta^1_2 = T_2 T_1 A_1 T_1 T_2.
\]

At the \( n \)th step define \( \beta^n \) by the formula

\[
\beta^n_t = \alpha_{s(t)} \cdot \beta^{n-1}_t \cdot \alpha_{s(t)}
\]
where \( s(t) = n + (n + 1 - t)/(t - n), s: [n, n + 1] \to [n, \infty) \). We define a continuous mapping

\[
\beta^n: [n, n + 1] \to \text{GL}(J)
\]

with \( \beta^n_n = \beta^n_{n+1} = T_{n+1} \cdots T_2 T_1 A_1 T_2 \cdots T_{n+1} \). It is easy to see that

\[
\beta^n_{n+1} = (P_{n+1} - P_n) + P_n A P_n + RQ_n A P_n + P_n A Q_n L + RQ_n A Q_n L
\]

or, in the matrix form,

\[
\beta^n_{n+1} = \begin{pmatrix}
a_{11} & \cdots & a_{1n} & 0 & a_{1,n+1} & a_{1,n+2} & \cdots \\
\vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a_{n1} & \cdots & a_{nn} & 0 & a_{n,n+1} & a_{n,n+2} & \cdots \\
0 & \cdots & 0 & 1 & 0 & 0 & \cdots \\
a_{n+1,1} & \cdots & a_{n+1,n} & 0 & a_{n+1,n+1} & a_{n+1,n+2} & \cdots \\
a_{n+2,1} & \cdots & a_{n+2,n} & 0 & a_{n+2,n+1} & a_{n+2,n+2} & \cdots \\
\vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & a_{11} & a_{12} & a_{13} & \cdots \\
0 & \cdots & 0 & a_{21} & a_{22} & a_{23} & \cdots \\
0 & \cdots & 0 & a_{31} & a_{32} & a_{33} & \cdots \\
0 & \cdots & 0 & a_{41} & a_{42} & a_{43} & \cdots \\
\end{pmatrix}
\]

We have \( \text{w- lim}_n \beta^n_{n+1} = A \), \( \text{w- lim}_t \alpha = 1 \), and \( \text{w- lim}_t \beta^n_{t} = A \) for \( n \to \infty \) and \( t \to \infty \).

To end the proof of Lemma 2 we define

\[
\beta: [0, \infty] \to \text{GL}(J)
\]

by the equality \( \beta_t = \beta^n_t \) for \( n \leq t < n + 1 \) and \( \beta_{\infty} = A \).

Proof of Theorem 1. Let \( \beta: [0, \infty] \to \text{GL}(J) \) be a mapping from Lemma 2. Then we have a continuous mapping \( \gamma: [0, 1] \to \text{GL}(J) \) with \( \gamma_0 = A \) and \( \gamma_1 = A_1 \):

\[
\gamma_t = \beta_{s(t)}, \quad \text{where} \quad s(t) = (1 - t)/t, s: [0, 1] \to [0, \infty].
\]

For \( t \in [n, n + 1] \) we define

\[
\gamma_t = P_n + R^n \cdot \gamma_{t-n} \cdot L^n.
\]

The operator \( \gamma_n \) has the following matrix form:

\[
\begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 1 \\
0 & \cdots & 0 & a_{11} & a_{12} & a_{13} & \cdots \\
0 & \cdots & 0 & a_{21} & a_{22} & a_{23} & \cdots \\
0 & \cdots & 0 & a_{31} & a_{32} & a_{33} & \cdots \\
\end{pmatrix}.
\]

The proof that we have a continuous mapping

\[
\gamma: [0, \infty] \to \text{GL}(J)
\]

with \( \gamma_0 = A \) and \( \gamma_{\infty} = 1|_J \) is completely analogous to the proof of Lemma 1. \( \square \)
3.

The above construction may be generalized.

**Definition.** Let $E$ and $B$ be Banach spaces. We say that $E$ is a $B$-divisible space if:

(a) there exist disjoint projections $F_k : E \to E$ such that

$$\sum_{k=1}^{\infty} F_k = 1|_E;$$

(b) there exist isomorphisms $\tau_k : \text{Im} F_k \to B$ such that right and left translations $R$ and $L$ are continuous (in norm topology) operators:

$$R = \sum_{k=1}^{\infty} i_{k+1} \tau_{k+1}^{-1} \tau_k F_k, \quad L = \sum_{k=2}^{\infty} i_{k-1} \tau_{k-1}^{-1} \tau_k F_k$$

where $i_k : \text{Im} F_k \subset E$—identity inclusions);

(c) $\sup_n \|R^n\| < \infty$ and $\text{w-lim}_n L^n = 0$.

**Theorem 2.** Any norm-bounded subset of the group $\text{GL}(E)$ of isomorphisms of a Banach space $E$ which is $B$-divisible for some $B$ is contractible in $\text{GL}(E)$ to identity operator $1|_E$ in pointwise convergence topology. Hence $\text{GL}(E)$ is a weak homotopy trivial set in this topology.

Examples of such Banach spaces $E$ are: $c_0, l_p, L_p, C[0, 1], C^k[0, 1]$, Orlicz spaces $l_M$ and $L_M$, and so on. In other words, all usual Banach spaces, with the exception of the space $l_\infty$, which is not known.

The definition of $B$-divisibility is similar to the definition of infinite divisibility (see [3]). In fact the construction of the proof of Theorem 1 is similar to the construction of Wong [4]. More precisely it is an analogue of the construction of Wong in the category of Banach spaces.

**References**


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